Algebraic structures on walks of graphs

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Outlines

1. Lawler’s loop erasing procedure and Giscard’s nesting product
   - Lawler’s procedure
   - Giscard’s product

2. Hopf algebras generated by walks of a given graph
   - copre-Lie coalgebra
   - Tensor and symmetric Hopf algebras
   - Hopf subalgebras of ladders and corollas

3. Walks ans cacti
   - Hopf algebras of cacti
   - Quotient Hopf algebras in the complete countable tatal graph
   - Walks vs cacti
Let $\Gamma$ be a connected graph, $\omega = w_1 \ldots w_m$ be a walk in $\Gamma$ and $\text{Nod}(\omega)$ the set of the nodes of $\Gamma$ visited by $\omega$.

1. The walk $\omega$ is called a self-avoiding walk if $\text{Nod}(\omega)$ is a set of cardinality $m$.

2. The walk $\omega$ is called a simple cycle if $w_1 = w_m$ and $\text{Nod}(\omega)$ is a set of cardinality $m - 1$.

Examples

$\rho = 15324 = 15324$ is a self-avoiding walk and $\mu = 245912 = 25432123$ is a simple cycle.
**Definition**

Let $\Gamma$ be a connected graph, $\omega = w_1 \ldots w_m$ be a walk in $\Gamma$ and $\text{Nod}(\omega)$ the set of the nodes of $\Gamma$ visited by $\omega$.

1. The walk $\omega$ is called a self-avoiding walk if $\text{Nod}(\omega)$ is a set of cardinality $m$.

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Definition

Let $\Gamma$ be a connected graph, $\omega = w_1 \ldots w_m$ be a walk in $\Gamma$ and $\text{Nod}(\omega)$ the set of the nodes of $\Gamma$ visited by $\omega$.

1. The walk $\omega$ is called a self-avoiding walk if $\text{Nod}(\omega)$ is a set of cardinality $m$.
2. The walk $\omega$ is called a simple cycle if $w_1 = w_m$ and $\text{Nod}(\omega)$ is a set of cardinality $m - 1$.

Examples

$\rho = 15324 = \begin{array}{c}
1 & 5 & 2 & 3 & 2 & 4 & 4
\end{array}$ is a self-avoiding walk and

$\mu = 245912 = \begin{array}{c}
4 & 2 & 3 & 9 & 2 & 1 & 4 & 5 & 1 & 2 & 5
\end{array}$ is a simple cycle.
Let consider the walk $\sigma = 12324522$. With the Lawler's loop erasing procedure we get: $\frac{4}{32}$. 
Let consider the walk $\sigma = 12324522 = \cdots$. 
Let consider the walk $\sigma = 12324522 = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9 \\
10 \\
11 \\
12 \end{array}$. With the Lawler’s loop erasing procedure we get:
Let consider the walk $\sigma = 12324522 = \text{Lawler’s procedure}$.

With the Lawler’s loop erasing procedure we get:
Let consider the walk \( \sigma = 12324522 = 12345672 \). With the Lawler’s loop erasing procedure we get:
Let consider the walk $\sigma = 12324522 = \ldots$. With the Lawler’s loop erasing procedure we get:
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\[
\begin{array}{cccc}
\text{Let consider the walk } & \sigma = 12324522 = & \ldots & \text{With the } \\
\text{Lawler’s loop erasing procedure we get: }
\end{array}
\]
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Let consider the walk $\sigma = 12324522$. With the Lawler’s loop erasing procedure we get:
Let consider the walk $\sigma = 12324522 = 55445232$. With the Lawler’s loop erasing procedure we get:

![Diagram of the walk and its Lawler’s loop erasing procedure result]
Let consider the walk $\sigma = 12324522 = \begin{array}{lllllll}
1 & 2 & 3 & 4 & 5 & 2 & 2 \\
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}$. With the Lawler’s loop erasing procedure we get:
Let consider the walk $\sigma = 12324522 = \begin{array}{c} 
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
2 \end{array}$. With the Lawler’s loop erasing procedure we get:
Let $\omega = w_1 \ldots w_m$ and $\tau = t_1 \ldots t_n$ be walks. We define the Giscard's nesting product $\circ$ of $\omega$ by $\tau$ as follow:

1. If $\tau$ is not a closed walk ($t_1 \neq t_n$) then $\omega \circ \tau = 0$.
2. If $\omega$ and $\tau$ are both closed walks such that $w_1 = t_1$ then $\omega \circ \tau = w_1 \ldots w_m t_2 \ldots t_n$.
3. If $\tau$ is a closed walk such that $t_1 \neq w_1$ or $t_1 \neq w_m$ then let $s$ be the greatest integer in $\{1, \ldots, m\}$ such that $w_s = t_1$:
   - either $s$ does not exist and $\omega \circ \tau = 0$,
   - or $s$ exists and $\omega \circ \tau = \begin{cases} w_1 \ldots w_s t_2 \ldots t_n w_{s+1} \ldots w_m & \text{if the letter } t_1 \text{ is the unique vertex both visited by } \tau \text{ and } w_1 \ldots w_s, \\ 0 & \text{otherwise.} \end{cases}$
Definition (Giscard)

Let $\omega = w_1 \ldots w_m$ and $\tau = t_1 \ldots t_n$ be walks. We define the Giscard's nesting product $\circ$ of $\omega$ by $\tau$ as follow:

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3. If $\tau$ is a closed walk such that $t_1 \neq w_1$ or $t_1 \neq w_m$ then let $s$ be the greatest integer in $\{1, \ldots, m\}$ such that $w_s = t_1$:
   - Either $s$ does not exist and $\omega \circ \tau = 0$,
   - Or $s$ exists and $\omega \circ \tau = w_1 \ldots w_s t_2 \ldots t_n w_s + 1 \ldots w_m$ if the letter $t_1$ is the unique vertex both visited by $\tau$ and $w_1 \ldots w_s$.
Definition (Giscard)

Let \( \omega = w_1 \ldots w_m \) and \( \tau = t_1 \ldots t_n \) be walks. We define the Giscard’s nesting product \( \odot \) of \( \omega \) by \( \tau \) as follow:

1. If \( \tau \) is not a closed walk (\( t_1 \neq t_n \)) then \( \omega \odot \tau = 0 \).
2. If \( \omega \) and \( \tau \) are both closed walks such that \( w_1 = t_1 \) then \( \omega \odot \tau = w_1 \ldots w_m t_2 \ldots t_n \).
3. If \( \tau \) is a closed walk such that \( t_1 \neq w_1 \) or \( t_1 \neq w_m \) then let \( s \) be the greatest integer in \( \{1, \ldots, m\} \) such that \( w_s = t_1 \):

   - either \( s \) does not exist and \( \omega \odot \tau = 0 \),
   - or \( s \) exists and
     \[
     \omega \odot \tau = \begin{cases} 
     w_1 \ldots w_s t_2 \ldots t_n w_{s+1} \ldots w_m & \text{if the letter } t_1 \text{ is the unique vertex both visited by } \tau \text{ and } w_1 \ldots w_s, \\
     0 & \text{otherwise.}
     \end{cases}
     \]

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   - either $s$ does not exist and $\omega \circ \tau = 0$,
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   $\omega \odot \tau = w_1 \ldots w_m t_2 \ldots t_n$.
3. If $\tau$ is a closed walk such that $t_1 \neq w_1$ or $t_1 \neq w_m$ then let $s$ be the greatest integer in $\{1, \ldots, m\}$ such that $w_s = t_1$:
   - either $s$ does not exist and $\omega \odot \tau = 0$,
   - or $s$ exists and $\omega \odot \tau =
     \begin{cases} 
     w_1 \ldots w_s t_2 \ldots t_n w_{s+1} \ldots w_m & \text{if the letter } t_1 \text{ is the unique vertex both visited by } \tau \text{ and } w_1 \ldots w_s, \\
     0 & \text{either.} 
     \end{cases}$
Example 1
Example 1
Example 2
Example 2
Theorem (P.-L. Giscard, S.J. Thwaite et D. Jaksch)

Let $\Gamma$ be a connected graph. Any walk $\omega = w_1 \ldots w_m$ on $\Gamma$ can be factorised into nesting products of self-avoiding walk and simple cycles on $\Gamma$. The factorisation is essentially unique.
Counter-example 1

\[\begin{align*}
(1 & \circ 1) \circ (2 & \circ 3) \\
= & (1 & \circ 1) \circ (2 & \circ 3) \\
= & 1 & \circ 3 \\
= & 0.
\end{align*}\]

Thus the product \(\circ\) is not associative.
Counter-example 1

\[
\begin{align*}
&\text{Lawler's procedure - Giscard's product} \\
&\text{Giscard's product}
\end{align*}
\]
Counter-example 1

Thus the product $\odot$ is not associative.
Counter-example 1

and
Counter-example 1

Thus the product $\odot$ is not associative.
Lawler’s procedure - Giscard’s product

Giscard’s product

Counter-example 2

1 = 1
2 ⊙ 1
3

And yet,

1 = −1
2 ⊙ 1
3

Thus the product ⊙ is neither a Lie nor a pre-Lie product.

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Counter-example 2
Counter-example 2

\[ 1 \odot 2 = 3 \quad \text{and} \quad 2 \odot 3 = 1 \]

And yet, \( 1 \odot 2 \neq -1 \odot 2 \).

Thus the product \( \odot \) is neither a Lie nor a pre-Lie product.
Counter-example 2

\[ 3 \quad 2 = 1 \quad 2 \quad \odot \quad 1 \quad 2 \quad \odot \quad 1 \quad 2 \]

\[ 3 \quad 2 = 1 \quad 2 \quad \odot \quad 1 \quad 2 \quad \odot \quad 1 \quad 2 \]

Thus the product \( \odot \) is neither a Lie nor a pre-Lie product.
Counter-example 2

\[
\begin{array}{l}
3 & 3 & 3 \\
2 & 2 & 2 \\
1 & 1 & 1 \\
\hline
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3 \\
\end{array}
\]

and

\[
\begin{array}{l}
3 & 3 & 3 \\
2 & 2 & 2 \\
1 & 1 & 1 \\
\hline
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3 \\
\end{array}
\]

And yet,

\[
\begin{array}{l}
3 & 3 & 3 \\
2 & 2 & 2 \\
1 & 1 & 1 \\
\hline
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3 \\
\end{array}
\]

Thus the product $\circ$ is neither a Lie nor a pre-Lie product.
Let \( \omega = w_1 \ldots w_m \) be a walk in a finite or countable connected graph \( \Gamma \).

We say that a walk \( \omega_{ll}' := w_l w_{l+1} \ldots w_{l'} \) is an admissible cut of \( \omega \) when it satisfies all of the following conditions:

1. \( \omega_{ll}' \neq \omega \) and \( \omega_{ll}' \neq () \) where () is the empty walk;
2. \( w_l = w_{l'} \), i.e. \( \omega_{ll}' \) is a closed walk;
3. \( \omega_{ll}' \) is a cycle erased by the Lawler's procedure;
4. Let \( \mathcal{L} \) be the set of loop-erased cycles of \( \omega \) containing \( \omega_{ll}' \). Either \( \mathcal{L} = \emptyset \) or the minimum element \( \omega_{kk}' \) for inclusion such that \( k' > l' \) satisfies the statement: the letter \( w_l \) does not appear in \( w_l \ldots w_{k'-1} \).

The set of admissible cuts of \( \omega \) is denoted \( \text{AdC}(\omega) \).
Definition

Let $\omega = w_1 \ldots w_m$ be a walk in a finite or countable connected graph $\Gamma$. We say that a walk $\omega''' := w_l w_{l+1} \ldots w_{l'}$ is an admissible cut of $\omega$ when it satisfies all of the following conditions:

1. $\omega''' \neq \omega$ and $\omega''' \neq ()$, where $(())$ is the empty walk;
2. $w_l = w_{l'}$, i.e. $\omega'''$ is a closed walk;
3. $\omega'''$ is a cycle erased by Lawler’s procedure;
4. Let $L$ be the set of loop-erased cycles of $\omega$ containing $\omega'''$. Either $L = \emptyset$ or the minimum element $\omega_{kk'}$ for inclusion such that $k' > l'$ satisfies the statement: the letter $w_l$ does not appear in $w_l w_{l+1} \ldots w_{k'}$.

The set of admissible cuts of $\omega$ is denoted $\text{AdC}(\omega)$. 


Definition

Let \( \omega = w_1 \ldots w_m \) be a walk in a finite or countable connected graph \( \Gamma \). We say that a walk \( \omega'' := w_l w_{l+1} \ldots w_{l'} \) is an admissible cut of \( \omega \) when it satisfies all of the following conditions:

1. \( \omega'' \neq \omega \) and \( \omega'' \neq () \) where ( ) is the empty walk;
Definition

Let \( \omega = w_1 \ldots w_m \) be a walk in a finite or countable connected graph \( \Gamma \). We say that a walk \( \omega''' := w_lw_{l+1} \ldots w_{l'} \) is an admissible cut of \( \omega \) when it satisfies all of the following conditions

1. \( \omega''' \neq \omega \) and \( \omega''' \neq () \) where ( ) is the empty walk;
2. \( w_l = w_{l'} \), i.e. \( \omega''' \) is a closed walk;
Definition

Let $\omega = w_1 \ldots w_m$ be a walk in a finite or countable connected graph $\Gamma$. We say that a walk $\omega'''' := w_l w_{l+1} \ldots w_{l'}$ is an admissible cut of $\omega$ when it satisfies all of the following conditions

1. $\omega'''' \neq \omega$ and $\omega'''' \neq ()$ where () is the empty walk;
2. $w_l = w_{l'}$, i.e. $\omega''''$ is a closed walk;
3. $\omega''''$ is a cycle erased by the Lawler’s procedure.
Definition

Let $\omega = w_1 \ldots w_m$ be a walk in a finite or countable connected graph $\Gamma$. We say that a walk $\omega''' := w_l w_{l+1} \ldots w_{l'}$ is an admissible cut of $\omega$ when it satisfies all of the following conditions

1. $\omega''' \neq \omega$ and $\omega''' \neq ()$ where () is the empty walk;
2. $w_l = w_{l'}$, i.e. $\omega'''$ is a closed walk;
3. $\omega'''$ is a cycle erased by the Lawler’s procedure;
4. Let $\mathcal{L}$ be the set of loop-erased cycles of $\omega$ containing $\omega'''$. 


Definition

Let \( \omega = w_1 \ldots w_m \) be a walk in a finite or countable connected graph \( \Gamma \). We say that a walk \( \omega^{ll'} := wlw_{l+1} \ldots w_{l'} \) is an admissible cut of \( \omega \) when it satisfies all of the following conditions

1. \( \omega^{ll'} \neq \omega \) and \( \omega^{ll'} \neq () \) where (()) is the empty walk;
2. \( w_l = w_{l'} \), i.e. \( \omega^{ll'} \) is a closed walk;
3. \( \omega^{ll'} \) is a cycle erased by the Lawler’s procedure
4. Let \( \mathcal{L} \) be the set of loop-erased cycles of \( \omega \) containing \( \omega^{ll'} \).
   a. Either \( \mathcal{L} = \emptyset \)
Definition

Let \( \omega = w_1 \ldots w_m \) be a walk in a finite or countable connected graph \( \Gamma \). We say that a walk \( \omega^{ll'} := w_l w_{l+1} \ldots w_{l'} \) is an admissible cut of \( \omega \) when it satisfies all of the following conditions

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3. \( \omega^{ll'} \) is a cycle erased by the Lawler’s procedure
4. Let \( \mathcal{L} \) be the set of loop-erased cycles of \( \omega \) containing \( \omega^{ll'} \).
   - Either \( \mathcal{L} = \emptyset \)
   - or the minimum element \( \omega^{kk'} \) for inclusion such that \( k' > l' \) satisfies
     the statement: the letter \( w_l \) does not appear in \( w_{l'+1} \ldots w_{k'} \).

The set of admissible cuts of \( \omega \) is denoted \( \text{AdC}(\omega) \).
Example 1

In the walk \( \pi = 12131 \), the subwalk \( \eta_{13} = 121 \) is not an admissible cut. The subwalk \( \eta_{35} = 3534 \) is an admissible cut.
Example 1

In the walk $\pi = 12131 = \begin{array}{c}
1 \\
2 \\
3 \\
1 \\
2 \\
3 \\
4 \\
1 \\
\end{array}$, the subwalk

$\eta^{13} = 121 = \begin{array}{c}
1 \\
2 \\
1 \\
\end{array}$

is not an admissible cut. The subwalk

$\eta^{35} = \begin{array}{c}
5 \\
4 \\
3 \\
\end{array}$

is an admissible cut.
Example 2

In the walk $\gamma = 12324345 = 234151234567$, the subwalk $\gamma_{36} = 3243 = 234345$ is not an admissible cut.
Example 2

In the walk

$$\gamma = 12324345 =$$

the subwalk

$$\gamma^{36} = 3243 =$$

is not an admissible cut.
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let $\Gamma$ be a finite or a countable connected graph and $\mathcal{W}(\Gamma)$ the vector space spanned by its walks. Let define the linear map $\Delta_{CP}$ by:

$$\Delta_{CP} : \begin{cases} \mathcal{W}(\Gamma) & \rightarrow \mathcal{W}(\Gamma) \otimes \mathcal{W}(\Gamma) \\ \omega & \mapsto \Delta_{CP}(\omega) = \sum_{\omega_{\|}\otimes \omega_{\|\|}, \omega_{\|\|} \in AdC(\omega)} \omega_{\|}\otimes \omega_{\|\|}, \end{cases}$$

where $\omega = w_1 \ldots w_m$ is a walk, $\omega_{\|}= w_1 \ldots w_{l}w_{l+1} \ldots w_m$ and the sum is taken over all the admissible cuts of $\omega$. Then the vector space $\mathcal{W}(\Gamma)$, equipped with the coproduct $\Delta_{CP}$, is a co-preLie (not co-unital) co-algebra.
Example

\[ \Delta_{\text{CP}} \left( \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array} \right) = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array} \otimes \begin{array}{c}
2 \\
3 \\
4 \\
5 \\
\end{array} + \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array} \otimes \begin{array}{c}
2 \\
3 \\
4 \\
5 \\
\end{array} \]
Hopf algebras on walks

Tensor and symmetric Hopf algebras
Definition

Let $\Gamma$ be a finite or a countable connected graph and $\omega = w_1 \ldots w_m$ be a walk in $\Gamma$. An extended admissible cut of $\omega$ is a sequence

$$1 \leq l_1 < l'_1 < l_2 < l'_2 < \cdots < l_s < l'_s \leq m$$

satisfying that $\omega^{l_1} \omega^{l'_1} \omega^{l_2} \omega^{l'_2} \cdots \omega^{l_s} \omega^{l'_s}$ is an admissible cut of $\omega$, for any $1 \leq k \leq s$. The set of extended admissible cuts of $\omega$ is denoted $EAdC(\omega)$. 
Hopf algebras on walks

We define the morphism of algebras $\Delta_H$ defined by:

$$\Delta_H : T\langle W(\Gamma) \rangle \rightarrow T\langle W(\Gamma) \rangle \otimes T\langle W(\Gamma) \rangle$$

$$\omega \mapsto \Delta_H(\omega) = \omega \otimes 1 + 1 \otimes \omega + \sum_{c \in EAdC(\omega)} \omega_{l_1 l_1'} \ldots \omega_{l_s l_s'},$$

where $\omega = w_1 \ldots w_m$ is a walk in $\Gamma$, the extended admissible cut $c$ is the sequence $1 \leq l_1 < l_1' < \cdots < l_s < l_s' \leq m$ and the sum is taken over all the extended admissible cuts of $\omega$. 
Definition

We define the morphism of algebras $\Delta_H$ defined by:

$$\Delta_H : \begin{cases} 
T \langle W(\Gamma) \rangle & \mapsto & T \langle W(\Gamma) \rangle \otimes T \langle W(\Gamma) \rangle \\
\omega & \mapsto & \Delta_H(\omega) = \omega \otimes 1 + 1 \otimes \omega \\
& & + \sum_{c \in EAdC(\omega)} \omega_{l_1 l_1', \ldots, l_s l_s'} \otimes \omega_{l_1 l_1'} | \ldots | \omega_{l_s l_s'},
\end{cases}$$

where $\omega = w_1 \ldots w_m$ is a walk in $\Gamma$, the extended admissible cut $c$ is the sequence $1 \leq l_1 < l_1' < \cdots < l_s < l_s' \leq m$ and the sum is taken over all the extended admissible cuts of $\omega$. 
<table>
<thead>
<tr>
<th>Hopf algebras on walks</th>
<th>Tensor and symmetric Hopf algebras</th>
</tr>
</thead>
</table>

**Example**

\[
\Delta H(i^j_k 1^3 5^2 4) = i^j_k 1^3 5^2 4 \otimes 1 + 1 \otimes i^j_k 1^3 5^2 4 + i^j_k 1^3 5^2 4 \otimes j^2_k 4 + \Delta CP(i^j_k 1^3 5^2 4).
\]
Example

\[ \Delta_H \left( \begin{array}{c}
2 \\
3 \\
4 \\
1 \\
5 \\
\end{array} \right) = \begin{array}{c}
2 \\
3 \\
4 \\
1 \\
5 \\
\end{array} \otimes 1 + 1 \otimes \begin{array}{c}
2 \\
3 \\
4 \\
1 \\
5 \\
\end{array} + \Delta_{CP} \left( \begin{array}{c}
2 \\
3 \\
4 \\
1 \\
5 \\
\end{array} \right). \]
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let $\Gamma$ a finite or countable connected graph. Consider the triple $(T\langle W(\Gamma) \rangle, \ast, \Delta_H)$. It is a Hopf algebra.

Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

In the graph $\Gamma$, we denote by $I$ the vector space spanned by the elements $\omega_1|...|\omega_s-\omega_\sigma(1)|...|\omega_\sigma(s)$ where $\omega_1|...|\omega_s \in T\langle W(\Gamma) \rangle$ and $\sigma$ is a permutation. Then, $I$ is a Hopf bi-ideal of $T\langle W(\Gamma) \rangle$. Thus, $(S\langle W(\Gamma) \rangle, \Box, \Delta_H)$ is a quotient Hopf algebra of $T\langle W(\Gamma) \rangle$. 
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let $\Gamma$ a finite or countable connected graph. Consider the triple $(\mathcal{T}(\mathcal{W}(\Gamma)), \star, \Delta_H)$. It is a Hopf algebra.
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let $\Gamma$ a finite or countable connected graph. Consider the triple $(\mathcal{T}\langle \mathcal{W}(\Gamma) \rangle, \ast, \Delta_H)$. It is a Hopf algebra.

Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

In the graph $\Gamma$, we denote by $\mathcal{I}$ the vector space spanned by the elements $\omega_1|\ldots|\omega_s - \omega_{\sigma(1)}|\ldots|\omega_{\sigma(s)}$ where $\omega_1|\ldots|\omega_s \in \mathcal{T}\langle \mathcal{W}(\Gamma) \rangle$ and $\sigma$ is a permutation. Then, $\mathcal{I}$ is a Hopf bi-ideal of $\mathcal{T}\langle \mathcal{W}(\Gamma) \rangle$. Thus, $(\mathcal{S}\langle \mathcal{W}(\Gamma) \rangle, \Box, \Delta_H)$ is a quotient Hopf algebra of $\mathcal{T}\langle \mathcal{W}(\Gamma) \rangle$. 
A ladder of basis $i_1i_2\ldots i_n$ is a closed walk $\omega = i_1i_2\ldots i_{n-1}i_n W_1W_2\ldots W_{n-1}W_n + 1 CYC_n$.
Definition

Let $\Gamma$ a connected graph. A ladder of basis $i_1$ is a closed walk

\[ \omega = \cdots \]

\[ i_2 \rightarrow W_1 \rightarrow i_1 \]

\[ i_{n-1} \rightarrow W_{n-1} \rightarrow \cdots \]

\[ i_n \rightarrow W_n \rightarrow \text{Cycl}_n \]

\[ i_{n+1} \rightarrow W_{n+1} \rightarrow \cdots \]
Hopf algebras on walks

Hopf subalgebras of ladders and corollas

Example

The walk $\eta = 256652 = 2^56^12^5 = (2^56^12^5 \circ 56^12^5)$ is a ladder.
Example

The walk

$$\eta = 256652 =$$

is a ladder.
Let $\Gamma$ be a connected graph. A corolla of basis $i$ in $\Gamma$ is a closed walk $\omega = i \text{Cycl}_1 \text{Cycl}_2 \cdots = \cdots (\text{Cycl}_1 \circ \text{Cycl}_2) \cdots \circ \text{Cycl}_n$ where $\text{Cycl}_1, \ldots, \text{Cycl}_n$ are simple cycles of basis $i$.

Examples

The walk $\mu = 11111 = 1123$ is a corolla.

The walk $\nu = 123412451 = 12345678$ is a corolla.
Definition

Let $\Gamma$ be a connected graph. A corolla of basis $i$ in $\Gamma$ is a closed walk

$$\omega = \cdots \left( \text{Cycl}_1 \circ \text{Cycl}_2 \right) \cdots \circ \text{Cycl}_n$$

where $\text{Cycl}_1, \ldots, \text{Cycl}_n$ are simple cycles of basis $i$. 

Examples

The walk $\mu = 11111 = 11234$ is a corolla.

The walk $\nu = 123412451 = 12345678$ is a corolla.
Definition

Let $\Gamma$ be a connected graph. A corolla of basis $i$ in $\Gamma$ is a closed walk

$$\omega = \ldots (\text{Cycl}_1 \circ \text{Cycl}_2) \ldots \circ \text{Cycl}_n$$

where $\text{Cycl}_1, \ldots, \text{Cycl}_n$ are simple cycles of basis $i$.

Examples
Definition

Let $\Gamma$ be a connected graph. A corolla of basis $i$ in $\Gamma$ is a closed walk

$$\omega = \ldots (\text{Cycl}_1 \odot \text{Cycl}_2) \ldots ) \odot \text{Cycl}_n$$

where $\text{Cycl}_1, \ldots, \text{Cycl}_n$ are simple cycles of basis $i$.

Examples

- The walk $\mu = 11111 = (1) \odot (2) \odot (3) \odot (4)$ is a corolla.
Definition

Let $\Gamma$ be a connected graph. A corolla of basis $i$ in $\Gamma$ is a closed walk

$$ \omega = \cdots (\text{Cycl}_1 \odot \text{Cycl}_2) \cdots ) \odot \text{Cycl}_n $$

where $\text{Cycl}_1, \ldots, \text{Cycl}_n$ are simple cycles of basis $i$.

Examples

- The walk $\mu = 11111 = \cdots$ is a corolla.
- The walk $\nu = 123412451 = \cdots$ is a corolla.
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let $\Gamma$ be a connected finite or countable graph. Let define $\text{Lad}(\Gamma)$, $\text{Cor}(\Gamma)$ and $\text{Cor}_i(\Gamma)$ the vector spaces spanned by ladders, corollas and corollas of base $i$ respectively. Then the spaces $(T\langle \text{Lad}(\Gamma) \rangle, \star, \Delta_H)$, $(T\langle \text{Cor}_i(\Gamma) \rangle, \star, \Delta_H)$, $(T\langle \text{Cor}(\Gamma) \rangle, \star, \Delta_H)$ are Hopf subalgebras of $(T\langle W(\Gamma) \rangle, \star, \Delta_H)$.

The spaces $(S\langle \text{Lad}(\Gamma) \rangle, \square, \Delta_H)$, $(S\langle \text{Cor}_i(\Gamma) \rangle, \square, \Delta_H)$, $(S\langle \text{Cor}(\Gamma) \rangle, \square, \Delta_H)$ are Hopf subalgebras of $(S\langle W(\Gamma) \rangle, \square, \Delta_H)$. 
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1. the spaces $(\mathcal{T}\langle \text{Lad}(\Gamma) \rangle, \star, \Delta_H)$, $(\mathcal{T}\langle \text{Cor}_i(\Gamma) \rangle, \star, \Delta_H)$, $(\mathcal{T}\langle \text{Cor}(\Gamma) \rangle, \star, \Delta_H)$ are Hopf subalgebras of $(\mathcal{T}\langle \text{W}(\Gamma) \rangle, \star, \Delta_H)$.

2. the spaces $(\mathcal{S}\langle \text{Lad}(\Gamma) \rangle, \Box, \Delta_H)$, $(\mathcal{S}\langle \text{Cor}_i(\Gamma) \rangle, \Box, \Delta_H)$, $(\mathcal{S}\langle \text{Cor}(\Gamma) \rangle, \Box, \Delta_H)$ are Hopf subalgebras of $(\mathcal{S}\langle \text{W}(\Gamma) \rangle, \Box, \Delta_H)$. 
Walks ans cacti

Definition

Let $\Gamma$ be a connected graph. A walk $\omega = w_1 \ldots w_m$ is a cactus if and only if for any $1 \leq l < l' \leq m$ the two following statements are equivalent:

- $w_l = w_{l'}$
- $\omega_{ll'}$ is a loop-erased cycle of $\omega$.

The vector space spanned by the cacti of $\Gamma$ is denoted by $\text{Cact}(\Gamma)$.

Example

The walk $\sigma = 12324522 = \sigma_{1234522}$ is a cactus.
Definition

Let $\Gamma$ be a connected graph. A walk $\omega = w_1 \ldots w_m$ is a cactus if and only if for any $1 \leq l < l' \leq m$ the two following statements are equivalent:

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Example

The walk $\sigma = 1234522 = \begin{array}{c} \begin{array}{c} 5 \end{array} \\ \begin{array}{c} 4 \end{array} \\ \begin{array}{c} 3 \end{array} \end{array} = \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \begin{array}{c} 4 \end{array} \begin{array}{c} 5 \end{array} \begin{array}{c} 6 \end{array} \begin{array}{c} 7 \end{array}$ is a cactus.
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

1. The space \((T\langle C_{\text{cact}}(\Gamma) \rangle, \star, \Delta_{\text{H}})\) is a Hopf subalgebra of \((T\langle W(\Gamma) \rangle, \star, \Delta_{\text{H}})\).

2. The spaces \((S\langle C_{\text{cact}}(\Gamma) \rangle, \boxtimes, \Delta_{\text{H}})\) is a Hopf subalgebras of \((S\langle W(\Gamma) \rangle, \boxtimes, \Delta_{\text{H}})\).
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let $\Gamma$ be a connected finite or countable graph.

1. The space $(T\langle C\text{act}(\Gamma) \rangle, \star, \Delta_H)$ is a Hopf subalgebra of $(T\langle W(\Gamma) \rangle, \star, \Delta_H)$.

2. The spaces $(S\langle C\text{act}(\Gamma) \rangle, \square, \Delta_H)$, is a Hopf subalgebras of $(S\langle W(\Gamma) \rangle, \square, \Delta_H)$. 
Walks ans cacti

Quotient Hopf algebras

Let consider $\Omega$ the complete graph such that $\text{Nod}(\Omega) = \mathbb{N}^*$ and $\text{Edg}(\Omega) = \{(i, j) \in (\mathbb{N}^*)^2\}$. Let $J_1$ and $J_2$ be the vector spaces defined by:

$$J_1 = \text{Span}(\omega_1 | \ldots | \omega_s - f_1(\omega_1) | \ldots | f_s(\omega_s), s \in \mathbb{N}^*, \forall i \in \{1, \ldots, s\}, \omega_i \in \text{Cact}(\Omega), f_i \in F)$$

and

$$J_2 = \text{Span}(\omega_1 \Box \ldots \Box \omega_s - f_1(\omega_1) \Box \ldots \Box f_s(\omega_s), s \in \mathbb{N}^*, \forall i \in \{1, \ldots, s\}, \omega_i \in \text{Cact}(\Omega), f_i \in F),$$

where $f$ is an admissible label of $\omega_i$. 

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**Definition**

Let consider $\Omega$ the complete graph such that $\text{Nod}(\Omega) = \mathbb{N}^*$ and $\text{Edg}(\Omega) = \{(i, j) \in (\mathbb{N}^*)^2\}$. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be the vector spaces defined by:

\[
\mathcal{J}_1 = \text{Span}\left(\omega_1 | \ldots | \omega_s - f_1(\omega_1) | \ldots | f_s(\omega_s), \right.
\]

\[
s \in \mathbb{N}^*, \quad \forall i \in \{1, \ldots, s\}, \quad \omega_i \in \text{Cact}(\Omega), \quad f_i \in \mathcal{F},
\]

and

\[
\mathcal{J}_2 = \text{Span}\left(\omega_1 \square \ldots \square \omega_s - f_1(\omega_1) \square \ldots \square f_s(\omega_s), \right.
\]

\[
s \in \mathbb{N}^*, \quad \forall i \in \{1, \ldots, s\}, \quad \omega_i \in \text{Cact}(\Omega), \quad f_i \in \mathcal{F},
\]

where $f$ is an admissible label of $\omega_i$. 
Let consider the walk $\sigma = 12324522$. The walk $\sigma_1 = 13532733$ or the walk $\sigma_2 = 23435633$ give admissible labels of $\sigma$. 

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Examples

Let consider the walk \( \sigma = 12324522 = \)

The walk \( \sigma_1 = 13532733 = \)

or the walk

\( \sigma_2 = 23435633 \)

give admissible labels of \( \sigma \).
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco) \[ T_{\langle W(\Omega) \rangle}^{J_1} \text{ (respectively } S_{\langle W(\Omega) \rangle}^{J_2} \text{)} \] is a Hopf algebra called the tensor Hopf algebra of unlabeled walks (respectively the symmetric Hopf algebra of unlabeled walks).

Remark To compute the coproduct of a walk \( \omega \) in \( T_{\langle W(\Omega) \rangle}^{J_1} \) or in \( S_{\langle W(\Omega) \rangle}^{J_2} \) it is sufficient to compute it by equipping \( \omega \) with an admissible label and after that forgot all vertices' labels.
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

The space \( \frac{T\langle W(\Omega) \rangle}{J_1} \) (respectively \( \frac{S\langle W(\Omega) \rangle}{J_2} \)) is a Hopf algebra called the tensor Hopf algebra of unlabeled walks (respectively the symmetric Hopf algebra of unlabeled walks).
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Example

Let consider the walk $\eta = 1312321$ in $T_{W(\Omega)}$. In $T_{W(\Omega)}$, we get:

$$\Delta H(123456) = 123456 \otimes 1 + 123456 \otimes 123456 + 123456 \otimes 456 + 123456 \otimes 36456.$$
Example

Let consider the walk $\eta = 1312321$ in $\mathcal{T}(\mathcal{W}(\Omega))$. In $\mathcal{T}(\mathcal{W}(\Omega))$ we get:

$$\Delta_H \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \right) = \begin{array}{c} 1 \\ 2 \end{array} \otimes 1 + 1 \otimes \begin{array}{c} 1 \\ 2 \end{array} + \begin{array}{c} 1 \\ 2 \end{array} \otimes \begin{array}{c} 4 \\ 5 \\ 6 \end{array} + \begin{array}{c} 1 \\ 2 \end{array} \otimes \begin{array}{c} 4 \\ 5 \\ 6 \end{array}.$$
Theorem (L. Foissy, P.L. Giscard, C. M., M. Ronco)

Let $\Gamma$ be a finite or a countable connected graph. Let $\omega = w_1 \ldots w_m$ a walk in $\Gamma$. There exist at least one admissible label $f$ making $f(\omega) = f(w_1) \ldots f(w_m)$ into a cactus. Let $\text{Forgot}(f(\omega))$ the unlabeled cactus where the labels of vertices are forgotten. Then, the algebra morphisms $\phi_1: \{T\langle W(\Gamma) \rangle \to T\langle W(\Omega) \rangle \}$ $\omega \mapsto \text{Forgot}(f(\omega))$ and $\phi_2: \{S\langle W(\Gamma) \rangle \to S\langle W(\Omega) \rangle \}$ $\omega \mapsto \text{Forgot}(f(\omega))$ are Hopf algebra morphisms.
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$$\varphi_1 : \begin{cases} T \langle W(\Gamma) \rangle & \rightarrow \frac{T \langle W(\Omega) \rangle}{J_1} \\ \omega & \mapsto \text{Forgot}(f(\omega)) \end{cases}$$

and

$$\varphi_2 : \begin{cases} S \langle W(\Gamma) \rangle & \rightarrow \frac{S \langle W(\Omega) \rangle}{J_2} \\ \omega & \mapsto \text{Forgot}(f(\omega)) \end{cases}$$

are Hopf algebra morphisms.
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<td>Example 1</td>
<td>$\phi_1(3, 4, 5, 6, 7, 8, 9, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 1)$</td>
</tr>
</tbody>
</table>
Example 1

\[ \varphi_1 \left( \begin{array}{c} 3 \rightarrow 4 \rightarrow 8 \rightarrow 6 \rightarrow 9 \rightarrow 7 \rightarrow 13 \end{array} \right) = \varphi_1 \left( \begin{array}{c} 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10 \rightarrow 11 \rightarrow 12 \rightarrow 13 \rightarrow 7 \end{array} \right) = \varphi_1 \left( \begin{array}{c} 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10 \rightarrow 11 \rightarrow 12 \rightarrow 13 \rightarrow 7 \end{array} \right) \]
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<tr>
<td>( \phi_1(1, 2, 3, 4, 5, 1, 2, 3, 4) )</td>
<td>( \phi_1(1, 2, 4, 5, 7, 3, 6, 7, 8) )</td>
</tr>
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</table>
Example 2

\[ \varphi_1 \left( \begin{pmatrix} 3 & 2 & 5 & 4 \\ 2 & 6 & 1 & 8 \\ 1 & 5 & 4 & 7 \\ 4 & 3 & 5 & 6 \end{pmatrix} \right) = \varphi_1 \left( \begin{pmatrix} 3 & 2 & 5 & 4 \\ 2 & 6 & 1 & 8 \\ 1 & 5 & 4 & 7 \\ 4 & 3 & 5 & 6 \end{pmatrix} \right) = \begin{pmatrix} 3 & 2 & 5 & 4 \\ 2 & 6 & 1 & 8 \\ 1 & 5 & 4 & 7 \\ 4 & 3 & 5 & 6 \end{pmatrix} . \]