A Profunctorial Finiteness Semantics

Species and Operads in Combinatorics and Semantics

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Quantitative semantics: programs as series

Girard 1988 : Normal functors: λ-terms $\longleftrightarrow$ power series

General idea:

- a program $P : A \Rightarrow B$ is interpreted as:

$$[P](x) = \sum_{n=0}^{+\infty} P_n \cdot x^n$$

  - P_n : nth-linearisation of P
  - n : number of times P_n uses the argument x

- a program that uses its argument once corresponds to a linear map

Ehrhard and Regnier 2003: Extension of linear logic and λ-calculus with differential constructions

Issues

- Functional types lead to infinite dimensional vector spaces
- The sums need not to converge

Enumerative Combinatorics

Combinatorial Species:

- Joyal 1986: Equivalence between species of structures and analytic functors
- Labelle 1986: Methods of resolution of differential equations of species of structures

Generalized Species:

- Fiore, Gambino, Hyland and Winskel 2008: Notion of generalized species of structure as a model of differential linear logic

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Combinatorial Species of Structure

category $\mathbb{B}$

Objects: finite sets
Morphisms: bijections

category $\mathbb{P}$

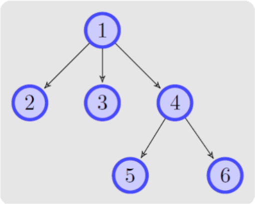
Objects: sets of the form $\underline{n} := \{1, \ldots n\}$
Morphisms: permutations

Definition

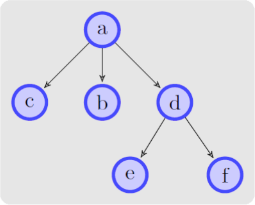
A *species of structure* is a functor $F : \mathbb{B} \to \mathbf{Set}$.

- Given a finite set of *labels* $\mathcal{U} \in \mathbb{B}$, an element $x \in F[\mathcal{U}]$ is called a *F-structure on $\mathcal{U}$*
- Given a bijection $\sigma : \mathcal{U} \xrightarrow{\sim} \mathcal{V} \in \mathbb{B}$, the bijection $F[\sigma] : F[\mathcal{U}] \xrightarrow{\sim} F[\mathcal{V}]$ is called the *transport of F-structures along σ*

$$\mathcal{U} = \{1, 2, 3, 4, 5, 6\} \quad \underset{\sigma}{\xrightarrow{\sim}} \quad \mathcal{V} = \{a, b, c, d, e, f\}$$



$\xrightarrow{\sim}$



Generating Series

Definition

A *generating series* of a species F is the formal series:

$$f : x \mapsto \sum_{n=0}^{+\infty} f_n \frac{x^n}{n!} \text{ where } f_n := |F[\underline{n}]| \qquad (\underline{n} := \{1, \ldots n\}).$$

- One $\mathbf{1} : \mathbb{B} \to \mathbf{Set}$
  $\mathcal{U} \mapsto \begin{cases} \{*\} & \text{if } \mathcal{U} = \emptyset \\ \emptyset & \text{otherwise} \end{cases}$
  generating series: $x \mapsto 1 + 0 \cdot x + 0 \cdot \frac{x^2}{2} + \cdots + 0 \cdot \frac{x^n}{n!} + \cdots = 1$
- Singleton $X : \mathbb{B} \to \mathbf{Set}$
  $\mathcal{U} \mapsto \begin{cases} \{*\} & \text{if } |\mathcal{U}| = 1 \\ \emptyset & \text{otherwise} \end{cases}$
  generating series: $x \mapsto 0 + 1 \cdot x + 0 \cdot \frac{x^2}{2} + \cdots + 0 \cdot \frac{x^n}{n!} + \cdots = x$

Examples

- Lists $L : \mathbb{B} \to \mathbf{Set}$

  $$\mathcal{U} \mapsto \{f : \underline{n} \xrightarrow{\sim} \mathcal{U} |\ n := |\mathcal{U}|\} \quad \text{(a list on } \mathcal{U} = \{a, b, c\} \text{ seen as } \{(1, a), (2, c), (3, b)\}\text{)}$$

  $\sigma : \mathcal{U} \to \mathcal{V} \mapsto (f \in L[\mathcal{U}] \mapsto \sigma \circ f)$

  generating series: $x \mapsto 1 + 1 \cdot x + 2 \cdot \frac{x^2}{2} + \cdots + n! \cdot \frac{x^n}{n!} + \cdots = \frac{1}{1-x}$

- Permutations $P : \mathbb{B} \to \mathbf{Set}$

  $$\mathcal{U} \mapsto \{f : \mathcal{U} \xrightarrow{\sim} \mathcal{U}\}$$

  $\sigma : \mathcal{U} \to \mathcal{V} \mapsto (f \in L[\mathcal{U}] \mapsto \sigma \circ f \circ \sigma^{-1})$

  generating series: $x \mapsto 1 + 1 \cdot x + 2 \cdot \frac{x^2}{2} + \cdots + n! \cdot \frac{x^n}{n!} + \cdots = \frac{1}{1-x}$

- Cycles $C : \mathbb{B} \to \mathbf{Set}$

  $$\mathcal{U} \mapsto \{f : \mathcal{U} \xrightarrow{\sim} \mathcal{U} |\ f \text{ is a cycle}\}$$

  $\sigma : \mathcal{U} \to \mathcal{V} \mapsto (f \in C[\mathcal{U}] \mapsto \sigma \circ f \circ \sigma^{-1})$

  generating series: $x \mapsto 0 + 1 \cdot x + 1 \cdot \frac{x^2}{2} + \cdots + (n-1)! \cdot \frac{x^n}{n!} + \cdots = \ln\left(\frac{1}{1-x}\right)$

Analytic Functors and Species of Structures

Definition

P : **Set** $\to$ **Set** is analytic if there exists a species $F : \mathbb{B} \to$ **Set** such that:

$$\forall X \in \mathbf{Set},\ P(X) \cong \sum_n \left(F[\underline{n}] \times X^n\right)/\mathfrak{S}_n$$

where $(x, f : \underline{n} \to X) \sim (F[\sigma](x), f \circ \sigma^{-1})$ for

$$\sigma \in \mathfrak{S}_n \text{ and } (x, f : \underline{n} \to X) \in F[\underline{n}] \times X^n$$

Sets $E : \mathbb{B} \to$ **Set**

$$\mathcal{U} \mapsto \{*\}$$

- Generating series:
  $e : x \mapsto 1 + 1 \cdot x + 1 \cdot \frac{x^2}{2} + \cdots + 1 \cdot \frac{x^n}{n!} + \cdots = \sum_{n \in \mathbb{N}} \frac{x^n}{n!} = e^x$
- Analytic functor:
  $P_E : X \mapsto \{*\} + (\{*\} \times X) + (\{*\} \times X^2)/\mathfrak{S}_2 + \cdots + (\{*\} \times X^n)/\mathfrak{S}_n + \ldots \cong \sum_{n \in \mathbb{N}} X^n/\mathfrak{S}_n$

Theorem: Analytic Functors

Functors of the form

$$P = \sum_{i \in I} \mathbf{Set}(X_i, -)/G_i$$

- X_i: finite sets
- G_i: subgroups of $Aut(X_i)$

Functors isomorphic to

$$P(X) \cong \sum_n X^n \times F[\underline{n}]/\mathfrak{S}_n$$

F: species of structure

Functors preserving

- filtered colimits
- weak wide pullbacks

Operations on Species of Structures

- Addition:

$$(F + G)[\mathcal{U}] := F[\mathcal{U}] \sqcup G[\mathcal{U}]$$

- Multiplication:

$$(F \cdot G)[\mathcal{U}] := \sum_{\mathcal{V} \sqcup \mathcal{W} = \mathcal{U}} F[\mathcal{V}] \times G[\mathcal{W}]$$

- Composition:

$$(F \circ G)[\mathcal{U}] := \sum_{\pi \in Part(\mathcal{U})} F[\pi] \times \prod_{\mathcal{V} \in \pi} G[\mathcal{V}]$$

Example

$L \cong \mathbf{1} + X \cdot L$

- A list is either empty or is constituted of an element followed by a list
- $\frac{1}{1-x} = 1 + x \cdot \frac{1}{1-x}$

Derivative of Species

Definition

The *derivative* of a species of structures F is the species F' given by:

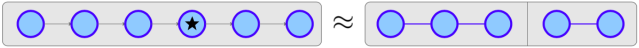
$$F'[\mathcal{U}] := F[\mathcal{U} \sqcup \{*\}]$$
$$F'[\sigma : \mathcal{U} \xrightarrow{\sim} \mathcal{V}] : F[\mathcal{U} \sqcup \{*\}] \xrightarrow{\sim} F[\mathcal{V} \sqcup \{*\}]$$
$$s \mapsto \begin{cases} \sigma(s) & \text{if } s \in \mathcal{U} \\ * & \text{if } s = * \end{cases}$$

Examples:

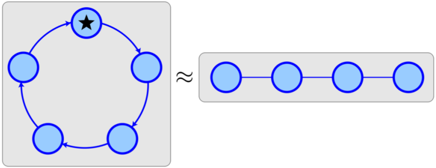
- $X' \cong \mathbf{1}$
  $(x)' = 1$
- $E' \cong E$
  $(e^x)' = e^x$

Examples

- $L' \cong L^2$
  - a list with a marked element $*$ determines a pair of lists
  - $\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$



- $C' \cong L$
  - a cycle with a marked element $*$ corresponds to a list
  - $\left(\ln\left(\frac{1}{1-x}\right)\right)' = \frac{1}{1-x}$



Relational model reminder

Definition

The *category of relations* (denoted **Rel**) is defined by

- **objects**: sets $A, B, \dots$
- **morphisms**: binary relations $\mathbf{Rel}(A, B) := \mathcal{P}(A \times B)$
- **identity**: $id_A := \{(a, a) \mid a \in A\} \in \mathbf{Rel}(A, A)$
- **composition**: for $\mathcal{R} \in \mathbf{Rel}(A, B)$ and $\mathcal{S} \in \mathbf{Rel}(B, C)$, $\mathcal{S} \circ \mathcal{R} \subseteq A \times C$ is defined by:

$$\mathcal{S} \circ \mathcal{R} := \{(a, c) \mid \exists b \in B, (a, b) \in \mathcal{R} \text{ and } (b, c) \in \mathcal{S}\}$$

or equivalently $(\mathcal{S} \circ \mathcal{R})_{(a,c)} = \bigvee_{b \in B} \mathcal{R}_{(a,b)} \wedge \mathcal{S}_{(b,c)}$

Rel as a model of linear logic

- Dual: $A^\perp := A$
- Tensor: $A \otimes B := A \times B$
- Unit: $\mathbb{1} := \{*\}$
- Linear implication: $A \multimap B := A \times B$
- Exponential: $!A := \mathcal{M}_{fin}(A)$ and for $\mathcal{R} \in \mathbf{Rel}(A, B)$,

$$!\mathcal{R} := \{([a_1, \dots, a_n], [b_1, \dots, b_n]) \mid \forall i \in \underline{n}, (a_i, b_i) \in \mathcal{R}\}$$

Key intuition: a morphism in $\mathbf{Rel}_!$ corresponds to the support of a power series.

Matrices and Normal Functors

Girard 1988, Hasegawa 2002

Definition

The *bicategory of matrices* (denoted **Mat**) with set-theoretic coefficients is defined by

objects sets $A, B, \ldots$

morphisms matrices $M \in \mathbf{Set}^{A \times B}$

identity $id_A : (a, a') \mapsto \delta_{a,a'} \in \mathbf{Set}^{A \times A}$

composition for $M \in \mathbf{Set}^{A \times B}$ and $N \in \mathbf{Set}^{B \times C}$,

$$N \circ M : (a, c) \mapsto \sum_{b \in B} M(a, b) \times N(b, c)$$

exponential $!A := \mathcal{M}_{fin}(A)$

Matrices and Normal Functors

Definition

A *normal functor* between two sets A and B is a functor $P : \mathbf{Set}^A \to \mathbf{Set}^B$ that preserves wide pullbacks and filtered colimits (i.e. a finitary polynomial functor).

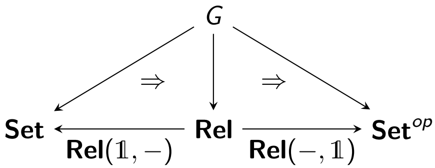
Matrices $M \in \mathbf{Set}^{\mathcal{M}_{fin}(A) \times B}$	$\longrightarrow$	Normal functors $P : \mathbf{Set}^A \to \mathbf{Set}^B$

Given a matrix M, the following functor $P : \mathbf{Set}^A \to \mathbf{Set}^B$ is normal :

$$P(X)_b \mapsto \sum_{m \in \mathcal{M}_{fin}(A)} M_{(m,b)} \times \mathbf{Set}^A(m, X)$$

Interactions in $\mathbf{Mat}_!$ are infinite

Relational Finiteness Spaces (short version)



The category **FinRel** of finiteness spaces is the tight orthogonality category (in the sense of Hyland-Schalk) obtained from **Rel** using the following orthogonality relation:

For $x \in \mathbf{Rel}(1, A) = \mathcal{P}(A)$ and $x' \in \mathbf{Rel}(A, 1) = \mathcal{P}(A)$,

$$x \perp x' \quad :\Leftrightarrow \quad x \cap x' \text{ is finite}$$

- Note: the standard model of coherence spaces is obtained by taking the orthogonality relation:

$$x \perp x' \quad :\Leftrightarrow \quad |x \cap x'| \leq 1$$

Relational Finiteness Spaces

For a countable set A and $\mathcal{F} \subseteq \mathcal{P}(A)$, define

$$\mathcal{F}^{\perp} := \{x \in \mathcal{P}(A) \mid \forall x' \in \mathcal{F}, x \perp x'\}$$

Definition

A relational *finiteness space* is a pair $(A, \mathcal{F}(A))$ where A is a countable set and $\mathcal{F}(A)$ is a subset of $\mathcal{P}(A)$ verifying $\mathcal{F}(A) = (\mathcal{F}(A))^{\perp\perp}$.

Elements of $\mathcal{F}(A)$ are called *finitary subsets* as they "behave" like finite sets:

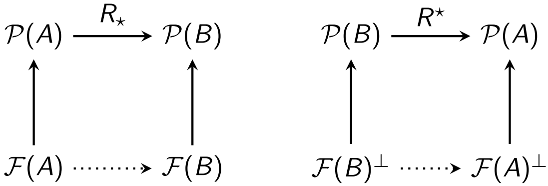
- closure under inclusion
- closure under finite unions

$\mathcal{P}_{fin}(A)$	$\subseteq$	$\mathcal{F}(A)$	$\subseteq$	$\mathcal{P}(A)$
smallest finiteness structure				largest finiteness structure

Relational Finiteness Spaces

For finiteness spaces $(A, \mathcal{F}(A))$ and $(B, \mathcal{F}(B))$, a relation $R \subseteq A \times B$ is *finitary* if:

- for all $x \in \mathcal{F}(A)$, $R_\star \cdot x \in \mathcal{F}(B)$
- for all $y \in \mathcal{F}(B)^\perp$, $R^\star \cdot y \in \mathcal{F}(A)^\perp$



Definition

The category **FinRel** has objects finiteness spaces and morphisms are relations that preserve the finitary structure.

Finiteness Spaces as a Model of Linear Logic

- Dual: $(A, \mathcal{F}(A))^\perp := (A, \mathcal{F}(A)^\perp)$
- Additives: $(A, \mathcal{F}(A)) \,\&\, (B, \mathcal{F}(B)) := (A \sqcup B, \mathcal{F}(A \oplus B))$ where $\mathcal{F}(A \,\&\, B) := \{x \sqcup y \mid x \in \mathcal{F}(A), y \in \mathcal{F}(B)\}$
- Linear implication: $A \multimap B := A \times B$ with $\mathcal{F}(A \multimap B) := \{R \subseteq A \times B \mid R \text{ is finitary}\}$
- Tensor: $(A, \mathcal{F}(A)) \otimes (B, \mathcal{F}(B)) := (A \times B, \mathcal{F}(A \otimes B))$ where
$$\mathcal{F}(A \otimes B) := \{x \times y \mid x \in \mathcal{F}(A), y \in \mathcal{F}(B)\}^{\perp\perp}$$
- Unit: $\mathbb{1} := (\{*\}, \mathcal{P}(\{*\}))$
- Exponential: $!(A, \mathcal{F}(A)) := (\mathcal{M}_{fin}(A), \mathcal{F}(!A))$ where
$$\mathcal{F}(!A) := \{\mathcal{M}_{fin}(x) \mid x \in \mathcal{F}(A)\}^{\perp\perp}$$

Linear Finiteness Spaces

Fix a ring $\mathcal{R}$, for a relational finiteness space $(A, \mathcal{F}(A))$, define the following vector space:

$$\mathcal{R}\langle A\rangle := \{X \in \mathcal{R}^A \mid \text{support}(X) \in \mathcal{F}(A)\}$$

Theorem (Ehrhard 2003)

$\mathcal{R}\langle A\rangle$ can be endowed with a topology $\mathcal{T}_A$ such that

- *A matrix $M \in \mathcal{R}\langle A \multimap B\rangle$ will correspond to a linear continuous map $\mathcal{R}\langle A\rangle \to \mathcal{R}\langle B\rangle$*
- *A matrix $M \in \mathcal{R}\langle !A \multimap B\rangle$ will correspond to an analytic map $\mathcal{R}\langle A\rangle \to \mathcal{R}\langle B\rangle$.*

We obtain a model of controlled non-determinism with iteration but no fixpoint.

From Relations to Profunctors

Recall

relations between two sets A and B ⟷ functions from A to $\mathcal{P}(B)$ ⟷ functions $A \times B \to 2$

Definition

Let $\mathbf{A}$ and $\mathbf{B}$ be two categories, a *profunctor* from $\mathbf{A}$ to $\mathbf{B}$ is a functor

$$F : \mathbf{A} \to \widehat{\mathbf{B}} \qquad \text{(also denoted } F : \mathbf{A} \nrightarrow \mathbf{B}\text{)}$$

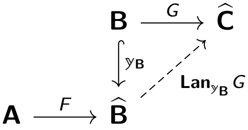
Profunctors between two categories $\mathbf{A}$ and $\mathbf{B}$ ⟷ functors $F : \mathbf{A} \to \widehat{\mathbf{B}}$ ⟷ functors $F : \mathbf{A} \times \mathbf{B}^{op} \to \mathbf{Set}$

Profunctor composition being not strictly associative, we need to work in the setting of bicategories

Bicategory of Profunctors

- **Objects:** small categories $\mathbf{A}, \mathbf{B}, \ldots$
- **1-cells:** profunctors $F : \mathbf{A} \nrightarrow \mathbf{B}$
- **2-cells:** natural transformations
- **Identity:** $id_{\mathbf{A}} : \mathbf{A} \to \widehat{\mathbf{A}}$ is the Yoneda embedding $a \mapsto \mathbf{Hom}_{\mathbf{A}}(-, a)$
- **Composition:** for $F : \mathbf{A} \nrightarrow \mathbf{B}$ and $G : \mathbf{B} \nrightarrow \mathbf{C}$,

$$G \circ F(a, c) = \int^{b \in \mathbf{B}} F(a, b) \times G(b, c)$$

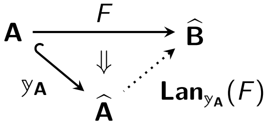


Linear Maps are Cocontinous Functors

Theorem

The bicategory of profunctors is biequivalent to the 2-category of cocontinous functors (functors preserving all colimits).

For a profunctor $F : \mathbf{A} \nrightarrow \mathbf{B}$, the functor $\mathbf{Lan}_{y_\mathbf{A}} F : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ is cocontinuous:



bicategory Prof

- 0-cells: small categories $\mathbf{A}$, $\mathbf{B}$
- 1-cells: profunctors $F : \mathbf{A} \nrightarrow \mathbf{B}$
- 2-cells: natural transformations

2-category Cocont

- 0-cells: small categories $\mathbf{A}$, $\mathbf{B}$
- 1-cells: cocontinuous functors $P : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$
- 2-cells: natural transformations

Generalized Species as a Model of Linear Logic

- **Tensor product:** $\mathbf{A} \otimes \mathbf{B} := \mathbf{A} \times \mathbf{B}$
- **Unit Object:** $\mathbb{1}$ (singleton category)
- **Dual:** $\mathbf{A}^{\perp} := \mathbf{A}^{op}$
- **Function Space:** $\mathbf{A} \multimap \mathbf{B} := \mathbf{A}^{op} \times \mathbf{B}$
- **Exponential:** $!\mathbf{A}$ is given by:
  - Objects: finite sequences $\langle a_1, \ldots, a_n \rangle$ of objects of $\mathbf{A}$.
  - Morphisms: pairs $(\sigma, (f_i)_{i \in \underline{n}}) : \langle a_1, \ldots, a_n \rangle \to \langle b_1, \ldots, b_n \rangle$ of a permutation $\sigma \in \mathfrak{S}_n$ and a finite sequence of morphisms $f_i : a_i \to b_{\sigma(i)}$ in $\mathbf{A}$.

Definition

Given $\mathbf{A}$ and $\mathbf{B}$ two small categories, an $(\mathbf{A}, \mathbf{B})$-*generalized species of structures* is a functor $\mathbb{F} : !\mathbf{A} \looparrowright \mathbf{B}$ (or equivalently a functor $\mathbb{F} : !\mathbf{A} \to \widehat{\mathbf{B}}$).

Examples

- Zero $\mathbf{0_A} : !\mathbf{A} \to\!\!\!\!\!+ \mathbf{A}$

$$(u, a) \mapsto \emptyset$$

Simple species:

$$\mathbf{0} : \mathbb{B} \to \mathbf{Set}$$
$$\mathcal{U} \mapsto \emptyset$$

- Singleton

$$X_{\mathbf{A}} : !\mathbf{A} \to\!\!\!\!\!+ \mathbf{A}$$
$$(u, a) \mapsto \mathbf{Hom}_{!\mathbf{A}}(\langle a \rangle, u)$$
$$\cong \begin{cases} \mathbf{Hom}_{\mathbf{A}}(a, a') & \text{if } u = \langle a' \rangle \\ \emptyset & \text{otherwise} \end{cases}$$

$$X : \mathbb{B} \to \mathbf{Set}$$
$$\mathcal{U} \mapsto \begin{cases} \{\mathcal{U}\} & \text{if } |\mathcal{U}| = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

- Lists

$$L_{\mathbf{A}} : !\mathbf{A} \to\!\!\!\!\!+ \mathbf{A}$$
$$(u, a) \mapsto \mathbf{Hom}_{!\mathbf{A}}(\langle a, \ldots, a \rangle, u)$$

$$L : \mathbb{B} \to \mathbf{Set}$$
$$\mathcal{U} \mapsto \{f : \underline{n} \xrightarrow{\sim} \mathcal{U}\} \quad (n := |\mathcal{U}|)$$

Operations on Generalized Species

$F, G : !\mathbf{A} \nrightarrow \mathbf{B}$, $H : !\mathbf{B} \nrightarrow \mathbf{C}$

- Addition:

$$(F + G)(u, b) := F(u, b) \sqcup G(u, b)$$

- Multiplication:

$$(F \cdot G)(u, b) := \int^{v,w \in !\mathbf{A}} F(v, b) \times G(w, b) \times \mathbf{Hom}_{!\mathbf{A}}(v \otimes w, u)$$

- Composition: $(H \circ G)(u, c) :=$

$$\int^{v=\langle b_1,\dots b_n\rangle \in !\mathbf{B}} H(v, b) \times \int^{u_1,\dots,u_n \in !\mathbf{A}} \prod_{i=1}^{n} G(u_i, b_i) \times \mathbf{Hom}_{!\mathbf{A}}(u_1 \otimes \cdots \otimes u_n, u)$$

Derivatives of Generalized Species of Structure

Definition

Given $F : !\mathbf{A} \nrightarrow \mathbf{B}$ a generalized species and a an object in $\mathbf{A}$, the *partial derivative of F wrt to a*, denoted $\frac{\delta F}{\delta a} : !\mathbf{A} \nrightarrow \mathbf{B}$ is defined by:

$$\begin{aligned} \frac{\delta F}{\delta a} &: !\mathbf{A} \nrightarrow \mathbf{B} \\ (u, b) &\mapsto F(u \otimes \langle a \rangle, b) \end{aligned}$$

Singleton $X_{\mathbf{A}} : (u, a) \mapsto Hom_{!\mathbf{A}}(\langle a \rangle, u)$

Differential species:
$$\begin{aligned} \frac{\delta X_A}{\delta a} &: !\mathbf{A} \nrightarrow \mathbf{A} \\ (u, a') &\mapsto Hom_{!\mathbf{A}}(\langle a' \rangle, u \otimes \langle a \rangle) \end{aligned}$$

Generalized Species and Analytic Functors

What is the series counterpart of generalized species?

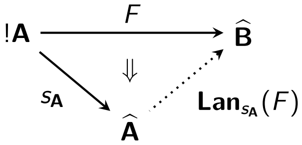
Definition

An generalized analytic functor between two small categories $\mathbf{A}$ and $\mathbf{B}$ is a functor

$$P : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$$

that preserves filtered colimits and weak wide pullbacks.

Given a generalized species $F : !\mathbf{A} \nrightarrow \mathbf{B}$, the functor $\mathbf{Lan}_{s_A} F : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ is analytic:



$$\text{where } s_{\mathbf{A}} : \langle a_1, \ldots a_n \rangle \mapsto \sum_{i=1}^{n} \mathbb{y}_{\mathbf{A}}(a_i)$$

Generalized Species and Analytic Functors

Theorem (Fiore 2013)

The bicategory of generalized species restricted to groupoids is equivalent to the 2-category of generalized analytic functors.

bicategory **Esp**		2-category **An**
0-cells: small groupoids $\mathbf{A}$, $\mathbf{B}$	$\longleftrightarrow$	0-cells: small groupoids $\mathbf{A}$,$\mathbf{B}$
1-cells: species $\mathbb{F} : !\mathbf{A} \nrightarrow \mathbf{B}$		1-cells: analytic functors $P : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$
2-cells: natural transformations		2-cells: weak cart. natural transformations

Orthogonality

Setting: we work with locally finite categories (homsets are finite)

- The yoneda embedding is valued in finite presheaves i.e. $\mathbb{y}_{\mathbf{A}} : \mathbf{A} \to [\mathbf{A}^{op}, \mathbf{FinSet}]$.
- Finitely presentable presheaves $[\mathbf{A}^{op}, \mathbf{Set}]_{fp}$ can be seen as a subcategory of finite presheaves $[\mathbf{A}^{op}, \mathbf{FinSet}]$.

Orthogonality relation

For $X \in \mathbf{Prof}(\mathbb{1}, \mathbf{A}) \cong [\mathbf{A}^{op}, \mathbf{Set}]$ and $X' \in \mathbf{Prof}(\mathbf{A}, \mathbb{1}) \cong [\mathbf{A}, \mathbf{Set}]$,

$$X' \perp X \quad :\Leftrightarrow \quad \int^{a \in \mathbf{A}} X'(a) \times X(a) \in \mathbf{FinSet}$$

Categorical Finiteness Spaces

Given a subcategory $\mathcal{F} \hookrightarrow [\mathbf{A}^{op}, \mathbf{FinSet}]$, let $\mathcal{F}^{\perp}$ be the full subcategory of $[\mathbf{A}, \mathbf{FinSet}]$ whose object set is

$$\{X' : \mathbf{A} \to \mathbf{FinSet} \mid \forall X \in \mathcal{F}, X' \perp X\}$$

Definition

A *categorical finiteness space* is a pair $(\mathbf{A}, \mathcal{F}(\mathbf{A}))$ of a locally finite category $\mathbf{A}$ and a full subcategory of $\mathcal{F}(\mathbf{A}) \hookrightarrow [\mathbf{A}^{op}, \mathbf{FinSet}]$ verifying $\mathcal{F}(\mathbf{A}) \cong (\mathcal{F}(\mathbf{A}))^{\perp\perp}$.

Finiteness Presheaves

Elements of $\mathcal{F}(\mathbf{A})$ are called *finiteness presheaves* and they "behave" like finitely presentable objects:

- closure under retractions: if X' is a retract of $X \in \mathcal{F}(\mathbf{A})$, then $X' \in \mathcal{F}(\mathbf{A})$.
- closure under finite colimits: a finite colimit of elements of $\mathcal{F}(\mathbf{A})$ is in $\mathcal{F}(\mathbf{A})$.

$[\mathbf{A}^{op}, \mathbf{Set}]_{fp}$	$\hookrightarrow$	$\mathcal{F}(\mathbf{A})$	$\hookrightarrow$	$[\mathbf{A}^{op}, \mathbf{FinSet}]$
smallest finiteness structure				largest finiteness structure

Finiteness Profuntors

Definition

For finiteness spaces $(\mathbf{A}, \mathcal{F}(\mathbf{A}))$ and $(\mathbf{B}, \mathcal{F}(\mathbf{B}))$, $F : \mathbf{A} \times \mathbf{B}^{op} \to \mathbf{FinSet}$ is a *finiteness profunctor* if $\mathbf{Lan}_{\mathbb{y}_{\mathbf{A}}}(F)$ can be factored as follows:

$$\begin{array}{ccc} \widehat{\mathbf{A}} & \xrightarrow{\mathbf{Lan}_{\mathbb{y}_{\mathbf{A}}}(F)} & \widehat{\mathbf{B}} \\ \uparrow & & \uparrow \\ \mathcal{F}(\mathbf{A}) & \dashrightarrow & \mathcal{F}(\mathbf{B}) \end{array}$$

Finiteness profunctors compose and we denote by **FinProf** the bicategory of finiteness spaces and finiteness profunctors.

FinProf as a Model of Linear Logic

- Dual: $(\mathbf{A}, \mathcal{F}(\mathbf{A}))^{\perp} := (\mathbf{A}^{op}, \mathcal{F}(\mathbf{A})^{\perp})$
- Additives: $(\mathbf{A}, \mathcal{F}(\mathbf{A})) \,\&\, (\mathbf{B}, \mathcal{F}(\mathbf{B})) := (\mathbf{A} + \mathbf{B}, \mathcal{F}(\mathbf{A} \,\&\, \mathbf{B}))$ where $\mathcal{F}(\mathbf{A} \,\&\, \mathbf{B}) := \{X + Y \mid A \in \mathcal{F}(\mathbf{A}), Y \in \mathcal{F}(\mathbf{B})\}$
- Tensor: $(\mathbf{A}, \mathcal{F}(\mathbf{A})) \otimes (\mathbf{B}, \mathcal{F}(\mathbf{B})) := (\mathbf{A} \times \mathbf{B}, \mathcal{F}(\mathbf{A} \otimes \mathbf{B}))$ where

$$\mathcal{F}(A \otimes B) := \{X \times Y \mid X \in \mathcal{F}(\mathbf{A}) \text{ and } Y \in \mathcal{F}(\mathbf{B})\}^{\perp\perp}$$

- Unit: $(\mathbb{1}, \mathbf{FinSet})$

FinProf as a Model of Linear Logic

For a finite presheaf $X : \mathbf{A}^{op} \to \mathbf{FinSet}$, its lifting $X^! : (!\mathbf{A})^{op} \to \mathbf{FinSet}$ is given by

$$X^! : \langle a_1, \ldots, a_n \rangle \in !\mathbf{A} \mapsto \prod_{i=1}^{n} X(a_i)$$

is also a finite presheaf.

Definition

For a finiteness structure $(\mathbf{A}, \mathcal{F}\mathbf{A})$, we define $!(\mathbf{A}, \mathcal{F}(\mathbf{A})) := (!\mathbf{A}, \mathcal{F}(!\mathbf{A}))$ where $\mathcal{F}!\mathbf{A} := \{X^! \mid X \in \mathcal{F}\mathbf{A}\}^{\perp\perp}$.

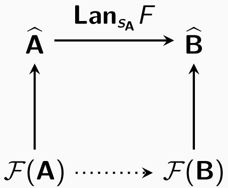
Theorem

FinProf *constitutes a model of differential linear logic. In particular,* $\mathbf{FinProf}_!$ *is a cartesian closed bicategory.*

Analytic functors

Proposition

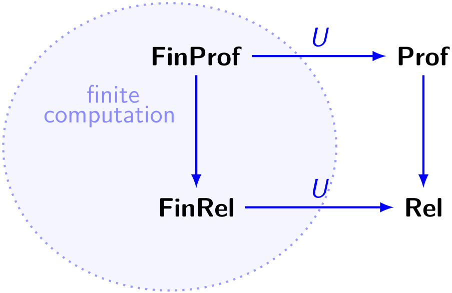
For a finiteness profunctor $F : !(\mathbf{A}, \mathcal{F}\mathbf{A}) \nrightarrow (\mathbf{B}, \mathcal{F}(\mathbf{B}))$, *its associated analytic functor* $\mathbf{Lan}_{s_\mathbf{A}}(F)$ *can be factored as follows:*



This property ensures that all computations remain finite: for any $X \in \mathcal{F}(\mathbf{A})$ and $b \in \mathbf{B}$,

$$\mathbf{Lan}_{s_\mathbf{A}}(F)(X, b) = \int^{u=\langle a_1,\ldots,a_n\rangle \in !\mathbf{A}} F(u, b) \times \prod_{i=1}^{n} X(a_i) \quad \text{is finite}$$

Conclusion and Next Steps



- Categorify the construction of orthogonality for a glueing bicategory.
- Do the same construction in the general enriched case (in particular for species enriched over vector spaces to stay in a finite dimensional setting).
- Replace finite presentability by absolute presentability to obtain a profunctorial model of totality.

Thank you for your attention