Kits, groupoids, and linear logic

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University of Oxford

Joint work with Marcelo Fiore & Zeinab Galal

Relational semantics

Relational semantics

Type / Formula A

Set A

Relational semantics

Type / Formula A	*Program of type A / Proof of A*
Set A	$x \in \mathscr{P}(A)$

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Type / Formula A

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$x \in \mathscr{P}(A)$

The category $\mathbf{Rel}$

objects = sets

morphisms $A \to B$ = relations $R \subseteq A \times B$

or $A \to \mathscr{P}(B)$

or $B \to \mathscr{P}(A)$

Relational semantics

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$x \in \mathscr{P}(A)$

this is a relation $1 \to A$

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Relational semantics, refined

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Set A **+ structure**	

Relational semantics, refined

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Set A **+ structure**	$x \in \mathscr{Q}(A) \subseteq \mathscr{P}(A)$

nice properties

Relational semantics, refined

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Set A + **structure**	$x \in \mathscr{Q}(A) \subseteq \mathscr{P}(A)$
e.g. a partial order $\leq$	
a binary relation on A	
the subset $\mathscr{Q}(A)$	**nice properties**

Relational semantics, refined

Type / Formula

Set A **+ structure**

e.g. a partial order $\leq$

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Program / Proof

$$x \in \mathscr{Q}(A) \subseteq \mathscr{P}(A)$$

nice properties

The category $\mathbf{QRel}$

objects = sets with structure

morphisms $A \to B$ = relations $R \subseteq A \times B$ such that

- $R : A \to \mathscr{P}(B)$ preserves $\mathscr{Q}$
- $R : B \to \mathscr{P}(A)$ "co-preserves" $\mathscr{Q}$

Relational semantics, refined

Type / Formula

Set A + **structure**

e.g. a partial order $\leq$

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the subset $\mathcal{Q}(A)$

Program / Proof

$$x \in \mathcal{Q}(A) \subseteq \mathscr{P}(A)$$

nice properties

The

EXAMPLES

finiteness spaces [Ehrhard 2005]

coherence spaces [Girard 1986]

totality spaces [Loader 1994]

bistructures [Curien-Plotkin-Winskel 1994]

...

Relational semantics, categorified

Type / Formula	*Program / Proof*

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Groupoid $\mathbb{A}$	

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Groupoid $\mathbb{A}$	$X \in \mathrm{PSh}(\mathbb{A}) = [\mathbb{A}^{\mathrm{op}}, \mathrm{Set}]$

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Groupoid $\mathbb{A}$	$X \in \mathsf{PSh}(\mathbb{A}) = [\mathbb{A}^{\mathsf{op}}, \mathsf{Set}]$

The bicategory **Prof**

objects = groupoids

morphisms $\mathbb{A} \to \mathbb{B}$ = profunctors $\mathbb{B}^{\mathsf{op}} \times \mathbb{A} \to$ **Set**,

or $\mathbb{A} \to \mathsf{PSh}(\mathbb{B})$

or $\mathbb{B}^{\mathsf{op}} \to \mathsf{PSh}(\mathbb{A}^{\mathsf{op}})$

This talk:

relational semantics, categorified + refined

1. **Kits**: groupoids with structure
2. Profunctors that “preserve and co-preserve” kits
3. Species, and a CCC of **stable** functors

(nice properties)

Type / Formula	*Program / Proof*
Groupoid $\mathbb{A}$ **+ ??**	$X \in \mathsf{Q}(\mathbb{A}) \hookrightarrow \mathsf{PSh}(\mathbb{A})$

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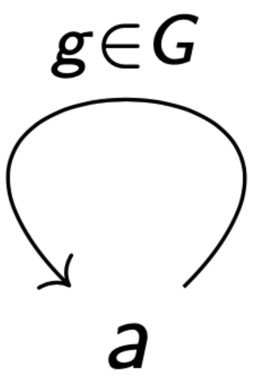
For a subgroup G



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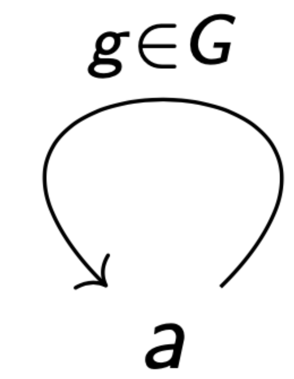

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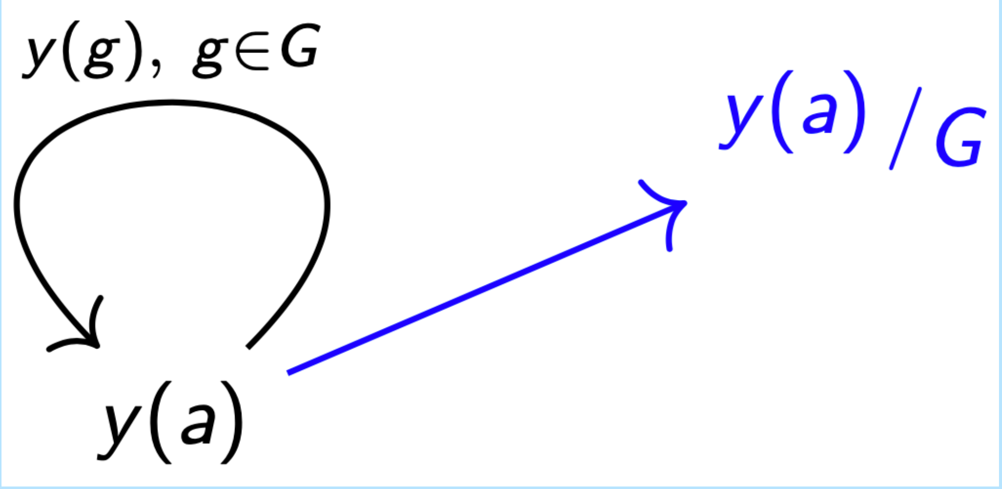

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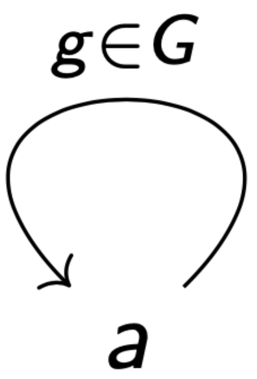

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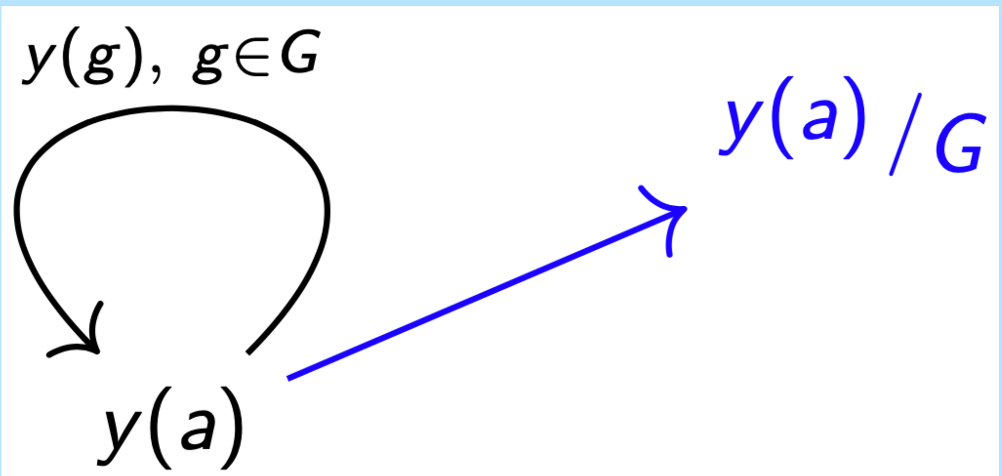

$$(y(a)/G)(a') = \mathbb{A}(a', a)/\sim$$

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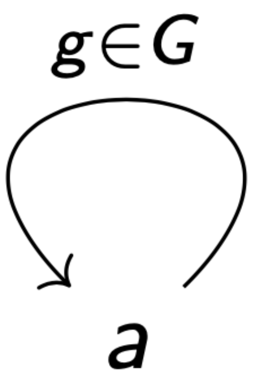

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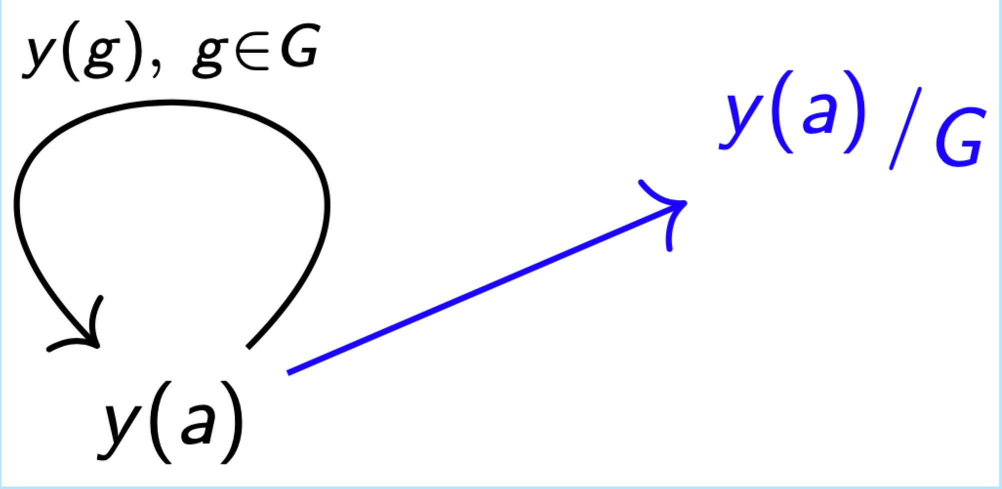

$(y(a)/G)(a') = \mathbb{A}(a',a)/\sim$

where $\alpha \sim \alpha'$ if $\alpha = g \circ \alpha'$

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A **kit** on a groupoid $\mathbb{A}$ is a family

$$\{\mathscr{A}(a)\}_{a\in\mathbb{A}}$$

where each $\mathscr{A}(a)$ is a set of subgroups of $\mathbb{A}(a,a)$

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Type / Formula	*Program / Proof*
Groupoid $\mathbb{A}$ **+ kit $\mathscr{A}$**	$X \in \mathsf{Q}(\mathbb{A}, \mathscr{A}) \hookrightarrow \mathsf{PSh}(\mathbb{A})$

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3. Species, and a CCC of **stable** functors

A profunctor $P : \mathbb{B}^{\mathsf{op}} \times \mathbb{A} \to \mathsf{Set}$

is equivalently $\mathbb{A} \to \mathsf{PSh}(\mathbb{B})$ **or** $\mathbb{B}^{\mathsf{op}} \to \mathsf{PSh}(\mathbb{A}^{\mathsf{op}})$

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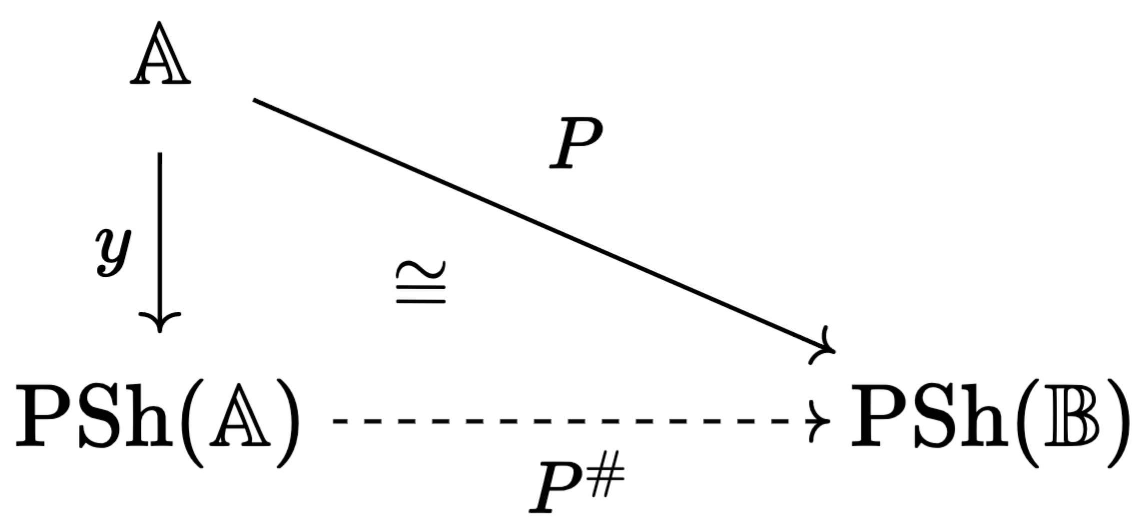
A profunctor induces functors between presheaf categories:



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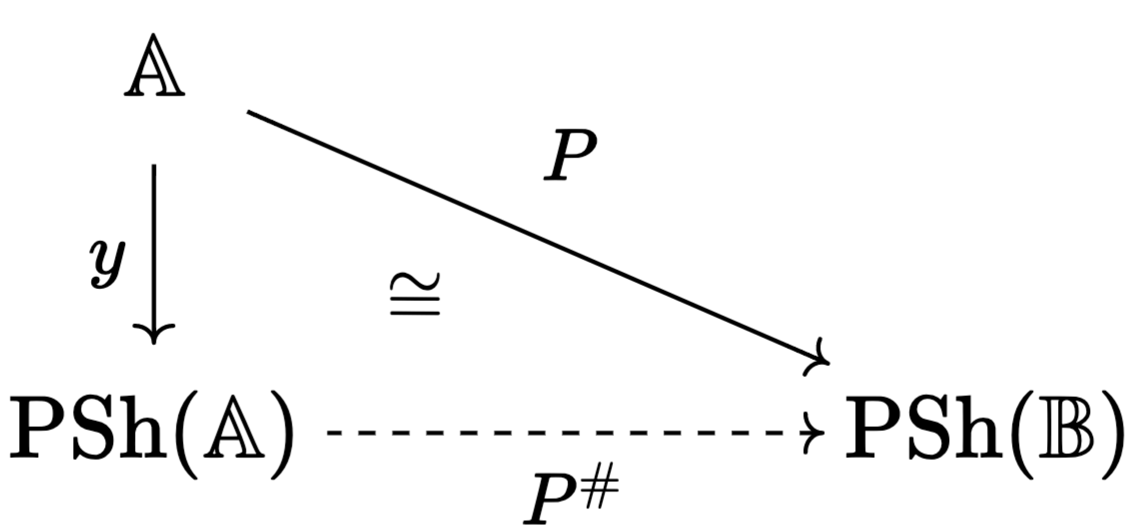

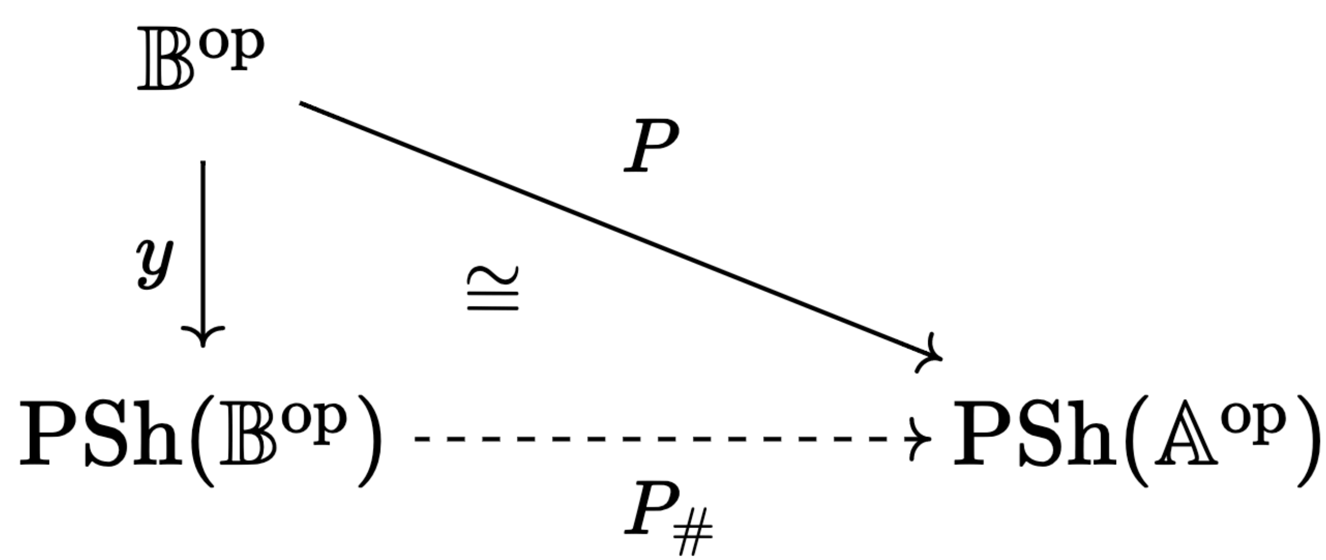

Profunctors between kits

Definition.

A **q-profunctor** $(\mathbb{A}, \mathscr{A}) \to (\mathbb{B}, \mathscr{B})$ **is a profunctor** $P : \mathbb{B}^{\mathrm{op}} \times \mathbb{A} \to \mathrm{Set}$ **such that**

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preserves Q(—)

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The bicategory $\mathbf{QProf}$

objects = groupoids **with kits**

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Tensor product of kits

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is the smallest kit containing all pairs

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QProf is symmetric monoidal closed

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$$A \multimap B = (A \otimes B^{\perp})^{\perp}$$

(In fact **QProf** is **-autonomous*)

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[Fiore, Gambino, Hyland, Winskel 08]

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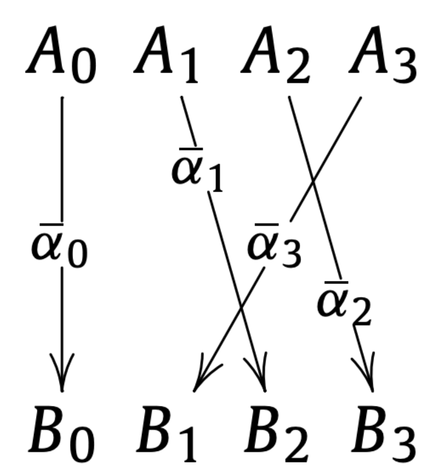

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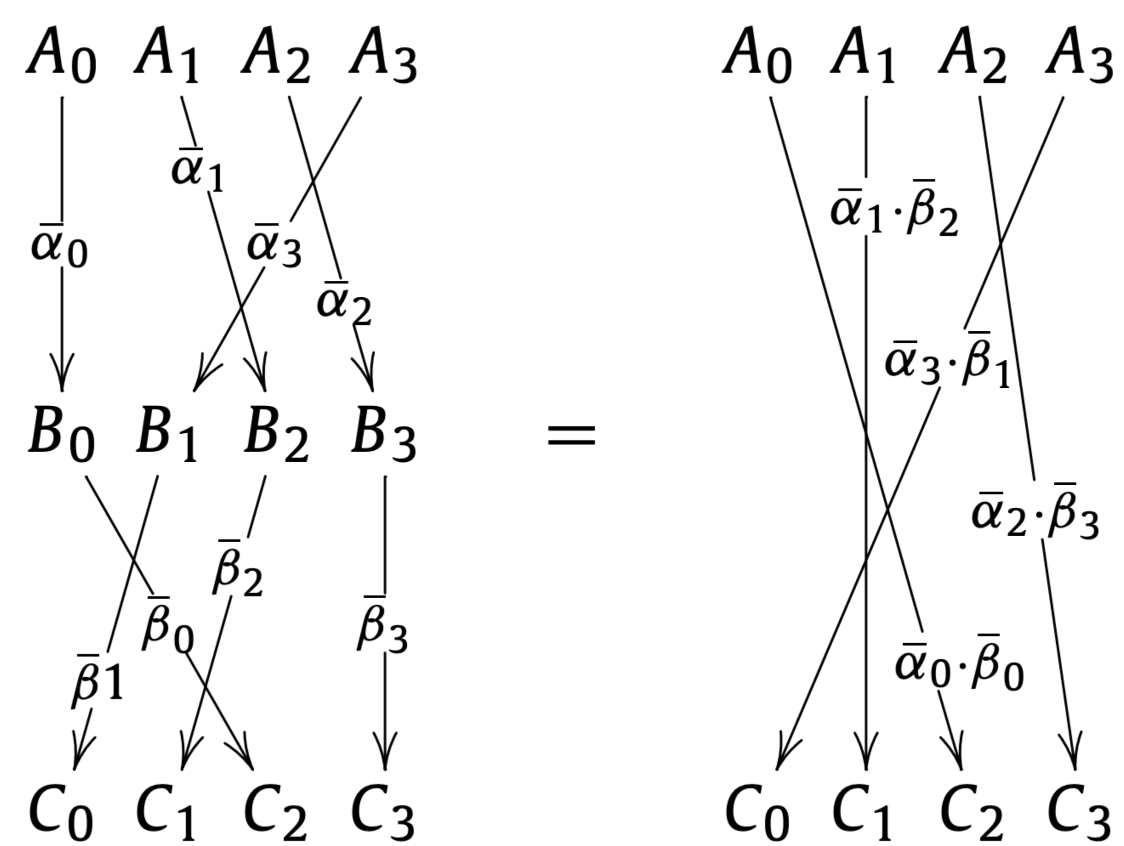

[Fiore 2013]

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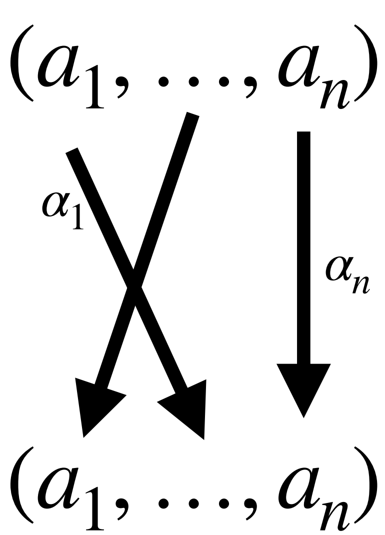

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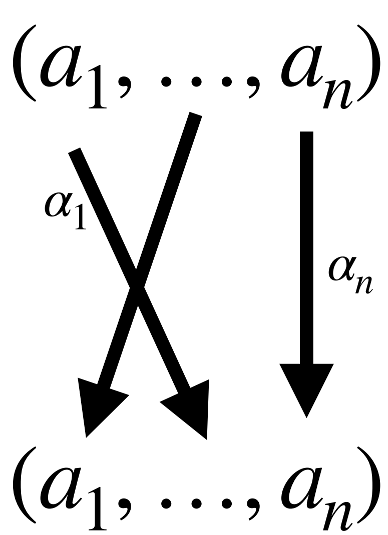

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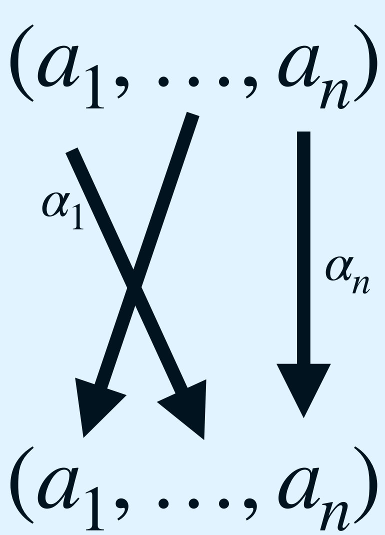
Definition.

A ***(generalised) q-species*** from $(\mathbb{A}, \mathscr{A})$ to $(\mathbb{B}, \mathscr{B})$ is a **q-profunctor** from $(!\mathbb{A}, !\mathscr{A})$ to $(\mathbb{B}, \mathscr{B})$

$!\mathscr{A}$ is the smallest kit containing all



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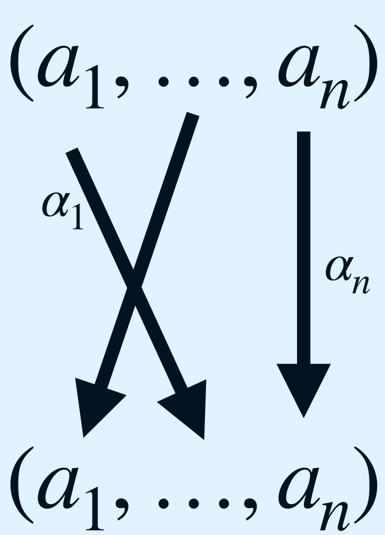

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such that each

$a_i \to a_{\sigma(i)} \to a_{\sigma^2(i)} \to \ldots \to a_i$

is in some subgroup

$G_i \in \mathscr{A}(a_i)$.

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Theorem.

QSpecies is cartesian closed

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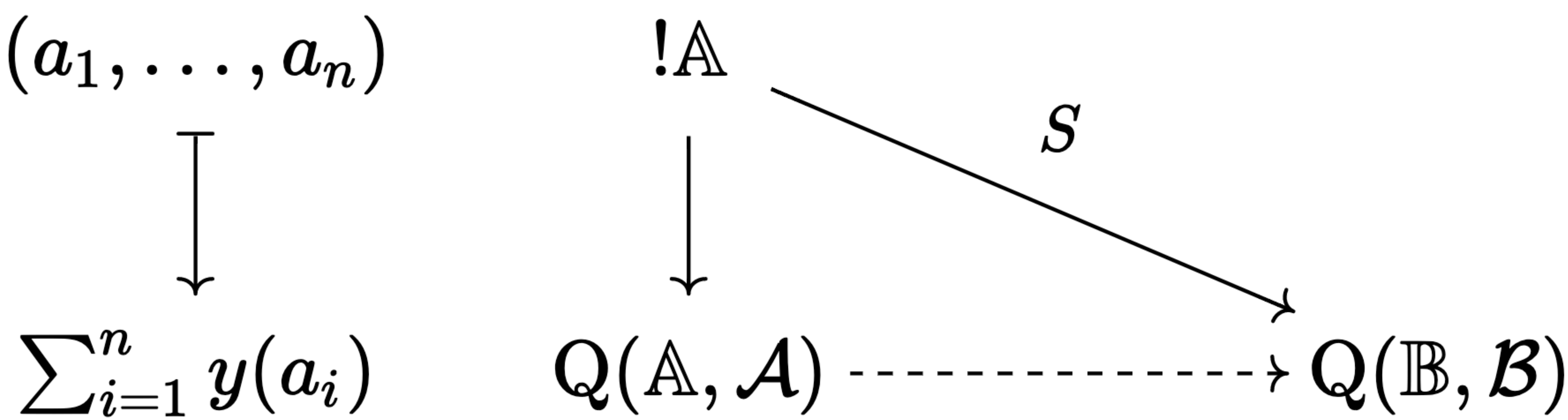
$$\begin{array}{c} (a_1, \ldots, a_n) \\ \big\downarrow \\ \sum_{i=1}^{n} y(a_i) \end{array} \qquad \begin{array}{ccc} !\mathbb{A} & \xrightarrow{\;S\;} & \mathbf{PSh}(\mathbb{B}) \\ \big\downarrow & \nearrow & \\ \mathbf{PSh}(\mathbb{A}) & \dashrightarrow & \mathbf{PSh}(\mathbb{B}) \end{array}$$

A q-species S can be extended to a functor **Q**$(\mathbb{A}, \mathscr{A}) \to$ **Q**$(\mathbb{B}, \mathscr{B})$**:**



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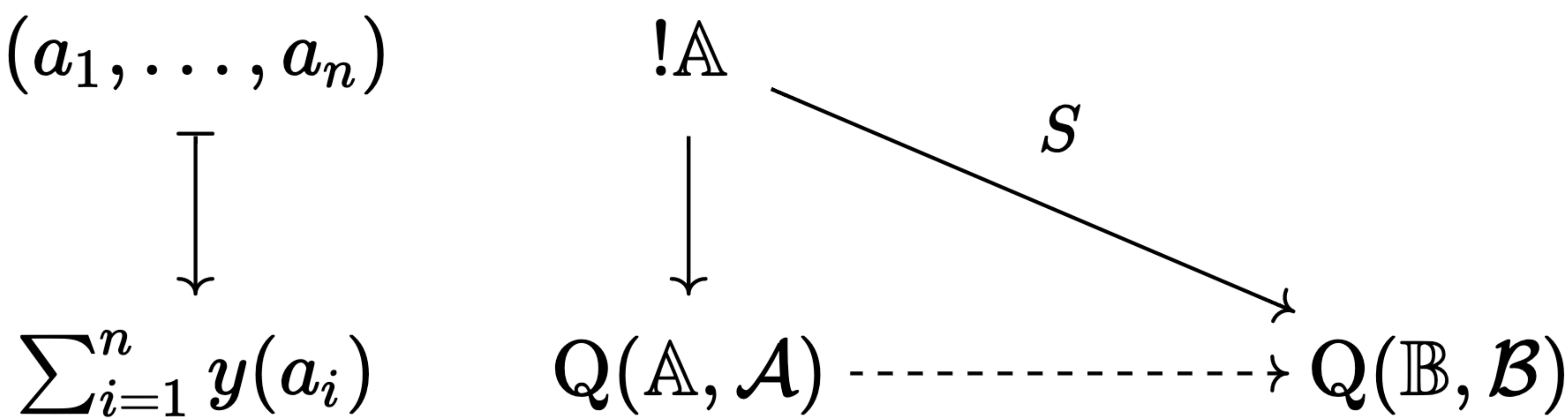

Thm.

These functors **Q**$(\mathbb{A}, \mathscr{A}) \to$ **Q**$(\mathbb{B}, \mathscr{B})$ are (up to iso) those which are:

— finitary

— parametric right adjoint (= right adjoint on each slice)

Stability

Computational intuition*: minimal data property*

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Defn. (for CPOs) [Berry 1978]

A continuous $f : X \to Y$ is **stable** if :

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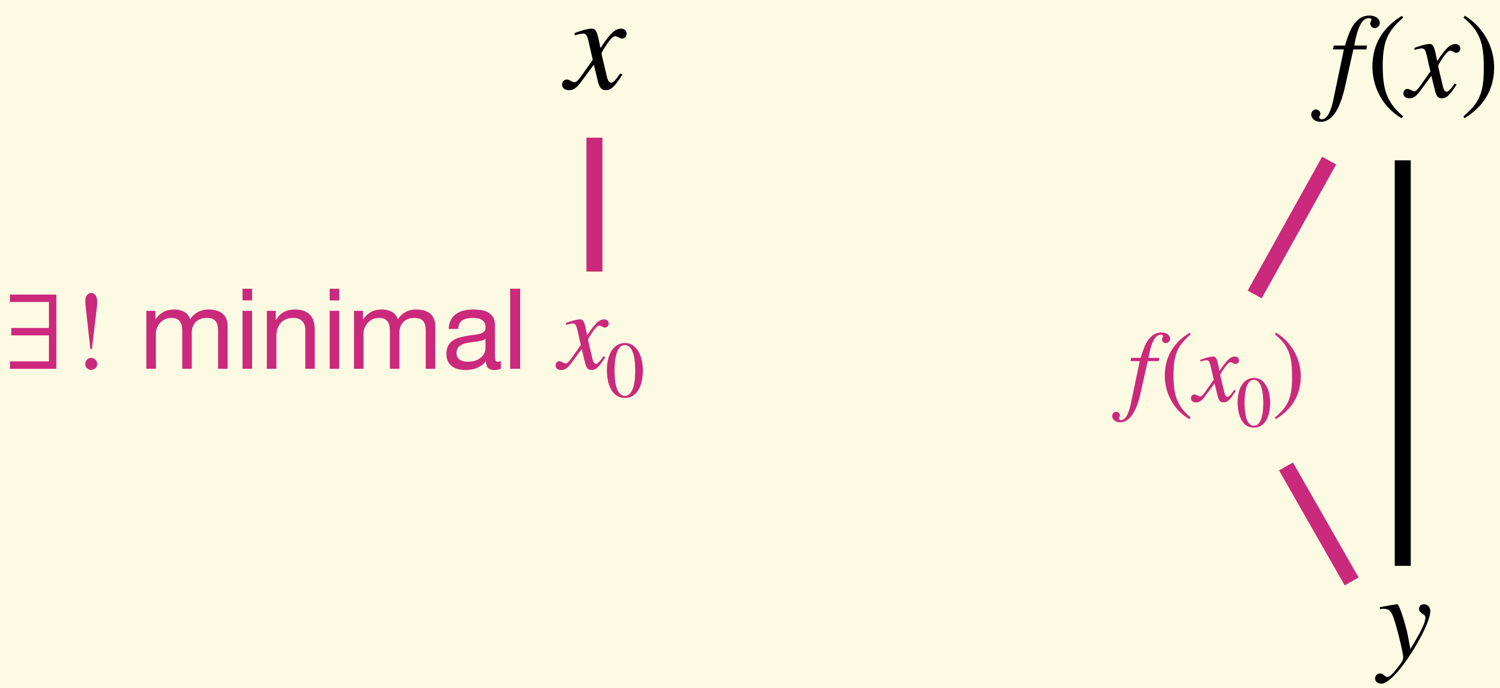

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A finitary functor $F : \mathbb{C} \to \mathbb{D}$ is **stable** if :

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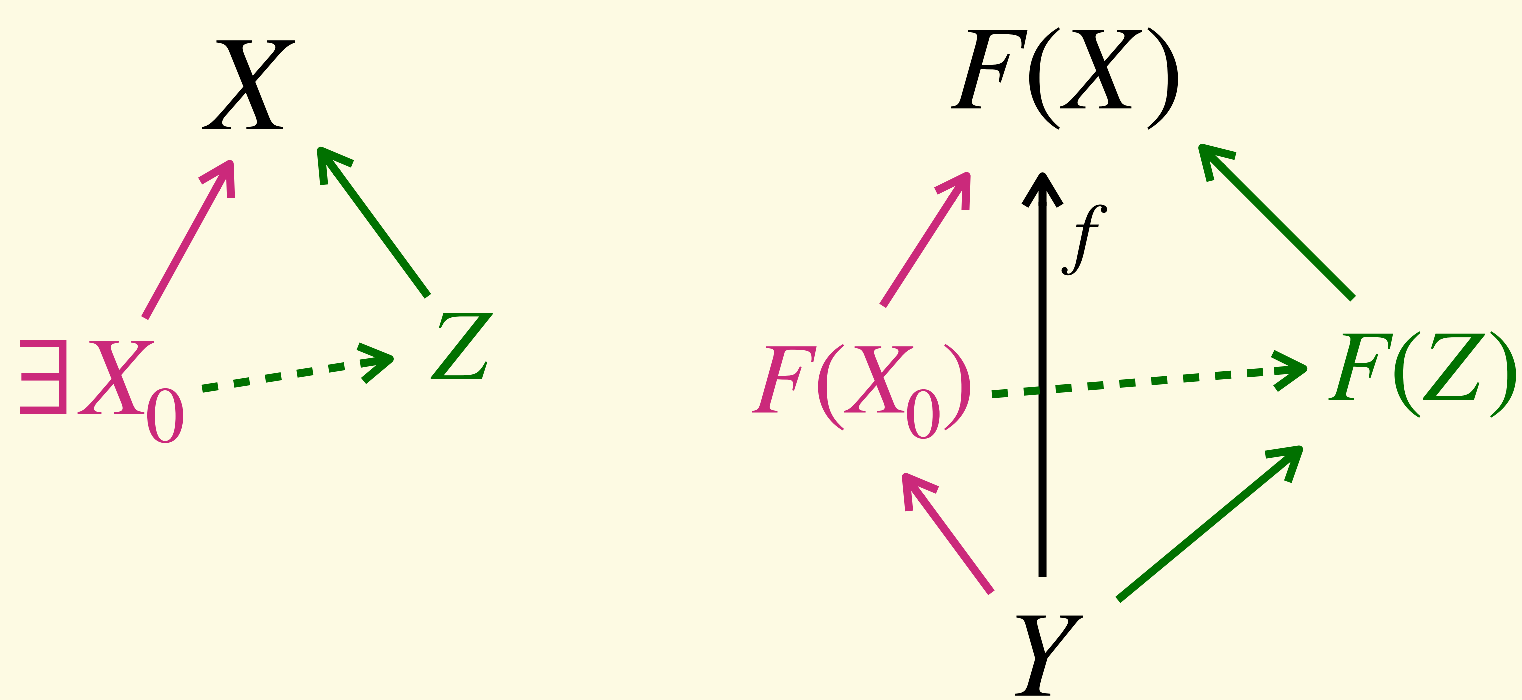

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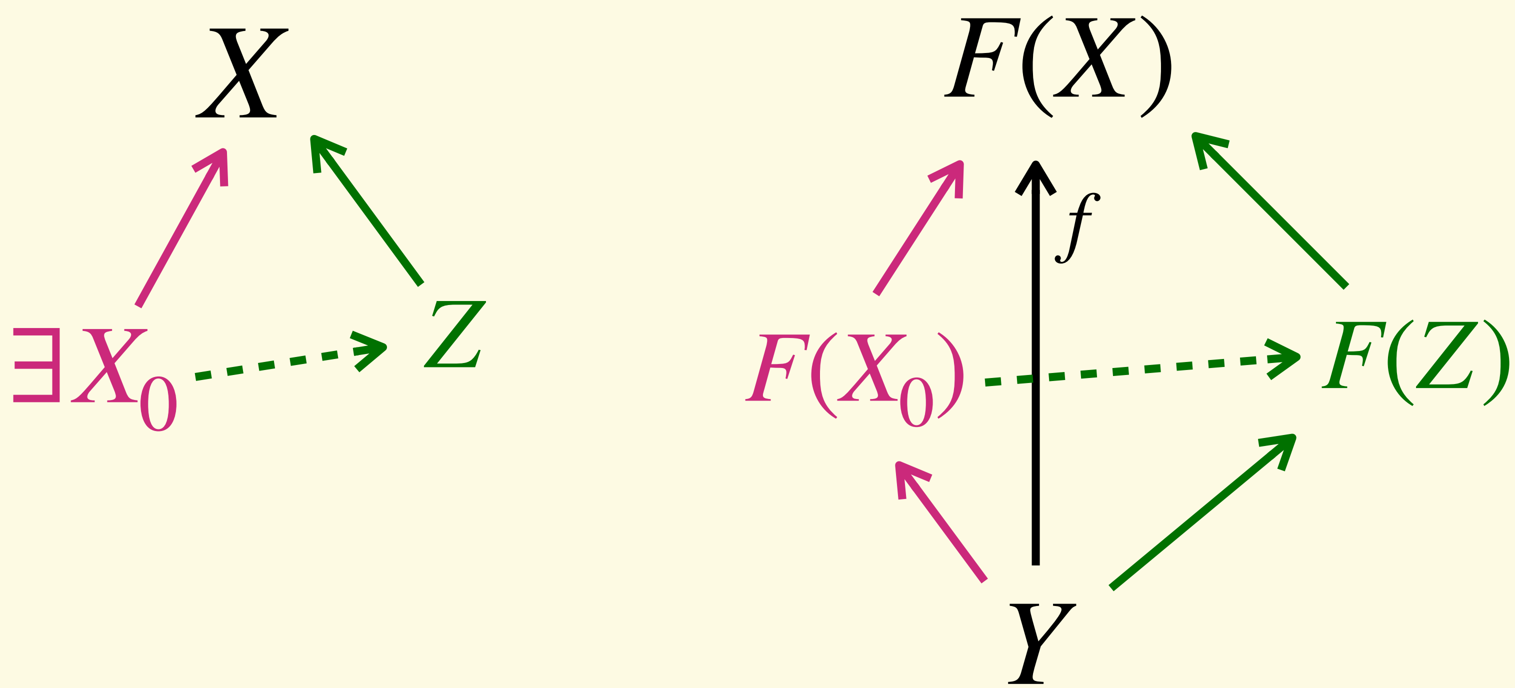

Prop. stable $\Longleftrightarrow$ parametric right adjoint

Conclusion

- Groupoids with kits and q-profunctors: a model of classical linear logic
- QSpecies: a cartesian closed bicategory of stable functors [Lamarche] [Taylor] [Girard] ..
- Applications