Mathematical Institute
of the Serbian Academy of Sciences and Arts

Minimal models for graphs-related operadic algebras

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arXiv:2002.06640

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The *triad* of an operadic category

level 1	$\mathtt{O}^+$-operads
	$\Downarrow$
level 0	$\mathtt{O}$-operads $= 1_{\mathtt{O}^+}$-algebras
	$\Downarrow$
level -1	algebras of $\mathtt{O}$-operads

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level -1 $\quad$ algebras of $\mathtt{O}$-operads

$\mathtt{Fin}^+$-operads $= \mathtt{RTr}$-operads

$\Downarrow$

$\mathtt{Fin}$-operads = classical operads $= 1_{\mathtt{RTr}}$-algebras

$\Downarrow$

algebras over classical operads

strongly homotopy operads $= \mathfrak{M}_{\mathtt{RTr}}$-algebras

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$\mathtt{Tr}^+$-operads

$\Downarrow$

$\mathtt{Tr}$-operads $= 1_{\mathtt{Tr}^+}$-algebras

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algebras of $\mathtt{Tr}$-operads $\ni$ cyclic operads $= 1_{\mathtt{Tr}}$-algebras

strongly homotopy cyclic operads $= \mathfrak{M}_{\mathtt{Tr}}$-algebras

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$\mathtt{ggGrc}^+$-operads

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$\mathtt{ggGrc}$-operads $= 1_{\mathtt{ggGrc}^+}$-algebras

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Goal & Methods used

Construct $\mathfrak{M}_{\mathtt{RTr}}$, $\mathfrak{M}_{\mathtt{Tr}}$ and $\mathfrak{M}_{\mathtt{ggGrc}}$.

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FREE OPERADS IN THE OPERADIC CATEGORY OF GRAPHS

Operadic categories: formal notions of fibres and fibre maps

An operadic category is a category $\mathtt{O}$ endowed with

- a chosen local terminal object U_c, for each $c \in \pi_0(\mathtt{O})$,
- a cardinality functor $|-| : \mathtt{O} \to \mathtt{Fin}$,
- for each $S \in \mathtt{O}$ and $i \in |S|$, a fibre functor $\varphi_{S,i} : \mathtt{O}/S \to \mathtt{O}$

$$T \xrightarrow{f} S \quad \mapsto \quad f^{-1}(i)$$

$$\begin{array}{ccc} T & \xrightarrow{f} & S \\ & {\scriptstyle gf}\searrow \quad \swarrow{\scriptstyle g} & \\ & R & \end{array} \quad \mapsto \quad f_i^g : (gf)^{-1}(i) \to g^{-1}(i),$$

subject to axioms ensuring that the fibres retain properties of fibres in $\mathtt{Fin}$.

Operadic category Fin of finite ordinals

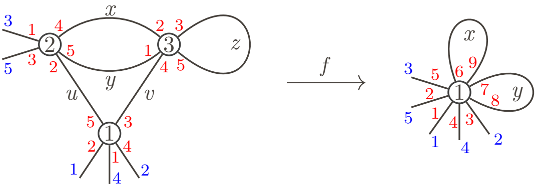
- The objects are $\underline{n} = \{1, \ldots, n\}$, for $n \in \mathbb{N}$.
- The morphisms are arbitrary functions.
- $\mathtt{Fin}$ has a unique terminal object $\underline{1}$.
- The cardinality functor is the identity.
- The action of the fibre functor $\varphi_{\underline{n},i} : \mathtt{Fin}/\underline{n} \to \mathtt{Fin}$ is given by

$$\underline{m} \xrightarrow{f} \underline{n} \quad \longmapsto \quad \{j \in \underline{m} \,|\, f(j) = i\}$$

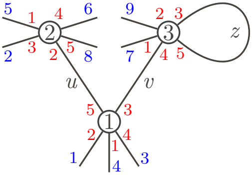
$$\begin{array}{ccc} \underline{m} & \xrightarrow{f} & \underline{n} \\ & {\scriptstyle gf}\searrow \quad \swarrow{\scriptstyle g} & \\ & \underline{l} & \end{array} \quad \longmapsto \quad \begin{array}{l} f_i^g : (gf)^{-1}(i) \to g^{-1}(i) \\ \qquad\qquad\qquad j \mapsto f(j) \end{array}$$

Operadic category Grc of connected directed graphs

- The objects are connected graphs with three kinds of indexing.



- The morphisms are given by contractions of subgraphs.
- Chosen local terminal objects are corollas with global order = local order.
- The cardinality functor associates to a graph its set of vertices.
- Fibers of a morphism are the subgraphs contracted to a vertex.

$f^{-1}(1) =$ 

Operads in the framework of operadic categories

An $\mathtt{O}$-operad is a $\mathtt{V}$-presheaf $\mathcal{M} : \mathtt{Iso}_{\mathtt{O}}^{op} \to \mathtt{V}$, together with

- units $\eta_c : \mathbb{k} \to \mathcal{M}(U_c)$, $c \in \pi_0(\mathtt{O})$, and
- structure maps

$$\circ_\phi : \mathcal{M}(S) \otimes \mathcal{M}(F) \to \mathcal{M}(T), \quad F \triangleright T \xrightarrow{\phi} S,$$

such that the units behave nicely, a natural equivariance axiom is satisfied, and so are the two associativity axioms.

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- The category of $\mathtt{Fin}$-operads is isomorphic to the category of classical symmetric operads.
- The category of $\mathtt{Grc}$-operads is isomorphic to the category of *hyperoperads* (without the genus grading) of Getzler & Kapranov.

Modular operads without the genus grading are algebras over $\mathtt{1}_{\mathtt{Grc}}$.

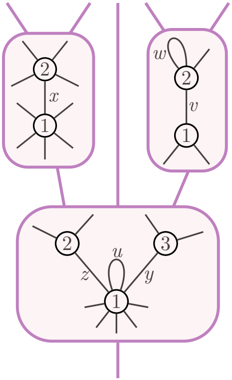
Algebras over O-operads

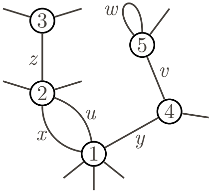
Modular operads without the genus grading are algebras over $1_{\mathtt{Grc}}$.

An algebra over an O-operad $\mathcal{P}$ is collection $A = \{A_c\}_{c\in\pi_0(\mathtt{O})}$, $A_c \in \mathtt{Vect}$, together with an O-operad map $\alpha : \mathcal{P} \to End_A^{\mathtt{O}}$.

$$End_A^{\mathtt{O}}(T) := \mathtt{Vect}\Big(\bigotimes\nolimits_{c\in\pi_0(s(T))} A_c, A_{\pi_0(T)}\Big)$$

Free operads in Grc

$T =$ 

$\Gamma =$ 

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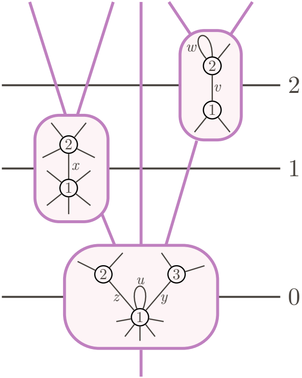
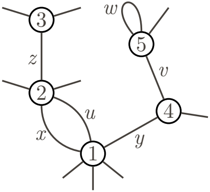
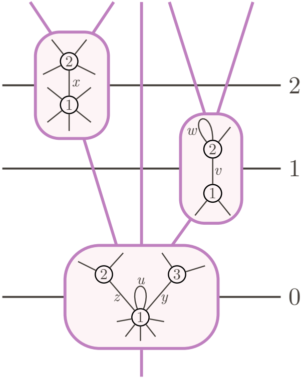
$\lambda_1 =$ 

$\Gamma =$ 

Free operads in `Grc`

$\lambda_2 =$



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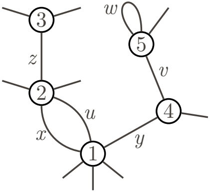

Free operads in $\mathtt{Grc}$

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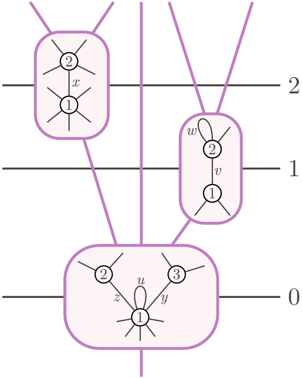

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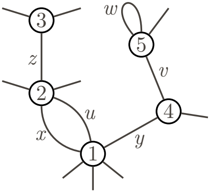

$$\mathbb{F}(E)(\Gamma) \cong \begin{cases} \bigoplus_{T\in \mathtt{gTr}(\Gamma)} \operatorname{colim}_{\lambda\in \mathtt{Lev}(T)} E(T,\lambda), & \text{if } |\mathrm{edg}(\Gamma)| \geq 1 \\ \Bbbk, & \text{if } |\mathrm{edg}(\Gamma)| = 0 \end{cases}$$

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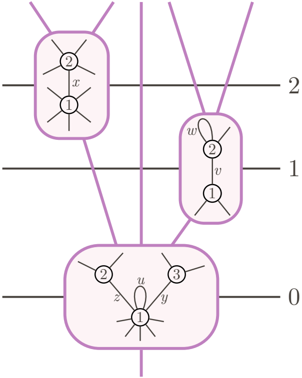
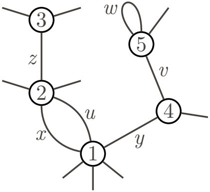
$\lambda_2 =$ 

$\Gamma =$ 

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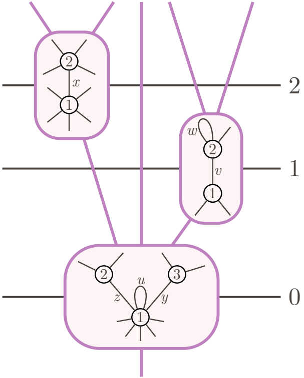
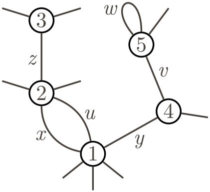
$\lambda_2 =$ 

$\Gamma =$ 

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$\circ_\phi : \mathbb{F}(E)(T) \otimes \mathbb{F}(E)(F) \to \mathbb{F}(E)(S)$, for $F \triangleright_i S \xrightarrow{\phi} T$, is the grafting

Algebras over $1_{\mathtt{Grc}}$

Theorem

Algebras over the terminal $\mathtt{Grc}$-operad $1_{\mathtt{Grc}}$, having $1_{\mathtt{Grc}}(\Gamma) = \Bbbk$ for each $\Gamma \in \mathtt{Grc}$ and with structure operations $\Bbbk \otimes \Bbbk \cong \Bbbk$, are modular operads without the genus grading.

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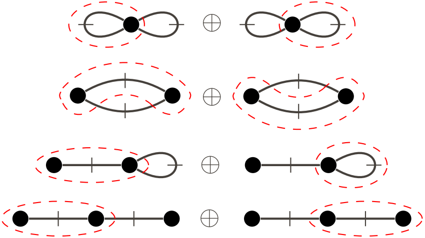
Proof. $1_{\mathtt{Grc}} \cong \mathbb{F}(E)/(R)$

$$E[\Gamma] := \begin{cases} \mathbb{k}, & \text{if } |\mathrm{edg}(\Gamma)| = 1 \\ 0, & \text{otherwise} \end{cases}$$



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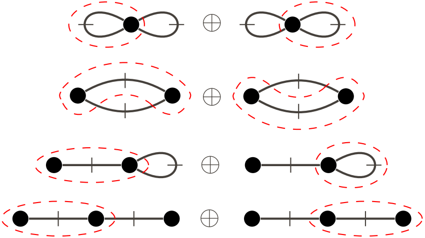


The operations $\alpha_\Gamma : E[\Gamma] \to End_M^{\texttt{Grc}}$ are non-trivial only for $\Gamma \in \{$  $\}$ and they satisfy the axioms $\tilde{\alpha}(r_i) = 0$.

The main theorem

The object $\mathfrak{M}_{\mathtt{Grc}} := (\mathbb{F}(D), \partial) \xrightarrow{\rho} (1_{\mathtt{Grc}}, 0)$, where

- $D(\Gamma) := \begin{cases} \bigwedge^{|\mathsf{edg}(\Gamma)|}(\mathsf{S}_{\Bbbk}(\mathsf{edg}(\Gamma))), & \text{if } |\mathsf{edg}(\Gamma)| \geq 1 \\ 0, & \text{if } |\mathsf{edg}(\Gamma)| = 0 \end{cases}$
- $\partial_\Gamma(e_1 \wedge \cdots \wedge e_n) := \bigoplus_{T \in \mathtt{gTr}^2(\Gamma)} (-1)^{|\mathsf{edg}(\Gamma)|-1}(e_1^b \wedge \cdots \wedge e_k^b) \otimes (e_1^t \wedge \cdots \wedge e_l^t)$
- $\rho|_{D(\Gamma)} := \begin{cases} 1_{\Bbbk} : D(\Gamma) = \Bbbk \to \Bbbk = 1_{\mathtt{Grc}}, & \text{if } |\mathsf{edg}(\Gamma)| = 1 \\ 0, & \text{if } |\mathsf{edg}(\Gamma)| \geq 2 \end{cases}$

is the minimal model of the terminal $\mathtt{Grc}$-operad $1_{\mathtt{Grc}}$.

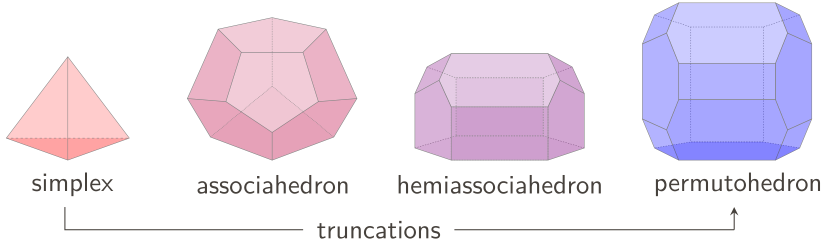
Algebras over $\mathfrak{M}_{\mathtt{Grc}}$ are strongly homotopy modular operads without the genus grading.

HYPERGRAPH POLYTOPES

Hypergraph polytopes

K. Došen, Z. Petrić
Hypergraph polytopes
Topology and its Applications 158, pp. 1405–1444, 2011

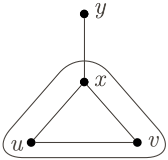
P.-L. Curien, J. Obradović, J. Ivanović
Syntactic aspects of hypergraph polytopes
Journal of Homotopy and Related Structures 14, pp. 235–279, 2019



Hypergraph terminology

$$H = \{x, y, u, v\}$$

$$\mathbf{H} = \{\{x\}, \{y\}, \{u\}, \{v\}, \{x, u\}, \{u, v\}, \{x, v\}, \{x, y\}, \{x, u, v\}\}$$



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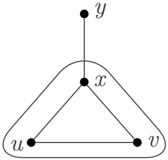

$\mathbf{H}$ is **saturated**: $(\forall X, Y \in \mathbf{H})\ X \cap Y \neq \emptyset \Rightarrow X \cup Y \in \mathbf{H}$

$$Sat(\mathbf{H}) = \mathbf{H} \cup \{\{x, y, u\}, \{x, y, v\}, \{x, y, u, v\}\}$$

A polytope from a hypergraph

Hemiassociahedron

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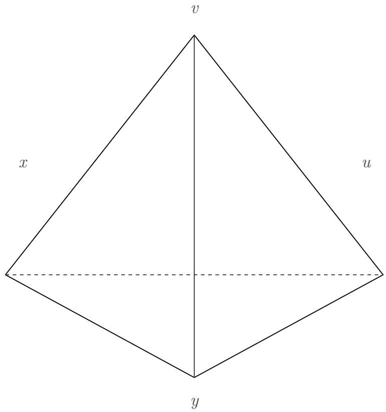

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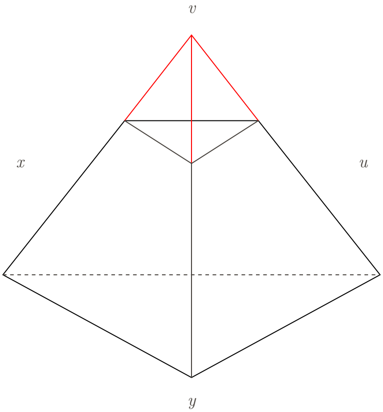

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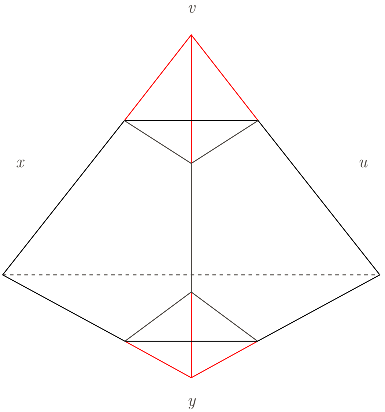

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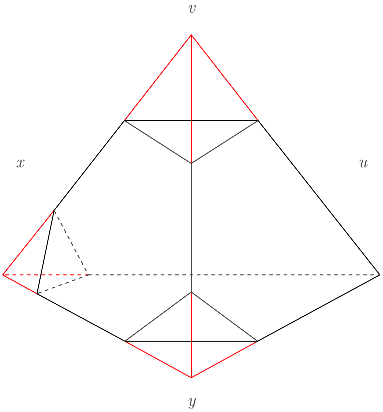

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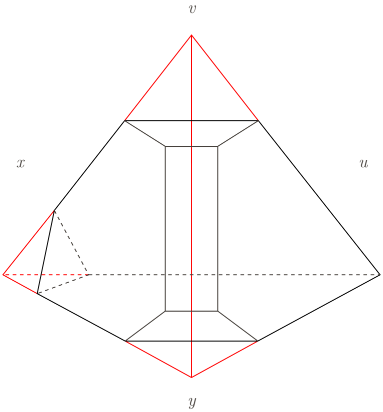

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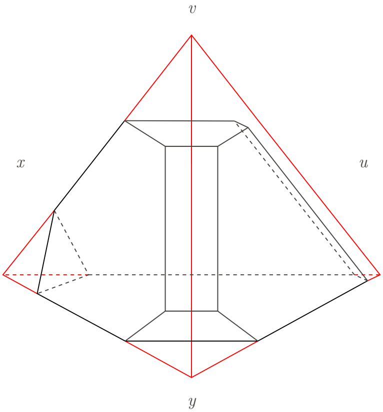

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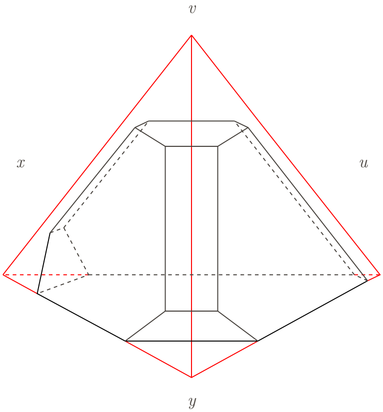

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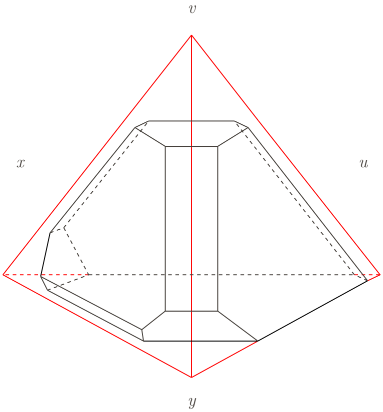

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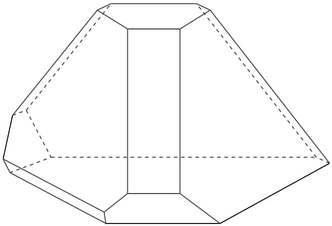

Combinatorial description of hypergraph polytopes: poset $\mathcal{A}(\mathbf{H})$

- Elements: constructs $C : \mathbf{H}$
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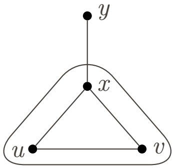

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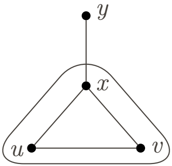

$\{x, y, u, v\}$

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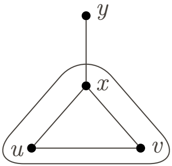

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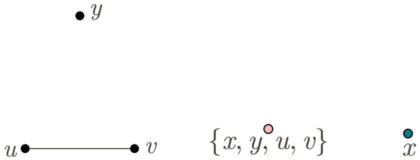

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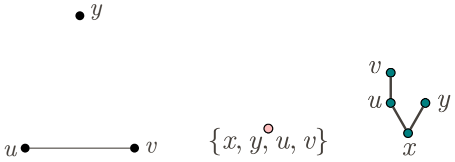

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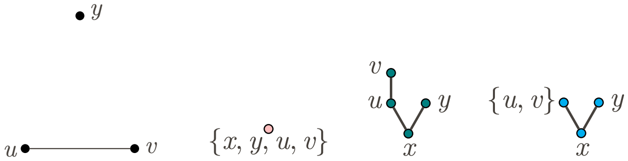

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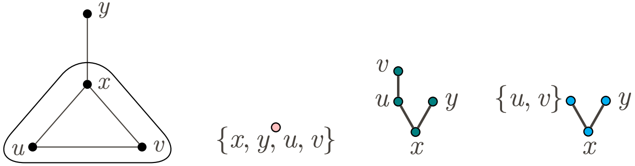

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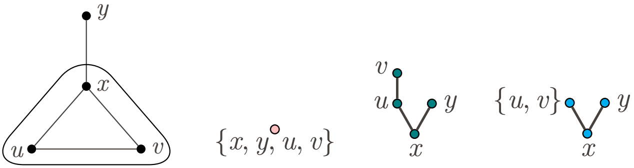
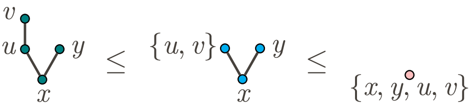


- Partial order: edge contraction



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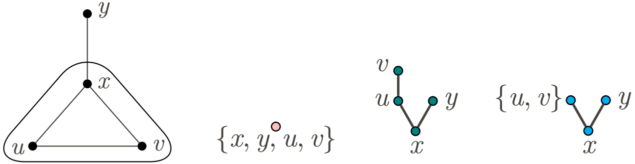

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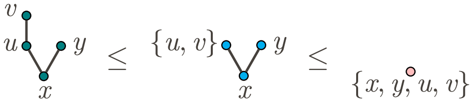
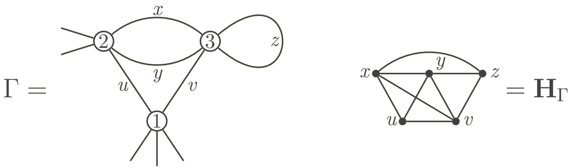


Theorem

$\mathcal{A}(\mathbf{H})$ is order-isomorphic to the face lattice of a convex polytope $\mathcal{G}(\mathbf{H})$ obtained as a truncation of the $(|H| - 1)$-dimensional simplex.

THE MAIN THEOREM: DEMYSTIFICATION AND PROOF

Constructs represent graph-trees



Theorem

There exists an order-iso $\alpha_\Gamma : \mathcal{A}(\mathbf{H}_\Gamma) \longrightarrow \mathtt{gTr}(\Gamma)$.

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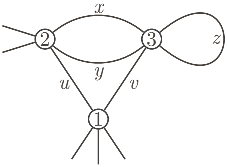
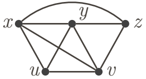
$\Gamma =$ 



$= \mathbf{H}_\Gamma$

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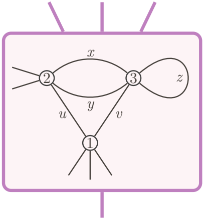
There exists an order-iso $\alpha_\Gamma : \mathcal{A}(\mathbf{H}_\Gamma) \longrightarrow \mathtt{gTr}(\Gamma)$.

Proof.

$$\alpha_\Gamma(\ \{x, y, \mathring{z}, u, v\}\ ) =$$



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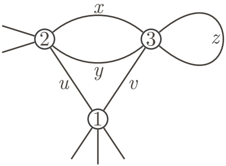
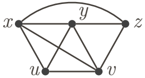
$\Gamma =$ 



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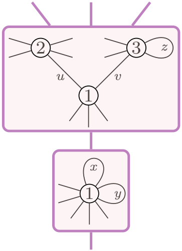
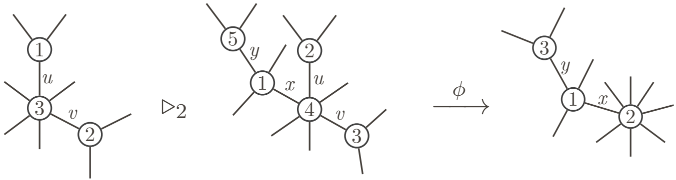
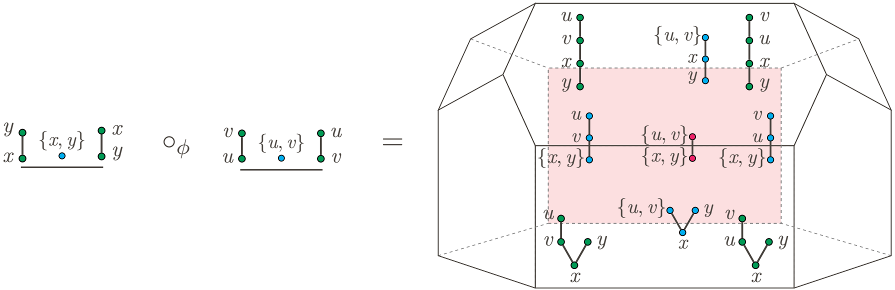
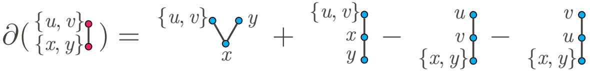
$\alpha_\Gamma\Big(\begin{smallmatrix}\{u, v, z\} \\ \{x, y\}\end{smallmatrix} \Big) =$ 

The main theorem *in English*

This is the minimal model of $1_{\mathtt{Grc}}$:







The ingenious lemma

The faces of $\mathcal{G}(\mathbf{H}_\Gamma)$ can be oriented, so that $(\bigoplus_{C:\mathbf{H}_\Gamma} \mathsf{S}_{\Bbbk}(\{e_C\}), \partial)$ is the cellular chain complex $(C_*(\mathcal{G}(\mathbf{H}_\Gamma)), \partial_*)$ of free abelian groups $C_k(\mathcal{G}(\mathbf{H}_\Gamma))$ generated by k-dimensional faces of $\mathcal{G}(\mathbf{H}_\Gamma)$, whose differential ∂_* is given by

$$\partial_*(\lambda) := \sum_{\delta \lessdot \lambda} \eta_\lambda^\delta \cdot \delta,$$

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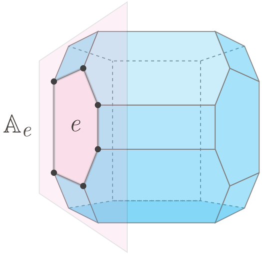

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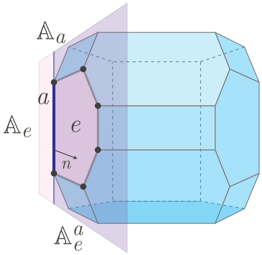

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$$\partial(h) = \eta' \cdot e' + \eta'' \cdot e'' + \cdots, \qquad \eta', \eta'' \in \{-1, +1\}$$

$$\partial(e') = \varepsilon' \cdot a + \cdots \qquad \partial(e'') = \varepsilon'' \cdot a + \cdots, \quad \varepsilon', \varepsilon'' \in \{-1, +1\}$$

$$\eta'\varepsilon' + \eta''\varepsilon'' = 0$$

The proof of the main theorem

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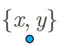
- We also calculate $\rho(\partial($ 

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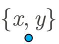
- We also calculate $\rho(\partial($ 

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M. Batanin, M. Markl and J. Obradović
Minimal models for graphs-related operadic algebras
arXiv:2002.06640, 2020

Thank you!

*This work has been supported by the Praemium Academiae of M. Markl and RVO:67985840.