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Algebraic complexity – Exercise session 2
Quantifiers, (non)uniformity

We investigate the links between the question $\mathrm{P}_M = \mathrm{NP}_M$ and the efficient elimination of quantifiers over the structure M.

We also study the notion of algebraic Turing machine and see the "equivalence" with uniform circuits.

Exercise 1 Elimination of quantifiers

1. Show that if $\mathrm{P}_M = \mathrm{NP}_M$ then the structure M has quantifier elimination.
2. Show that $\mathbb{P}_M = \mathbb{NP}_M$ if and only if there exists a tuple $\bar{c} \in M$ of parameters, a polynomial $p(n)$ and a family (C_n) of circuits of size $p(n)$ such that every existential formula $f(\bar{x}) = (\exists \bar{y})\phi(\bar{x}, \bar{y})$ of size n is equivalent to the formula $C_n(f, \bar{x}, \bar{c}) = 1$.
3. Prove a similar statement for the uniform version "$\mathrm{P}_M = \mathrm{NP}_M$".

Exercise 2 Algebraic Turing machine

1. Let M be a structure. Define a Turing machine over M in a similar way as boolean Turing machines, but able to manipulate elements of M.
2. Define the class of languages over M recognized by an algebraic Turing machine working in polynomial time. Show that this class is P_M (i.e. show the equivalence with a uniform family of algebraic circuits of polynomial size).