Algebraic structures on walks of graphs

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joint work with Foissy, Giscard and Ronco

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Outlines

1. Lawler's loop erasing procedure and Giscard's nesting product
   - Lawler's procedure
   - Giscard's product
2. Hopf algebras generated by walks of a given graph
   - copre-Lie coalgebra
   - Tensor and symmetric Hopf algebras
   - Hopf subalgebras of ladders and corollas
3. Walks ans cacti
   - Hopf algebras of cacti
   - Quotient Hopf algebras in the complete countable tatal graph
   - Walks vs cacti

Definition

Let Γ be a connected graph, $\omega = w_1 \dots w_m$ be a walk in Γ and $\mathrm{Nod}(\omega)$ the set of the nodes of Γ visited by ω.

1. The walk ω is called a self-avoiding walk if $\mathrm{Nod}(\omega)$ is a set of cardinality m.
2. The walk ω is called a simple cycle if $w_1 = w_m$ and $\mathrm{Nod}(\omega)$ is a set of cardinality $m - 1$.

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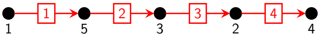
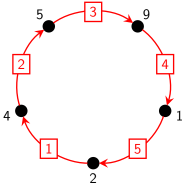
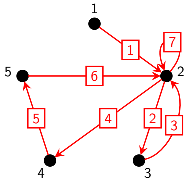
Examples

$\rho = 15324 =$ 

is a self-avoiding walk and

$\mu = 245912 =$ 

a simple cycle.

Let consider the walk $\sigma = 12324522 =$ 

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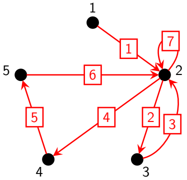
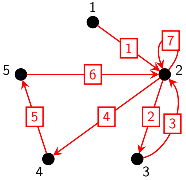
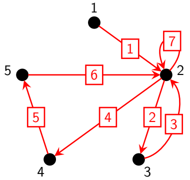
Let consider the walk $\sigma = 12324522 =$ 

. With the Lawler's loop erasing procedure we get:



Let consider the walk $\sigma = 12324522 =$ 

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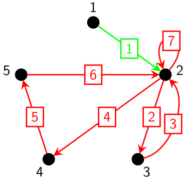
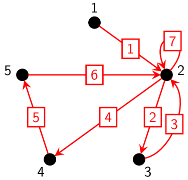


Let consider the walk $\sigma = 12324522 =$ 

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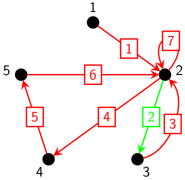
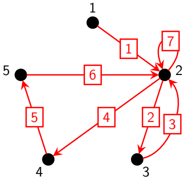


Let consider the walk $\sigma = 12324522 =$ 

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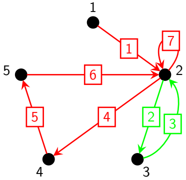
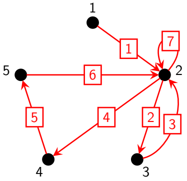


Let consider the walk $\sigma = 12324522 =$ 

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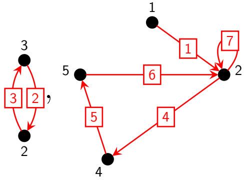
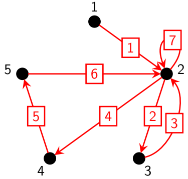


Let consider the walk $\sigma = 12324522 =$ 

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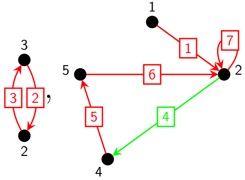
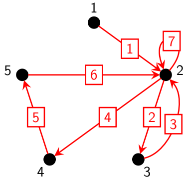
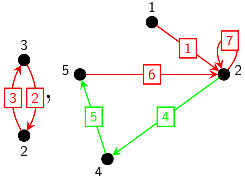
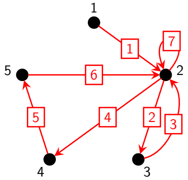


Let consider the walk $\sigma = 12324522 =$  . With the Lawler's loop erasing procedure we get:



Let consider the walk $\sigma = 12324522 =$ 

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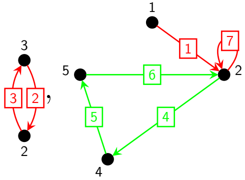
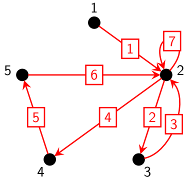


Let consider the walk $\sigma = 12324522 =$ 

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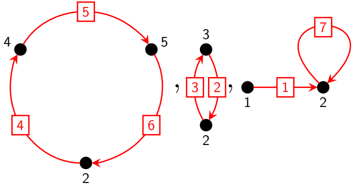
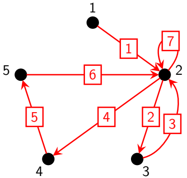


Let consider the walk $\sigma = 12324522 =$ 

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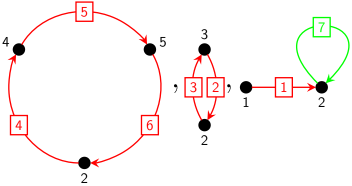
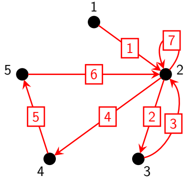


Let consider the walk $\sigma = 12324522 =$ 

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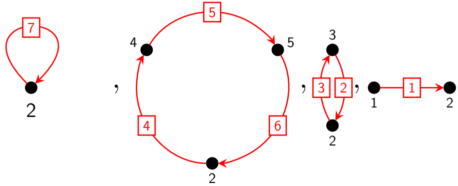

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- or s exists and $\omega \odot \tau =$

$$\begin{cases} w_1 \dots w_s t_2 \dots t_n w_{s+1} \dots w_m & \text{if the letter } t_1 \text{ is the unique vertex both visited by } \tau \text{ and } w_1 \dots w_s, \\ 0 & \text{either.} \end{cases}$$

Example 1

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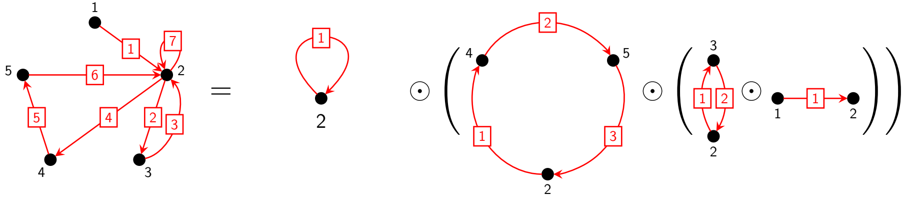

Example 2

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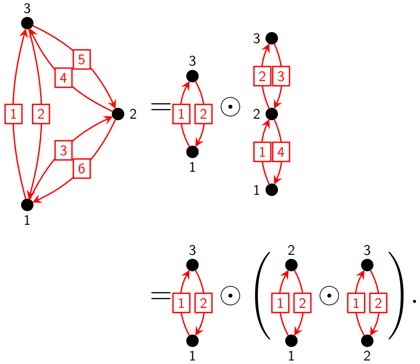

Theorem (P.-L. Giscard, S.J. Thwaite et D. Jaksch)

Let Γ be a connected graph. Any walk $\omega = w_1 \dots w_m$ *on Γ can be factorised into nesting products of self-avoiding walk and simple cycles on Γ. The factorisation is essentially unique.*

Counter-example 1

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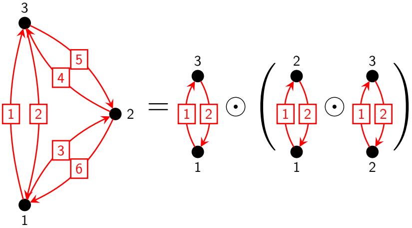

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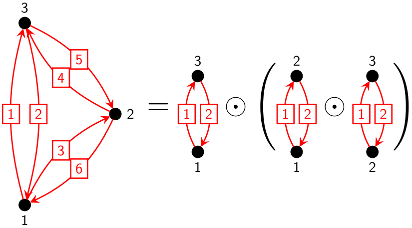

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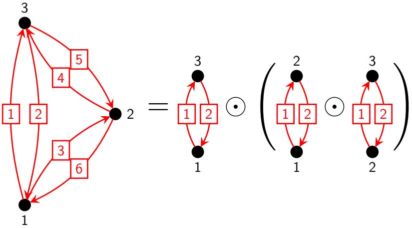

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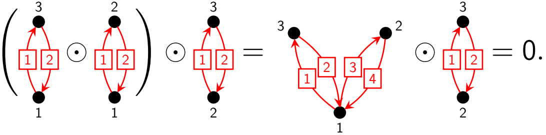

Thus the product $\odot$ is not associative.

Counter-example 2

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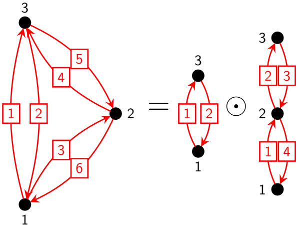

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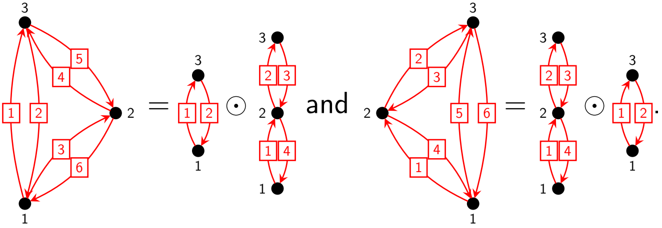

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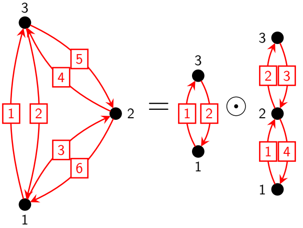

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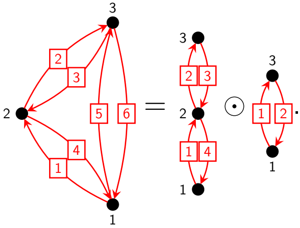
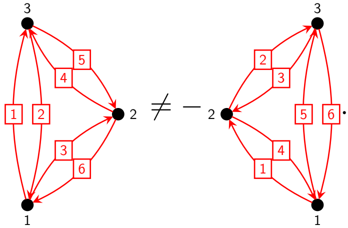


. And yet,



Thus the product $\odot$ is neither a Lie nor a pre-Lie product.

Definition

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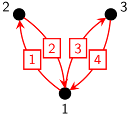
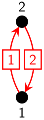
4. Let $\mathcal{L}$ be the set of loop-erased cycles of ω containing $\omega^{ll'}$.
    - Either $\mathcal{L} = \emptyset$
    - or the minimum element $\omega^{kk'}$ for inclusion sush that $k' > l'$ satisfies the statement: the letter w_l does not appear in $w_{l'+1} \dots w_{k'}$.

The set of admissible cuts of ω is denoted $AdC(\omega)$.

Example 1

In the walk $\pi = 12131 =$ 

, the subwalk

$$\eta^{13} = 121 =$$ 

is not an admissible cut. The subwalk

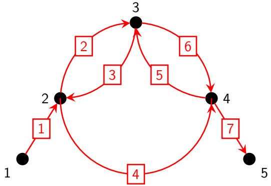
$$\eta^{35} =$$ 

is an admissible cut.

Example 2

In the walk

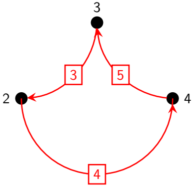
$$\gamma = 12324345 =$$



,

the subwalk

$$\gamma^{36} = 3243 =$$



is not an admissible cut.

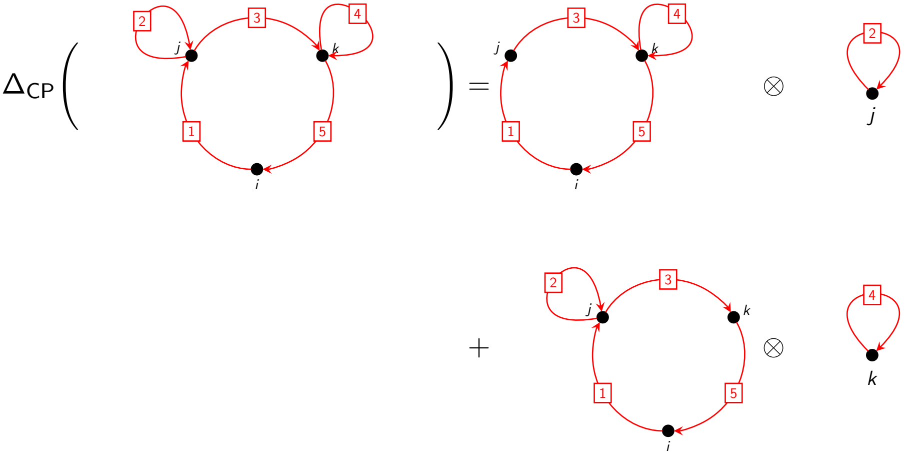
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let Γ be a finite or a countable connected graph and $\mathcal{W}(\Gamma)$ the vector space sanned by its walks. Let define the linear map Δ_{CP} by:

$$\Delta_{CP} : \left\{ \begin{array}{ccc} \mathcal{W}(\Gamma) & \longrightarrow & \mathcal{W}(\Gamma) \otimes \mathcal{W}(\Gamma) \\ \omega & \mapsto & \Delta_{CP}(\omega) = \displaystyle\sum_{\omega^{ll'} \in AdC(\omega)} \omega_{ll'} \otimes \omega^{ll'}, \end{array} \right.$$

where $\omega = w_1 \dots w_m$ is a walk, $\omega_{ll'} = w_1 \dots w_l w_{l'+1} \dots w_m$ and the sum is taken over all the admissible cuts of ω. Then the vector space $\mathcal{W}(\Gamma)$, equipped with the coproduct Δ_{CP}, is a co-preLie (not co-unital) co-algebra.

Example



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Let Γ be a finite or a countable connected graph and $\omega = w_1 \dots w_m$ be a walk in Γ. An extended admissible cut of ω is a sequence

$$1 \leq l_1 < l'_1 < l_2 < l'_2 < \cdots < l_s < l'_s \leq m$$

satisfying that $\omega^{l_k l'_k}$ is an admissible cut of ω, for any $1 \leq k \leq s$. The set of extended admissible cuts of ω is denoted $EAdC(\omega)$.

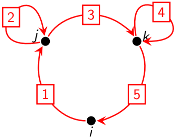
Definition

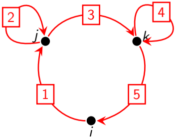
We define the morphism of algebras Δ_{H} defined by:

$$\Delta_{\mathrm{H}} : \begin{cases} \mathcal{T}\langle\mathcal{W}(\Gamma)\rangle & \longrightarrow \quad \mathcal{T}\langle\mathcal{W}(\Gamma)\rangle \otimes \mathcal{T}\langle\mathcal{W}(\Gamma)\rangle \\ \qquad\omega & \mapsto \quad \Delta_{\mathrm{H}}(\omega) = \omega \otimes 1 + 1 \otimes \omega \\ & \qquad + \displaystyle\sum_{c \in EAdC(\omega)} \omega_{l_1 l_1', \dots, l_s l_s'} \otimes \omega^{l_1 l_1'} | \dots | \omega^{l_s l_s'}, \end{cases}$$

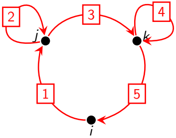
where $\omega = w_1 \dots w_m$ is a walk in Γ, the extended admissible cut c is the sequence $1 \leq l_1 < l_1' < \cdots < l_s < l_s' \leq m$ and the sum is taken over all the extended admissible cuts of ω.

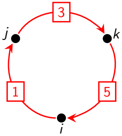
Example

$\Delta_H\Big($ 

$\Big) =$ 

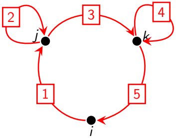
$\otimes 1$

$+ 1 \otimes$ 

$+$ 

$\otimes$ 

$|$ 

$+ \Delta_{CP}\Big($ 

$\Big).$

Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let Γ *a finite or countable connected graph. Consider the triple* $(\mathcal{T}\langle\mathcal{W}(\Gamma)\rangle, \star, \Delta_H)$. *It is a Hopf algebra.*

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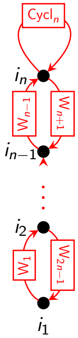
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

In the graph Γ, we denote by $\mathcal{I}$ the vector space spanned by the elements $\omega_1|\dots|\omega_s - \omega_{\sigma(1)}|\dots|\omega_{\sigma(s)}$ where $\omega_1|\dots|\omega_s \in \mathcal{T}\langle\mathcal{W}(\Gamma)\rangle$ and σ is a permutation. Then, $\mathcal{I}$ is a Hopf bi-ideal of $\mathcal{T}\langle\mathcal{W}(\Gamma)\rangle$. Thus, $(\mathcal{S}\langle\mathcal{W}(\Gamma)\rangle, \square, \Delta_H)$ is a quotient Hopf algebra of $\mathcal{T}\langle\mathcal{W}(\Gamma)\rangle$.

Definition

Let Γ a connected graph. A ladder of basis i_1 is a closed walk

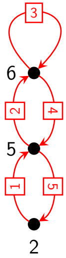
$\omega =$



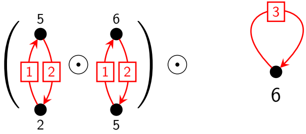
Example

The walk

$\eta = 256652 =$



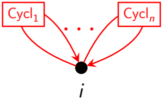
$=$



is a ladder.

Definition

Let Γ be a connected graph. A corolla of basis i in Γ is a closed walk

$\omega =$ 

$= \dots \Big(\mathrm{Cycl}_1 \odot \mathrm{Cycl}_2 \Big) \dots \Big) \odot \mathrm{Cycl}_n$ where Cycl_1, $\dots$, Cycl_n are simple cycles of basis i.

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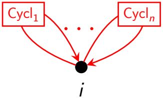
$\omega =$ 

$= \dots \left(\mathrm{Cycl}_1 \odot \mathrm{Cycl}_2 \right) \dots \Big) \odot \mathrm{Cycl}_n$ where $\mathrm{Cycl}_1, \dots, \mathrm{Cycl}_n$ are simple cycles of basis i.

Examples

Definition

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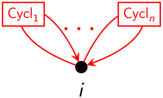
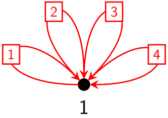
$\omega =$ 

$= \ldots \left(\mathrm{Cycl}_1 \odot \mathrm{Cycl}_2 \right) \ldots \Big) \odot \mathrm{Cycl}_n$ where Cycl_1, $\ldots$, Cycl_n are simple cycles of basis i.

Examples

- The walk $\mu = 11111 =$  is a corolla.

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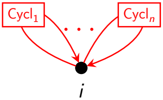
$\omega =$ 

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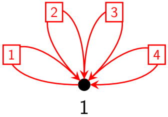
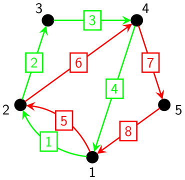
Examples

- The walk $\mu = 11111 =$ 

is a corolla.
- The walk $\nu = 123412451 =$ 

is a corolla.

Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let Γ be a connected finite or countable graph. Let define $Lad(\Gamma)$, $Cor(\Gamma)$ and $Cor_i(\Gamma)$ the vector spaces spanned by ladders, corollas and corollas of base i respectively. Then

1. *the spaces* $(\mathcal{T}\langle Lad(\Gamma)\rangle, \star, \Delta_H)$, $(\mathcal{T}\langle Cor_i(\Gamma)\rangle, \star, \Delta_H)$, $(\mathcal{T}\langle Cor(\Gamma)\rangle, \star, \Delta_H)$ *are Hopf subalgebras of* $(\mathcal{T}\langle \mathcal{W}(\Gamma)\rangle, \star, \Delta_H)$.
2. *the spaces* $(\mathcal{S}\langle Lad(\Gamma)\rangle, \square, \Delta_H)$, $(\mathcal{S}\langle Cor_i(\Gamma)\rangle, \square, \Delta_H)$, $(\mathcal{S}\langle Cor(\Gamma)\rangle, \square, \Delta_H)$ *are Hopf subalgebras of* $(\mathcal{S}\langle \mathcal{W}(\Gamma)\rangle, \square, \Delta_H)$.

Definition

Let Γ be a connected graph. A walk $\omega = w_1 \ldots w_m$ is a cactus if and only if for any $1 \leq l < l' \leq m$ the two following statements are equivalent:

- $w_l = w_{l'}$,
- $\omega^{ll'}$ is a loop-erased cycle of ω.

The vector space spanned by the cacti of Γ is denoted by $\mathrm{Cact}(\Gamma)$.

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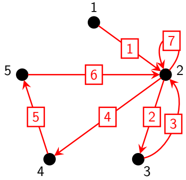
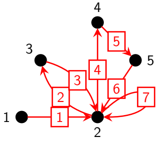
Example

The walk $\sigma = 12324522 =$ 

$=$ 

is a cactus.

Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let Γ be a connected finite or countable graph.

1. *The space* $(\mathcal{T}\langle Cact(\Gamma)\rangle, \star, \Delta_H)$ *is a Hopf subalgebra of* $(\mathcal{T}\langle \mathcal{W}(\Gamma)\rangle, \star, \Delta_H)$.
2. *The spaces* $(\mathcal{S}\langle Cact(\Gamma)\rangle, \square, \Delta_H)$, *is a Hopf subalgebras of* $(\mathcal{S}\langle \mathcal{W}(\Gamma)\rangle, \square, \Delta_H)$.

Definition

Let consider Ω the complete graph such that $\mathrm{Nod}(\Omega) = \mathbb{N}^*$ and $\mathrm{Edg}(\Omega) = \{(i,j) \in (\mathbb{N}^*)^2\}$. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be the vector spaces defined by:

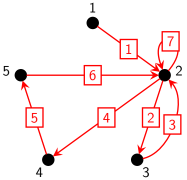
$$\mathcal{J}_1 = \mathrm{Span}\Big(\omega_1|\ldots|\omega_s - f_1(\omega_1)|\ldots|f_s(\omega_s),\\ s \in \mathbb{N}^*, \forall i \in \{1,\ldots,s\}, \omega_i \in \mathrm{Cact}(\Omega), f_i \in \mathcal{F}\Big),$$

and

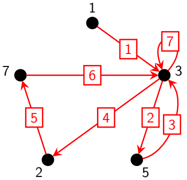
$$\mathcal{J}_2 = \mathrm{Span}\Big(\omega_1\square\ldots\square\omega_s - f_1(\omega_1)\square\ldots\square f_s(\omega_s),\\ s \in \mathbb{N}^*, \forall i \in \{1,\ldots,s\}, \omega_i \in \mathrm{Cact}(\Omega), f_i \in \mathcal{F}\Big),$$

where f is an admissible label of ω_i.

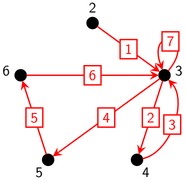
Examples

Let consider the walk $\sigma = 12324522 =$ 

. The walk

$\sigma_1 = 13532733 =$ 

or the walk

$\sigma_2 = 23435633$ 

give admissible labels of σ.

Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

The space $\frac{\mathcal{T}\langle\mathcal{W}(\Omega)\rangle}{\mathcal{J}_1}$ *(respectively* $\frac{\mathcal{S}\langle\mathcal{W}(\Omega)\rangle}{\mathcal{J}_2}$*) is a Hopf algebra called the tensor Hopf algebra of unlabeled walks (respectively the symmetric Hopf algebra of unlabeled walks).*

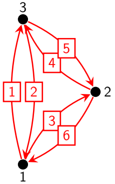
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

The space $\frac{\mathcal{T}\langle\mathcal{W}(\Omega)\rangle}{\mathcal{J}_1}$ *(respectively* $\frac{\mathcal{S}\langle\mathcal{W}(\Omega)\rangle}{\mathcal{J}_2}$*) is a Hopf algebra called the tensor Hopf algebra of unlabeled walks (respectively the symmetric Hopf algebra of unlabeled walks).*

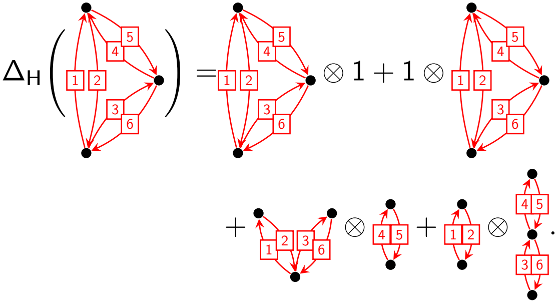
Remark

To compute the coproduct of a walk ω in $\frac{\mathcal{T}\langle\mathcal{W}(\Omega)\rangle}{\mathcal{J}_1}$ or in $\frac{\mathcal{S}\langle\mathcal{W}(\Omega)\rangle}{\mathcal{J}_2}$ it is sufficient to compute it by equipping ω with an admissible label and after that forgot all vertices' labels.

Example

Let consider the walk $\eta = 1312321 =$  in $\mathcal{T}\langle\mathcal{W}(\Omega)\rangle$. In

$\dfrac{\mathcal{T}\langle\mathcal{W}(\Omega)\rangle}{\mathcal{J}_1}$ we get:



Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let Γ be a finite or a countable connected graph. Let $\omega = w_1 \dots w_m$ a walk in Γ. There exist at least one admissible label f making $f(\omega) = f(w_1)\dots f(w_m)$ into a cactus. Let Forgot$(f(\omega))$ the unlabeled cactus where the labels of vertices are forgotten. Then, the algebra morphisms

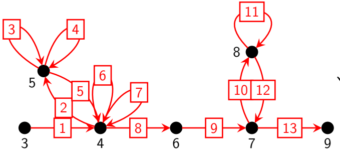
$$\varphi_1 : \begin{cases} \mathcal{T}\langle\mathcal{W}(\Gamma)\rangle & \longrightarrow & \frac{\mathcal{T}\langle\mathcal{W}(\Omega)\rangle}{\mathcal{J}_1} \\ \omega & \mapsto & \mathit{Forgot}(f(\omega)) \end{cases}$$

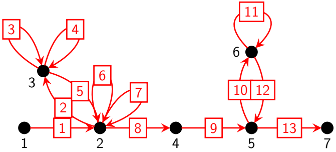
and

$$\varphi_2 : \begin{cases} \mathcal{S}\langle\mathcal{W}(\Gamma)\rangle & \longrightarrow & \frac{\mathcal{S}\langle\mathcal{W}(\Omega)\rangle}{\mathcal{J}_2} \\ \omega & \mapsto & \mathit{Forgot}(f(\omega)) \end{cases}$$

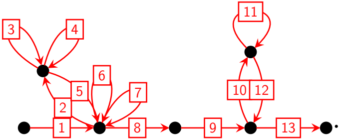
are Hopf algebra morphisms.

Example 1

$\varphi_1\Bigg($ 

$\Bigg) = \varphi_1\Bigg($ 

$\Bigg)$

$=$ 

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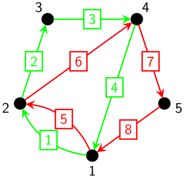
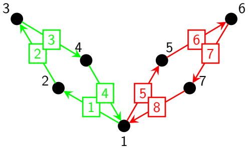
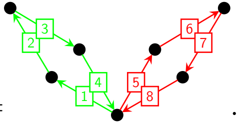
Example 2

$\varphi_1\Bigg($ 

$\Bigg) = \varphi_1\Bigg($ 

$\Bigg)$

$=$ 

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