The opetopic nerve of operads (and categories and combinads and . . .)

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Species and Operads in Combinatorics and Semantics

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The category of opetopes

We have seen that the category of opetopes $\mathbb{O}$ can be defined by generators and relations (just like the category of simplices Δ). It has some nice properties.

1. $\mathbb{O}$ is a direct category,

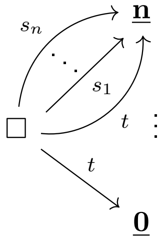
1' $\mathbb{O}$ is a Reedy category, all of whose non-identity morphism increase dimension,

2. $\mathbb{O}$ is locally finite ($\mathbb{O}/\omega$ is finite for each ω in $\mathbb{O}$).

Let $\mathbb{O}_{m,n}$ be the full subcategory of $\mathbb{O}$ of objects of dimensions $m \leq i \leq n$. Then

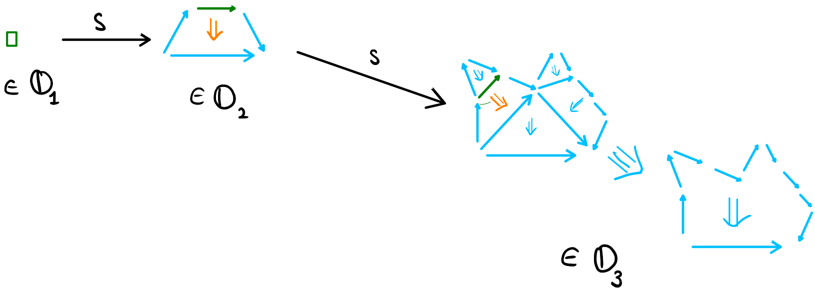
$$\mathbb{O}_{0,1} = \{ \ \diamond \ \overset{s}{\underset{t}{\rightrightarrows}} \ \square \ \}$$

$\mathbb{O}_{1,2} =$



So we have $\mathcal{G}\mathrm{ph} = \widehat{\mathbb{O}_{0,1}}$ (directed graphs) and $\mathcal{C}\mathrm{oll} = \widehat{\mathbb{O}_{1,2}}$ (coloured planar collections).

- example of a sequence of morphisms in $\mathbb{O}_{1,3}$:



The category $\widehat{\mathbb{O}_{1,3}}$ is the category of *coloured combinatorial patterns* of Loday [Lod12].

We have monadic adjunctions $\widehat{\mathbb{O}_{0,1}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{C}\mathrm{at}$ (small categories) and $\widehat{\mathbb{O}_{1,2}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{O}\mathrm{pd}$ (planar coloured $\mathcal{S}\mathrm{et}$-operads).

We also have a monadic adjunction $\widehat{\mathbb{O}_{1,3}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{C}\mathrm{omb}$ (planar coloured combinads [Lod12]).

Opetopic nerve theorem

Theorem [HTLS19]

$\mathcal{C}\mathrm{at}, \mathcal{O}\mathrm{pd}, \mathcal{C}\mathrm{omb}$ are *reflective* subcategories of $\widehat{\mathbb{O}}$.

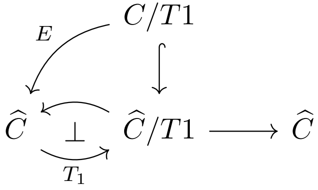
Parametric right adjoint monads on $\widehat{\mathbb{O}_{m,n}}$

Let $\mathcal{C}$ have a terminal object 1, and let $T : \mathcal{C} \longrightarrow \mathcal{D}$. Then

$$T : \ \mathcal{C}/1 \xrightarrow{T_1} \mathcal{D}/T1 \longrightarrow \mathcal{D}.$$

T is a *parametric right adjoint* (p.r.a.) if T_1 has a left adjoint.

If $\mathcal{C} = \mathcal{D} = \widehat{C}$ then T is uniquely determined by $T1 \in \widehat{C}$ and the restriction $E : C/T1 \longrightarrow \widehat{C}$ of the left adjoint of T_1 along the Yoneda embedding.



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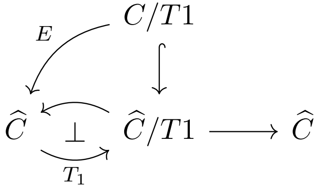


T is a **p.r.a. monad** if it is a monad on $\widehat{C}$ whose unit and multiplication are cartesian natural transformations. The image $\mathrm{im}(E) \hookrightarrow \widehat{C}$ of E provides arities for T.

Opetopic nerve functor

For every m, n, we define a p.r.a. monad $\mathfrak{Z}$ on $\widehat{\mathbb{O}_{m,n}}$ and a dense functor $h : \mathbb{O}_{m,n+2} \to \mathfrak{Z}\text{-}\mathcal{A}\mathrm{lg}$. Since $\mathbb{O}_{m,n+2} \subset \mathbb{O}$, we obtain a composite of fully faithful right adjoints

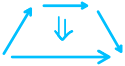
$$\mathfrak{Z}\text{-}\mathcal{A}\mathrm{lg} \hookrightarrow \widehat{\mathbb{O}_{m,n+2}} \hookrightarrow \widehat{\mathbb{O}},$$

called the **opetopic nerve functor** for $\mathfrak{Z}\text{-}\mathcal{A}\mathrm{lg}$.

For $(m, n) = (0, 1)$ (respectively, $(1, 2), (1, 3)$) we have $\mathfrak{Z}\text{-}\mathcal{A}\mathrm{lg} = \mathcal{C}\mathrm{at}$ (respectively, $\mathcal{O}\mathrm{pd}, \mathcal{C}\mathrm{omb}$).

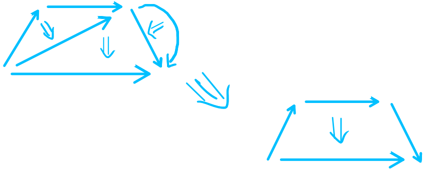
Why $n + 2$?

E.g. $\mathbb{O}_{0,1}$ (3-Alg = Cat)



composition of 3 arrows

$\in \mathbb{O}_2$



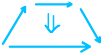
One of the "associativity" 3-opetopes

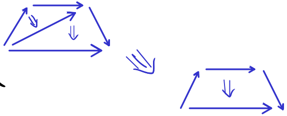
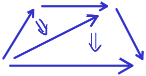
$\in \mathbb{O}_3$

$n+1$-opetopes encode *composition* operations for trees of n-opetopes, thus $n+2$-opetopes encode the associativity relations for the composition of trees of n-opetopes.

Segal conditions

E.g. $\mathbb{O}_{0,1}$ (3-Alg = Cat)

$\Lambda\big($  $\big)$ =  $\in \widehat{\mathbb{O}}_{0,1}$

$\Lambda\big($  $\big)$ =  $\in \widehat{\mathbb{O}}_{0,2}$

Groth.-Segal inclusions = $\{ \Lambda(\omega) \hookrightarrow \omega \mid \omega \in \mathbb{O}_{\geq 2} \}$

The essential image of opetopic nerve functors are characterised by Segal conditions/Grothendieck-Segal colimits.

Reflections on opetopes and species

Species

Recall that a $\mathcal{S}\mathrm{et}$-species ($\mathcal{E}\mathrm{ns}$-*espèce*) is a functor $X : \mathbb{B} \to \mathcal{S}\mathrm{et}$ ($\mathbb{B}$ is the groupoid of finite sets $\underline{n}$ and bijections).

Similarly, a planar $\mathcal{S}\mathrm{et}$-species is a functor $X : \mathbb{N} \to \mathcal{S}\mathrm{et}$ ($\mathbb{N}$ is the discrete set of natural numbers n).

Each such species gives an endofunctor on $\mathcal{S}\mathrm{et}$ by left Kan extension along $\mathbb{B} \to \mathcal{S}\mathrm{et}$ and $\mathbb{N} \to \mathcal{S}\mathrm{et}$ (the faithful functors mapping $\underline{n}$ and n to $\{1, \ldots, n\}$).

The endofunctors in the image of $\mathcal{S}\mathrm{et}^{\mathbb{B}}$ are called *analytic* and those in the image of $\mathcal{S}\mathrm{et}^{\mathbb{N}}$ are called *polynomial over* $\mathbb{N}$.

Left Kan extension is monoidal, sending $- \boxtimes -$ to $- \circ -$.

Operads are sent to analytic monads and planar operads are sent to polynomial monads over $\mathbb{N}$.

These monads are all finitary, namely their underlying endofunctors are left Kan extensions of the form

$$\begin{array}{ccc} \mathcal{F}\mathrm{in} & \xrightarrow{\;T\;} & \mathcal{S}\mathrm{et} \\ {\scriptstyle i}\big\downarrow & \cong & \nearrow {\scriptstyle T} \\ \mathcal{S}\mathrm{et} & & \end{array}$$

Pra monads and species

P.r.a. monads on presheaf categories are examples of *monads with arities*. In particular, their endofunctors can be calculated as left Kan extensions :

$$\begin{array}{ccc} \Theta_0 & \xrightarrow{T} & \widehat{C} \\ {\scriptstyle i}\big\downarrow & \cong & \nearrow {\scriptstyle T} \\ \widehat{C} & & \end{array}$$

Question

Does this give interesting examples that generalise species?

Example

Recall that $\widehat{\mathbb{O}_{0,1}}$ is the category of directed graphs. We have seen that the free-category monad is p.r.a.

Consider the category Λ_0 whose objects are finite linear graphs $\underrightarrow{n}$ and whose morphisms are given by

$$\begin{aligned}\Lambda_0(\underrightarrow{n}, \underrightarrow{n}) &= \{*\}\\ \Lambda_0(\underrightarrow{m}, \underrightarrow{n}) &= \mathcal{G}\mathrm{ph}(\underrightarrow{m}, \underrightarrow{n}) \text{ if } m \leq 1\\ &= \emptyset \text{ otherwise.}\end{aligned}$$

Consider the functor $X_{\mathcal{C}\mathrm{at}} : \Lambda_0 \to \mathcal{G}\mathrm{ph}$ that sends $\underset{\to}{m}$ to the graph $\circlearrowright x \longrightarrow y \circlearrowleft$ for $m \geq 1$ and sends $\underset{\to}{0}$ to the graph $x \circlearrowleft$.

Then the left Kan extension of $X_{\mathcal{C}\mathrm{at}}$ along the obvious functor $\Lambda_0 \to \mathcal{G}\mathrm{ph}$ is the free-category endofunctor.

Remark

Λ_0 is almost the category $\mathbb{O}_{0,2}$ (recall that $\mathcal{G}\mathrm{ph} = \widehat{\mathbb{O}_{0,1}}$).

Question

Is $\mathcal{G}\mathrm{ph}^{\Lambda_0}$ an interesting category of "generalised" species?

Cédric Ho Thanh and Chaitanya Leena Subramaniam.

Opetopic algebras I: Algebraic structures on opetopic sets.

arXiv preprint arXiv:1911.00907, 2019.

Jean-Louis Loday.

Algebras, operads, combinads.

2012.

Slides of a talk given at HOGT Lille on 23th of March (2012).