ALGEBRAIC COMPLEXITY – EXERCISE SESSION 2
Quantifiers, (non)uniformity

We investigate the links between the question $P_M = NP_M$ and the efficient elimination of quantifiers over the structure $M$.

We also study the notion of algebraic Turing machine and see the “equivalence” with uniform circuits.

**Exercise 1** Elimination of quantifiers

1. Show that if $P_M = NP_M$ then the structure $M$ has quantifier elimination.

2. Show that $P_M = NP_M$ if and only if there exists a tuple $\bar{c} \in M$ of parameters, a polynomial $p(n)$ and a family $(C_n)$ of circuits of size $p(n)$ such that every existential formula $f(\bar{x}) = (\exists \bar{y}) \phi(\bar{x}, \bar{y})$ of size $n$ is equivalent to the formula $C_n(f, \bar{x}, \bar{c}) = 1$.

3. Prove a similar statement for the uniform version “$P_M = NP_M$”.

**Exercise 2** Algebraic Turing machine

1. Let $M$ be a structure. Define a Turing machine over $M$ in a similar way as boolean Turing machines, but able to manipulate elements of $M$.

2. Define the class of languages over $M$ recognized by an algebraic Turing machine working in polynomial time. Show that this class is $P_M$ (i.e. show the equivalence with a uniform family of algebraic circuits of polynomial size).