Intersection Type Distributors

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Outline

Intersection Types

- Multiple typing becomes relevant (Coppo-Dezani 1978):

$$A, B ::= a \mid A \Rightarrow B \mid A \cap B \mid \Omega$$

- $A \cap B$ can be associative, commutative, idempotent.
- When $A \cap A \neq A$ the system becomes *resource sensitive.*

Denotational semantics

$$[\![M]\!] = \{(\Gamma, A) \mid \Gamma \vdash M : A\}$$

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A Simple Model of Linear Logic

- The category of sets and relations (Rel).

$$\mathrm{ob}(\mathrm{Rel}) = \mathrm{ob}(\mathrm{Set}) \qquad \mathrm{Rel}(X, Y) = \wp(X \times Y)$$

- Resources are *multisets*. (Rel + lists is not CCC).
  Daniel de Carvalho. “Semantique de la logique lineaire et temps de calcul”. In: PhD thesis, Aix-Marseille Université, 2007
- Syntactic presentation *via* an intersection types.
- Generalized to Kleisli categories of preorders (PoIr).
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Proof-Irrelevance of Polr

$$[\![M]\!](\Delta, a) = \begin{cases} 1 & \text{if } \Delta \vdash M : a \\ 0 & \text{otherwise.} \end{cases}$$

The structure

$$(\!|M|\!)(\Delta, a) = \left\{ \begin{array}{c} \pi \\ \vdots \\ \Delta \vdash M : a \end{array} \right\}$$

is neither a relation nor a *denotational semantics*.

Reasoning on type derivations happens *outside* the model.

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Taylor Expansion of λ-Terms

- *Quantitative semantics*: Types are some special kind of *vector spaces* and programs are *analytic functions*.
- The *differential* λ-calculus (Ehrhard-Regnier 2003).

$$\tau(MN) = \sum_{n=0}^{\infty} \frac{1}{n!}(D^n M \cdot N^n)0.$$

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What We Want

Set-Theoretic	Category-Theoretic
sets	categories
elements of sets	objects of categories
equations between elements	isomorphisms between objects
functions	functors
functions equations	functors natural isomorphisms

Relations $\Rightarrow$ Distributors

- General semantic framework where we can talk about both IT and Taylor expansion in a proof-relevant way.
- A *dynamic* denotational semantics.

$$M \to N \quad \Rightarrow \quad \beta : [\![M]\!] \cong [\![N]\!]$$

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- Melliès and Zeilberger: type systems as functors.
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- Mazza, Pellissier and Vial: operadic approach. Application: parametric intersection types constructions. Damiano Mazza, Luc Pellissier, and Pierre Vial. "Polyadic approximations, fibrations and intersection types". In: *PACMPL*. 2018
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Our Results

- *Proof-relevant* semantics :

$$[\![M]\!](\Delta, a) \cong \left\{ \begin{array}{c} \tilde{\pi} \\ \vdots \\ \Delta \vdash M : a \end{array} \right\} \cong \left\{ \begin{array}{r} \tilde{\varphi} \text{ approximates } M \\ \text{and } \Delta \vdash \varphi : a \end{array} \right\}$$

- *Parametric* generalization of IT and Taylor expansion.
- The natural isomorphism

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is given by the *evaluation* of approximants.

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Outline

The Idea Behind the Curtain

- Polr + comonad T;
- resources are *multisets*;
- standard subtyping;
- proof-irrelevant and "static" semantics.

- Dist + pseudomonad S;
- resources are *lists*;
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Distributors and Relations I

Relations

Let $A, B \in \mathrm{Set}$. We can see $f \subseteq A \times B$ as

$$\chi_f : A \times B \to \{0, 1\} \qquad \chi_f(a, b) = \begin{cases} 1 & \text{if } (a, b) \in f \\ 0 & \text{otherwise} \end{cases}$$

Distributors (aka Profunctors)

Let $A, B \in \mathrm{Cat}$. A distributor $F : A \nrightarrow B$ is a functor $F : B^o \times A \to \mathrm{Set}$. CAT is CC:

$$\frac{F : B^o \times A \to \mathrm{Set}}{F^\lambda : A \to \hat{B}}$$

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Distributors and Relations II

Composition of relations

Let $f \subseteq A \times B$, $g \subseteq B \times C$. Then

$$g \circ_{\mathrm{Rel}} f = \{(a, c) \mid \exists b \in B \text{ s.t } (a, b) \in f \text{ and } (b, c) \in g\}$$

Composition of Distributors

For $F : A \nrightarrow B$ and $G : B \nrightarrow C$ the composition:

$$(G \circ_{\mathrm{Dist}} F)(c, a) = \int^{b \in B} G(c, b) \times F(b, a) \cong \sum_{b \in B} H(b, b)/\sim$$

where $H(-,-) = G(c,-) \times F(-, a)$.

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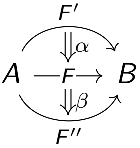
The Bicategory of Distributors



$$(H \circ G) \circ F \cong H \circ (G \circ F) \qquad F \circ 1 \cong 1 \circ F \cong F$$

Distributors:

- *0-cells* small categories $A, B, C \dots$;
- *1-cells* $F : A \nrightarrow B$ are distributors $F : B^o \times A \to \mathrm{Set}$;
- *2-cells* $\alpha : F \Rightarrow G$ are natural transformations.

$$A \otimes B = A \times B \qquad A \multimap B = A^o \times B \qquad A \,\&\, B = A \sqcup B.$$

Gian Luca Cattani and Glynn Winskel. “Profunctors, open maps and bisimulation”. In: *MSCS* (2005)

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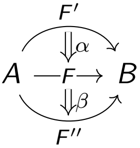

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Interpreting the Exponential

- Standard categorical semantics: a *comonad* $\langle !, \mathrm{der}, \mathrm{dig} \rangle$.

$$! : C \to C \qquad \mathrm{der}_X : !X \to X \qquad \mathrm{dig}_X : !X \to !!X$$

- The *relational case*:

$$X \mapsto M_f X$$

(the underlying set of) the free commutative monoid.

- *Idempotency*:

$$\mathcal{X} \mapsto L\mathcal{X}$$

(the underlying preorder of) the free bounded $\wedge$-semilattice.

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Resource Monads

2-monads over Cat that model resource management.

$A \mapsto SA$ SA is the free S-monoidal strict category over A

	linear	semicartesian (affine)	relevant	cartesian
Symmetry	yes	yes	yes	yes
Terminal object	no	yes	no	yes
Diagonals	no	no	yes	yes

Objects:

$$\vec{a}, \vec{b} ::= \langle a_1, \ldots, a_k \rangle \qquad a_i \in A$$

Morphisms:

$$\frac{\alpha \in [n]^{[m]} \qquad f_1 : a_{\alpha(1)} \to b_1 \ldots f_m : a_{\alpha(m)} \to b_m}{\langle \alpha, f_1, \ldots, f_m \rangle : \langle a_1, \ldots, a_n \rangle \to \langle b_1, \ldots, b_m \rangle}$$

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Lifting of Resource Monads to Distributors

M. Fiore et al. "Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures". In: *Selecta Mathematica* (2017)

- The presheaf construction gives a *relative pseudomonad* over the inclusion $J : \mathrm{Cat} \to \mathrm{CAT}$.
- $\mathrm{Dist} = \mathrm{Kleisli}(\hat{-})$:

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Let $(\mathcal{S}\text{-Dist})^{op} ::= \mathcal{S}\text{-CatSym}$.

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Nicola Gambino and André Joyal. "On operads, bimodules and analytic functors". In: *M. of the American Mathematical Society* (2017)

Theorem (O. 2020)

If the tensor product of SA is symmetric then $\mathcal{S}\text{-CatSym}$ is a CC bicategory.

The Parametric Bicategory $\mathcal{S}\text{-CatSym}$

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Non-Extensional Models for pure λ-calculus

Let A be a small category. We define by induction a family of small categories as follows:

$$D_0 = A \qquad D_{n+1} = (SD_n^o \times D_n) \sqcup A$$

Then we set $D_A = \varinjlim_{n \in \mathbb{N}} D_n$.

D_A is just the free algebra on A for the (unpointed) endofunctor $(S-)^o \times - : \mathrm{Cat} \to \mathrm{Cat}$.

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D_A is just the free algebra on A for the (unpointed) endofunctor $(S-)^o \times - : \mathrm{Cat} \to \mathrm{Cat}$.

The Category of Types D_A

Types:

$$a := x \in A \mid \langle a_1, \dots, a_k \rangle \Rightarrow a$$

Morphisms:

$$\frac{f \in A(x, x')}{f : x \to x'} \qquad \frac{\langle \sigma, \vec{f} \rangle : \vec{a}' \to \vec{a} \qquad f : b \to b'}{\langle \alpha, \vec{f} \rangle \Rightarrow f : (\vec{a} \Rightarrow b) \to (\vec{a}' \Rightarrow b')}$$

$$\frac{\alpha : [k'] \to [k] \quad f_1 : a_{\alpha(1)} \to a'_1 \cdots f_{k'} : a_{\alpha(k')} \to a'_{k'}}{\langle \alpha, f_1, \dots, f_{k'} \rangle : \langle a_1, \dots, a_k \rangle \to \langle a'_1, \dots, a'_{k'} \rangle}$$

A Tale of Two Systems

Relations (System R_X^T)

$$\frac{\bar{a}_i \leq [a] \quad \forall j \neq i, \bar{a}_j \leq []}{x_1 : \bar{a}_1, \ldots, x_i : \bar{a}_i, \ldots, x_n : \bar{a}_n \vdash x_i : a}$$

$$\frac{\Gamma, x : \bar{a} \vdash M : a}{\Gamma \vdash \lambda x.M : \bar{a} \Rightarrow a}$$

$$\frac{\Gamma_0 \vdash M : \bar{a} \Rightarrow a \qquad (\Gamma_i \vdash N : a_i)_{i \in [k]}}{\Delta \vdash MN : a}$$

where $\Delta \leq \sum_{j=0}^{k} \Gamma_j$.

Distributors (System E_A^S)

$$\frac{f_i : \vec{a}_i \to \langle a \rangle \quad \forall j \neq i, f_j : \vec{a}_j \to \langle\rangle}{x_1 : \vec{a}_1, \ldots, x_i : \vec{a}_i, \ldots, x_n : \vec{a}_n \vdash x_i : a}$$

$$\frac{\Gamma, x : \vec{a} \vdash M : a}{\Gamma \vdash \lambda x.M : \vec{a} \Rightarrow a}$$

$$\frac{\Gamma_0 \vdash M : \vec{a} \Rightarrow a \qquad (\Gamma_i \vdash N : a_i)_{i \in [k]}}{\Delta \vdash MN : a}$$

where $\eta : \Delta \to \sum_{j=0}^{k} \Gamma_j$.

Denotations are Distributors

Let $f : a \to a'$, $\eta : \Delta' \to \Delta$.

$$\begin{array}{c} \pi \\ \vdots \\ \Delta \vdash M : a \end{array} \quad \rightsquigarrow \quad \begin{array}{c} [f]\ \pi\ \{\eta\} \\ \vdots \\ \Delta' \vdash M : a' \end{array}$$

$$T_D(M)_{\vec{x}}(\Delta, a) = \left\{ \begin{array}{c} \tilde{\pi} \\ \vdots \\ \vec{x} : \Delta \vdash M : a \end{array} \right\}$$

Theorem (O. 2020. Points are Type Derivations)

$$[\![M]\!]_{\vec{x}} \cong T_D(M)_{\vec{x}}$$

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Congruence on Type Derivations I

$$\dfrac{\begin{array}{c}\pi_0\\ \vdots\\ \Gamma_0 \vdash M : \vec{b} \Rightarrow a\end{array} \quad \left(\begin{array}{c}[f_i]\pi_{\alpha(i)}\\ \vdots\\ \Gamma_{\alpha(i)} \vdash N : b_i\end{array}\right)_{i=1}^{k'} \quad (1 \otimes \alpha^\star) \circ \eta : \Delta \to \Gamma_0 \otimes \bigotimes_{i=1}^{k'} \Gamma_{\alpha(i)}}{\Delta \vdash MN : a}$$

$$\sim$$

$$\dfrac{\begin{array}{c}[\langle \alpha, \vec{f} \rangle \Rightarrow 1]\pi_0\\ \vdots\\ \Gamma_0 \vdash M : \vec{a} \Rightarrow a\end{array} \quad \left(\begin{array}{c}\pi_i\\ \vdots\\ \Gamma_i \vdash N : a_i\end{array}\right)_{i=1}^{k} \quad \eta : \Delta \to \bigotimes_{j=0}^{k} \Gamma_j}{\Delta \vdash MN : a}$$

Where $\langle \alpha, f_1, \ldots, f_{k'} \rangle : \vec{a} = \langle a_1, \ldots, a_k \rangle \to \vec{b} = \langle b_1, \ldots, b_{k'} \rangle$.

Example I

$$\dfrac{\dfrac{\dfrac{\color{red}{f} : a' \to a}{x : \langle a' \rangle \vdash x : a}}{\vdash \lambda x.x : \langle a' \rangle \Rightarrow a} \quad \dfrac{1 : a' \to a'}{z : \langle a' \rangle \vdash z : a'} \quad 1}{z : \langle a' \rangle \vdash (\lambda x.x)z : a}$$

$$\sim$$

$$\dfrac{\dfrac{\dfrac{1 : a \to a}{x : \langle a \rangle \vdash x : a}}{\vdash \lambda x.x : \langle a \rangle \Rightarrow a} \quad \dfrac{\color{red}{f} : a' \to a}{z : \langle a' \rangle \vdash z : a} \quad 1}{z : \langle a' \rangle \vdash (\lambda x.x)z : a}$$

Congruence on Type Derivations II

$$\frac{\begin{array}{c}\pi_0\{\theta_0\}\\ \vdots \\ \Gamma_0 \vdash M : \vec{a} \Rightarrow a\end{array} \qquad \left(\begin{array}{c}\pi_i\{\theta_i\}\\ \vdots \\ \Gamma_i \vdash N : a_i\end{array}\right)^k_{i=1} \qquad \eta : \Delta \to \bigotimes_{j=0}^k \Gamma_j}{\Delta \vdash MN : a}$$

$$\sim$$

$$\frac{\begin{array}{c}\pi_0\\ \vdots \\ \Gamma'_0 \vdash M : \vec{a} \Rightarrow a\end{array} \qquad \left(\begin{array}{c}\pi_i\\ \vdots \\ \Gamma'_i \vdash N : a_i\end{array}\right)^k_{i=1} \qquad (\bigotimes_{j=0}^k \theta_j) \circ \eta : \Delta \to \bigotimes_{j=0}^k \Gamma'_j}{\Delta \vdash MN : a}$$

where $\theta_j : \Gamma_j \to \Gamma'_j$.

Example II

$$\dfrac{\dfrac{1 : \langle a \rangle \Rightarrow a \to \langle a \rangle \Rightarrow a}{x : \langle \langle a \rangle \Rightarrow a \rangle \vdash x : \langle a \rangle \Rightarrow a} \quad \dfrac{1 : a \to a}{x : \langle a \rangle \vdash x : a} \quad \langle (1,2), f, 1 \rangle}{x : \langle a', \langle a \rangle \Rightarrow a \rangle \vdash xx : a}$$

$$\sim$$

$$\dfrac{\dfrac{1 : (\langle a \rangle \Rightarrow a) \to (\langle a \rangle \Rightarrow a)}{x : \langle \langle a \rangle \Rightarrow a \rangle \vdash x : \langle a \rangle \Rightarrow a} \quad \dfrac{f : a' \to a}{x : \langle a' \rangle \vdash x : a} \quad \langle (1,2), 1, 1 \rangle}{x : \langle a', \langle a \rangle \Rightarrow a \rangle \vdash xx : a}$$

Type Derivations under Reduction I

$$\pi_1 = \cfrac{\cfrac{\cfrac{h \circ f : a \to b}{x : \langle a \rangle \vdash x : b}}{\vdash \lambda x.x : \langle a \rangle \Rightarrow b} \qquad \cfrac{g : c \to a}{y : \langle c \rangle \vdash y : a} \quad 1}{y : \langle c \rangle \vdash (\lambda x.x)y : b}$$

$$\pi_2 = \cfrac{\cfrac{\cfrac{h \circ f' : d \to b}{x : \langle a \rangle \vdash x : b}}{\vdash \lambda x.x : \langle d \rangle \Rightarrow b} \qquad \cfrac{g' : c \to d}{y : \langle c \rangle \vdash y : d} \quad 1}{y : \langle c \rangle \vdash (\lambda x.x)y : b}$$

suppose that $f \circ g = f' \circ g'$ and $h : b \to b, f : a \to b, f' : d \to b$.

Type Derivations under Reduction II

We have that $\pi_1 \sim \pi_2$. Indeed

$$\pi_1 \sim \dfrac{\dfrac{\dfrac{h : b \to b}{x : \langle b \rangle \vdash x : b}}{\vdash \lambda x.x : \langle b \rangle \Rightarrow b} \qquad \dfrac{f \circ g : c \to b}{y : \langle c \rangle \vdash y : b} \quad 1}{y : \langle c \rangle \vdash (\lambda x.x)y : b}$$

$$\pi_2 \sim \dfrac{\dfrac{\dfrac{h : b \to b}{x : \langle b \rangle \vdash x : b}}{\vdash \lambda x.x : \langle b \rangle \Rightarrow b} \qquad \dfrac{f' \circ g' : c \to b}{y : \langle c \rangle \vdash y : b} \quad 1}{y : \langle c \rangle \vdash (\lambda x.x)y : b}$$

and by the hypothesis that $f \circ g = f' \circ g'$ we can conclude by transitivity.

Type Derivations under Reduction III

Consider the following type derivation of y :

$$\pi_3 = \frac{h \circ (f \circ g) : c \to b}{y : \langle c \rangle \vdash y : b}$$

By an easy inspection of the definitions we have that given

$$\varphi_{\langle c \rangle, b} : [\![(\lambda x.x)y]\!]_{\langle y \rangle}(\langle c \rangle, b) \cong [\![y]\!]_{\langle y \rangle}(\langle c \rangle, b)$$

$\varphi_{\langle c \rangle, b}(\tilde{\pi_1}) = \pi_3.$

- There is then a nice correspondence between *substitution* on the term side and *composition* on the morphism side.
- The former "reduction step" on type derivations is an instance of the *Yoneda Lemma* (for coends, aka the Density Formula).

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Outline

The Idea Behind the Curtain

- Resource calculus / System R;
- Taylor expansion;
- $\mathrm{nf}(\tau(M)) \rightsquigarrow [\![M]\!]_{\mathrm{Rel}}$;
- $\mathrm{nf}(\tau(M))$ is a DS.

- S-approximants / System E_A^S;
- Rigid S-expansion;
- $\mathcal{T}_{\mathrm{rig}}(M) \cong [\![M]\!]_{S\text{-CatSym}}$;
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Following the TAO

- Polyadic calculus (Melliès 2006, Mazza 2012):

$$p ::= x \mid \lambda\langle x_1, \dots, x_k\rangle.p \mid p\vec{q} \mid \bot \qquad \vec{q} = \langle q_1, \dots, q_k\rangle$$

$$(\lambda\vec{x}.p)\vec{q} \to \begin{cases} p\{\vec{q}/\vec{x}\} & \text{if } \mathrm{len}(\vec{x}) = \mathrm{len}(\vec{q}) \\ \bot & \text{otherwise.} \end{cases}$$

- Rigid Taylor expansion (Tsukada-Asada-Ong 2017):

$$\mathcal{T}_{\mathrm{rig}}(\Gamma \vdash M : A) : [\![\Gamma]\!] \nrightarrow [\![A]\!]$$

$$\langle a \lhd A, \Delta \lhd \Gamma\rangle \mapsto \{\tilde{p} \mid p \lhd M \text{ and } \Delta \vdash p : a\}$$

Theorem (Tsukada-Asada-Ong 2017)

Let $M \to N$. Then $\mathcal{T}_{\mathrm{rig}}(\Gamma \vdash M : A) \cong \mathcal{T}_{\mathrm{rig}}(\Gamma \vdash N : A)$.

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Failure of Direct Generalization

Two problems:

- The rigid Taylor expansion is not well-defined in general.

$$x : A \vdash x : A \qquad A = (o \Rightarrow o) \Rightarrow o$$

$$f : \overbrace{\langle\langle\rangle \Rightarrow \star, \langle\star\rangle \Rightarrow \star\rangle \Rightarrow \star}^{a} \cong \overbrace{\langle\langle\star\rangle \Rightarrow \star, \langle\rangle \Rightarrow \star\rangle \Rightarrow \star}^{b}$$

$$a, b \lhd A$$

$$\mathcal{T}_{\mathrm{rig}}(x : A \vdash x : A)(x : \langle a\rangle, a) = \langle x\rangle$$

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- *Failure of subject reduction* for the system E^S_A.

$$(\lambda\langle x\rangle.x\langle\lambda\langle\rangle.y_1\langle\rangle, \lambda\langle f\rangle.y_2\langle f\rangle\rangle)\langle\lambda\langle z_1, z_2\rangle.z_1\langle z_2\rangle\rangle \to^* \bot$$

it's typable!

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The Solution

Subtyping-Aware Polyadic Terms

$$\frac{f_1 : \vec{a}_1 \to \langle\rangle, \ldots, f_i : \vec{a}_i \to \langle a \rangle, \ldots, f_n : \vec{a}_n \to \langle\rangle}{\vec{x}_1 : f_1 : \vec{a}_1, \ldots, \vec{x}_i : f_i : \vec{a}_i, \ldots, \vec{x}_n : f_n : \vec{a}_n \vdash x_{i,\mathrm{sm}(f_i)(1)} : a}$$

$$\frac{\zeta \oplus \langle \vec{z} \rangle : \eta \oplus \langle f \rangle : \Delta \oplus \langle \vec{a} \rangle \vdash p : a}{\zeta : \eta : \Delta \vdash \lambda \vec{z} : f.p : \vec{a} \Rightarrow a}$$

$$\frac{\zeta_0 : \eta_0 : \Gamma_0 \vdash p : \langle a_1, \ldots, a_k \rangle \Rightarrow a \qquad (\zeta_i : \eta_i : \Gamma_i \vdash q_i : a_i)_{i=1}^{k}}{\zeta : (\bigotimes_{j=0}^{k} \eta_j) \circ \eta : \Delta \vdash (p\langle q_1, \ldots, q_k \rangle)^{\eta} : a}$$

where $\eta : \zeta : \Delta \to \bigotimes_{j=0}^{k} \zeta_j : \bigotimes_{j=0}^{k} \Gamma_j$.

Rigid Expansion

- Subtyping-aware polyadic terms:

$$\begin{array}{c}\pi\\ \vdots\\ \zeta : \eta : \Delta \vdash p : a\end{array}$$

- Rigid expansion:

$$\mathcal{T}_{\mathrm{rig}}(M)_{\vec{x}} : D \nrightarrow SD^n$$

$$\langle \zeta : \Delta, a \rangle \mapsto \{ \widetilde{\langle \eta, p \rangle} \mid \zeta \lhd \vec{x} \vdash p \lhd M \text{ and } \zeta : \eta : \Delta \vdash p : a \}$$

Theorem (O. 2020. Approximants are Type Derivations)

$$\mathcal{T}_{\mathrm{rig}}(M)_{\vec{x}} \cong T_D(M)_{\vec{x}}$$

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Our Results

- *Proof-relevant* semantics :

$$[\![M]\!](\Delta, a) \cong \left\{ \begin{array}{c} \tilde{\pi} \\ \vdots \\ \Delta \vdash M : a \end{array} \right\} \cong \left\{ \begin{array}{r} \tilde{\varphi} \text{ approximates } M \\ \text{and } \Delta \vdash \varphi : a \end{array} \right\}$$

- *Parametric* generalization of IT and Taylor expansion.
- The natural isomorphism

$$M \to N \qquad \Rightarrow \qquad \beta : [\![M]\!] \cong [\![N]\!]$$

is given by the *evaluation* of approximants.

Ongoing Work and Perspectives

There is a preprint about about some this stuff on the Arxiv.

Federico Olimpieri. *Intersection Type Distributors*. 2020

- Extention to (untyped) call-by push value: the bang calculus (with G. Guerrieri). Giulio Guerrieri and Federico Olimpieri. "Categorifying Non-Idempotent Intersection Types". In: *CSL*. 2021
- Study of the relationship between the linear resource monad and the cartesian one (with Z. Galal). Looking for a "collapse" result *à la* Ehrhard.
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Thank You!