Algebraic complexity – Exercise session 1
NP-completeness

The goal of the following exercises is to study two usual NP-complete problems over the structures $(\mathbb{C},+,−,\times,=)$ and $(\mathbb{R},+,−,\times,=,\leq)$.

Let $HN_{\mathbb{C}}$ (sometimes called FEAS$_{\mathbb{C}}$) be the following problem over the structure $(\mathbb{C},+,−,\times,=)$.

- Input: $k$ multivariate polynomials $f_1,\ldots,f_k \in \mathbb{C}[X_1,\ldots,X_n]$.
- Problem: decide whether the polynomials $f_i$ share a common complex root.

**Exercise 1**

1. The name $HN_{\mathbb{C}}$ comes from Hilbert’s Nullstellensatz. Why?
2. Show that the problem $HN_{\mathbb{C}}$ is decidable over $(\mathbb{C},+,−,\times,=)$ (i.e. can be decided by a uniform family of algebraic circuits—no matter their size).
3. Show that $HN_{\mathbb{C}}$ is in $NP_{\mathbb{C}}$.
4. Let $x$ be a variable. Define via a conjunction of polynomial equalities (with possibly existential quantifiers) a variable $z$ whose value is 1 if $x = 0$ and 0 otherwise.
5. Show that $HN_{\mathbb{C}}$ is $NP_{\mathbb{C}}$-hard.

Let $4FEAS$ be the following problem over the structure $(\mathbb{R},+,−,\times,=,\leq)$.

- Input: a multivariate polynomial $f \in \mathbb{R}[X_1,\ldots,X_n]$ of degree $\leq 4$.
- Problem: decide whether the polynomial $f$ has a real root.

**Exercise 2**

1. Show that the problem $4FEAS$ is decidable over $(\mathbb{R},+,−,\times,=,\leq)$.
2. Show that $4FEAS$ is in $NP_{\mathbb{R}}$.
3. Let $x$ be a variable. Define via a conjunction of polynomial identities (with possibly existential quantifiers) a variable $z$ whose value is 1 if $x \geq 0$ and 0 otherwise.
4. Show that $4FEAS$ is $NP_{\mathbb{R}}$-hard. Hints: compared to the problem $HN_{\mathbb{C}}$, why is only one polynomial enough? How to reduce the degree to 4?