Pushdown compression

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Motivations

- New compression algorithms for structured documents (XML): behaviour depending on the current tag → use of a stack to push and pop tags.
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- Very simple algorithms → pushdown automata.

- Easy to compress and decompress.

  Practical tests: better performances than zip (Lempel-Ziv).
Motivations

- New compression algorithms for structured documents (XML): behaviour depending on the current tag → use of a stack to push and pop tags.

- Very simple algorithms → pushdown automata.

- Easy to compress and decompress. **Practical tests**: better performances than zip (Lempel-Ziv).

- Need for a theoretical study.
Outline

1. Introduction (LZ, FS)

2. Pushdown compression

3. Pushdown beats LZ

4. LZ beats pushdown

5. Conclusion
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Compression

- Lossless compression.
Lossless compression.

Compressor: injective and computable function \( f : \{0, 1\}^* \rightarrow \{0, 1\}^* \).
Compression

- Lossless compression.

- Compressor: injective and computable function \( f : \{0, 1\}^* \rightarrow \{0, 1\}^* \).

- Compression ratio on a finite word \( x \):
  \[
  \rho_f(x) = \frac{|f(x)|}{|x|}.
  \]

Compression ratio on an infinite sequence \( S \):
\[
\rho_f(S) = \lim_{n \to \infty} \sup \rho_f(S[1..n]).
\]
Lempel-Ziv

Text to be compressed:

```
0 1 0 0 0 1 0 1 1 1 0 1 0 0 1 0 0 0 0 0 1 1 1
```

Compression result:
Lempel-Ziv

Text to be compressed:

0

ε/ 0 1 0 0 0 1 0 1 1 1 0 1 0 0 1 0 0 0 0 0 1 1 1

Compression result:

ε;
Lempel-Ziv

Text to be compressed:

0 1

\(\epsilon/0/1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\)

Compression result:

\(\epsilon;(0, 0);\)
Text to be compressed:
0 1 2
ε/0/1/0 0 0 1 0 1 1 1 0 1 0 0 1 0 0 0 0 0 1 1 1

Compression result:
ε;(0, 0);(0, 1);
Text to be compressed:

0 1 2 3

$\epsilon/0/1/0 0/0 1 0 1 1 1 0 1 0 0 1 0 0 0 0 0 1 1 1$

Compression result:

$\epsilon;(0, 0);(0, 1);(1, 0);$
Lempel-Ziv

Text to be compressed:

\[0 \ 1 \ 2 \ 3 \ 4\]

\[\epsilon/0/1/0 \ 0/0 \ 1/0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1\]

Compression result:

\[\epsilon;(0, 0);(0, 1);(1, 0);(1, 1)\]
Lempel-Ziv

Text to be compressed:

0 1 2 3 4 5

ε/0/1/0 0/0 1/0 1 1/1 0 1 0 0 1 0 0 0 0 0 1 1 1

Compression result:

ε;(0, 0);(0, 1);(1, 0);(1, 1);(4, 1);
Lempel-Ziv

Text to be compressed:

0 1 2 3 4 5 6 7 8 9 10

\( \epsilon/0/1/0 \ 0/0 \ 1/0 \ 1/1 \ 0/1 \ 0 \ 0/1 \ 0 \ 0 \ 0/0 \ 0 \ 1/1 \ 1 \)

Compression result:

\( \epsilon;(0, 0);(0, 1);(1, 0);(1, 1);(4, 1);(2, 0);(6, 0);(7, 0);(3, 1);(2, 1) \)
Lempel-Ziv

Text to be compressed:

0 1 2 3 4 5 6 7 8 9 10
ε/0/1/0 0/0 1/0 1 1/1 0/1 0 0/1 0 0 0/0 0 1/1 1

Compression result:

ε;(0, 0);(0, 1);(1, 0);(1, 1);(4, 1);(2, 0);(6, 0);(7, 0);(3, 1);(2, 1)

Lemma

▶ If p is the number of phrases, then $|LZ(x)| = p \log p$.
▶ For all x, the compression ratio $\rho_{LZ}(x)$ satisfies

$$\frac{\log |x|}{\sqrt{|x|}} \leq \rho_{LZ}(x) \leq 1 + o(1).$$
Finite-state compression (1)

Finite-state **transducer**: finite-state automaton that outputs letters at each transition → function \( f : \{0, 1\}^* \rightarrow \{0, 1\}^* \).

\[
\begin{align*}
0 & \mapsto 0, \\
01 & \mapsto 01, \\
1 & \mapsto 011.
\end{align*}
\]

Example: 00 00 01 1 1 00 \(\mapsto\) 0 0 01 011 011 0.
Finite-state compression (1)

Finite-state **transducer**: finite-state automaton that outputs letters at each transition

→ function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$.

```
0 / ε → 0, 01 / 0 → 01, 1 / 011
```

Example: 00 00 01 1 1 00 → 0 0 01 011 011 0.

Finite-state **compressor**: injective finite-state transducer (taking into account the final state).
Finite-state compression (2)

For a finite-state compressor $C$: compression ratio of an infinite sequence $S$

$$\rho_C(S) = \lim_{n \to \infty} \frac{|C(S[1..n])|}{n}.$$
For a finite-state compressor $C$: compression ratio of an infinite sequence $S$

$$\rho_C(S) = \limsup_{n \to \infty} \frac{|C(S[1..n])|}{n}.$$ 

Finite-state compression ratio:

$$\rho_{FS}(S) = \inf_{C \in FS} \rho_C(S).$$
Theorem (Lempel, Ziv, 1979)

On every infinite sequence $S \in \{0, 1\}^\mathbb{N}$, Lempel-Ziv is better than any finite-state compressor, that is,

$$\rho_{LZ}(S) \leq \rho_{FS}(S).$$
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Pushdown transducers

Pushdown compressor = finite-state transducer with a stack.

The transition is done according both to the symbol read and to the topmost symbol of the stack.

Each transition either pushes or pops symbols from the stack.
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PD compression ratio: $\rho_{PD}(S) = \inf_{C \in PD} \rho_C(S)$.
Pushdown transducers

Pushdown compressor = finite-state transducer with a stack.

The transition is done according both to the symbol read and to the topmost symbol of the stack.

Each transition either pushes or pops symbols from the stack.

PD compression ratio: \( \rho_{PD}(S) = \inf_{C \in PD} \rho_C(S) \).

Two variants: with or without endmarkers
\( \rightarrow C(x) \) or \( C(x\#) \) (enables to empty the stack).
### Proposition

Let $S = 0^\infty$.

- The compression ratio on $S$ of a finite-state compressor with $k$ states is $\geq 1/k$.
- There exists a pushdown compressor with $k$ states whose compression ratio on $S$ is $\leq 1/k^2$ (with endmarkers).
Example

Proposition

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- There exists a pushdown compressor with $k$ states whose compression ratio on $S$ is $\leq 1/k^2$ (with endmarkers).

Proof.

1. Let $C$ be a FS compressor with $k$ states.
Then $C$ must output at least one symbol every $k$ letters.
Otherwise there would exist $u$ such that for all $i_0 \leq i \leq i_0 + k$, all the $u[1..i]$ have the same image.
Since there are only $k$ states, this contradicts injectivity.
Let $S = 0^\infty$.

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- There exists a pushdown compressor with $k$ states whose compression ratio on $S$ is $\leq 1/k^2$ (with endmarkers).

**Proof.**

2. Let $C$ be the following pushdown compressor on input $0^n$:
   - it pushes $0^{n/k}$ on the stack (by counting modulo $k$);
   - at the end it pops the stack and outputs one symbol every $k$ (by counting modulo $k$).
Remarks

- Same result as FS for pushdown without endmarkers.
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- LZ on $S = 0^\infty$ has compression ratio 0...
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- Same result as FS for pushdown without endmarkers.
- LZ on $S = 0^\infty$ has compression ratio $0\ldots$
- but FS also!

$$\rho_{FS}(S) = \inf_{C \in FS} \rho_C(S) \leq 1/k \text{ for all } k.$$
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There exists a sequence $S$ such that

$$\rho_{PD}(S) = \frac{1}{2} \ (\text{without endmarkers})$$

and

$$\rho_{LZ}(S) = 1.$$
The idea

Pushdown compresses palindromes with ratio $\approx 1/2. \ldots$

but LZ not always.
The idea

Pushdown compresses *palindromes* with ratio $\approx 1/2.\ldots$

but LZ not always.

→ build a sequence of the form

$$S = u_1 \bar{u}_1 u_2 \bar{u}_2 \ldots$$

with well-chosen words $u_i$ (here $\bar{u}$ stands for the mirror of $u$).
Proof

Let $E_n \subset \{0, 1\}^n$ be the set of words of size $n$ that are not palindromes. Let $u_1, \ldots, u_{|E_n|/2}$ be $|E_n|/2$ words of $E_n$ such that $\forall i, j, u_i \neq \bar{u}_j$. Then

$$u_1 \ldots u_{|E_n|/2} \bar{u}_{|E_n|/2} \ldots \bar{u}_1$$

is LZ-incompressible but $1/2$-PD-compressible.

$\to$ repeat this for all sizes $n$ to obtain the infinite sequence $S$. $\square$
1. Introduction (LZ, FS)

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Theorem

There exists a sequence $S$ such that

$$\rho_{LZ}(S) = 0$$

and

$$\rho_{PD}(S) = 1 \text{ (with endmarkers).}$$
The idea

LZ compresses repetitions very well (ratio tends to 0)...

but pushdown not always.
The idea

LZ compresses repetitions very well (ratio tends to 0)…

but pushdown not always.

- Show that some repetitions are not compressed by pushdown (→ pumping lemma);
- build a sequence of the form

\[ S = u_1^{n_1} u_2^{n_2} \ldots \]

for well chosen \( u_i \) and \( n_i \).
Lemma

Let $u$ be a word. The compression ratio of LZ on $u^n$ is $O\left(\frac{\log n}{\sqrt{n}}\right)$ (and thus tends to 0 when $n \to \infty$).
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Proof.
For all $k$, there are at most $|u|$ different words of size $k$ in $u^n$. Call $p$ the number of phrases in the parsing of $u^n$ by LZ algorithm. Let $t_k$ be the number of phrases of size $k$. We have:

$$|u^n| = \sum_{k \geq 1} t_k \geq \sum_{k=1}^{p/|u|} k|u| \geq \frac{p^2}{2|u|}.$$

Thus $p = O\left(\sqrt{n}\right)$ and $|LZ(x)| = p \log p$. □
Let $C$ be a pushdown compressor.

Suppose there is a **pumping lemma**: on input $uv^n w$, $C$ has each time the same behaviour on $v$.

If $v$ is not compressible, then $C(uv^n w) \geq n|v|$, thus $\rho_C(uv^n w) \to 1$. 
Pumping lemma

**Theorem**

Let $A$ be a pushdown transducer (working without endmarkers). There exist two constants $\alpha, \beta > 0$ such that all word $w$ can be cut in three pieces $w = tuv$ satisfying:

- $|u| \geq \lfloor \alpha |w|^{\beta} \rfloor$;
- if $C(tuv) = xyz$ then $C(tu^n) = xy^n$. 
Acceptors: reminder

Equivalence pushdown automata / context-free grammars.
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Equivalence pushdown automata / context-free grammars.
Lifetime of a column: never go below its top symbol.

Equivalent columns: \( c' \) in the lifetime of \( c \) and same state/top symbol.

State: \( q_1 \ q_2 \ q_3 \ q_1 \)

Lifetime of column (1)

Lifetime of column (2)
- Lifetime of a column: never go below its top symbol.
- Equivalent columns: $c'$ in the lifetime of $c$ and same state/top symbol.
Transducers: proof (2)

- $p$: number of pairs state/top symbol;
- $k$: max number of symbols pushed by one rule;
- $L(p, k, d)$: maximum lifetime of a column during which no pair of equivalent columns are at distance $\geq d$. 

\[
L(p + 1, k, d) = d + kdL(p, k, d)
\]
Transducers: proof (2)

- $p$: number of pairs state/top symbol;
- $k$: max number of symbols pushed by one rule;
- $L(p, k, d)$: maximum lifetime of a column during which no pair of equivalent columns are at distance $\geq d$.
- $L(p + 1, k, d) = d + kdL(p, k, d)$.
The endmarker

Theorem

Let A be a pushdown transducer (working with endmarkers). There exist two constants $\alpha, \beta > 0$ such that all word $w$ can be cut in three pieces $w = tuv$ satisfying:

- $|u| \geq \lfloor \alpha |w| \beta \rfloor$;
- there are five words $x, x', y, y', z$ such that $C(tu^ny\#) = xy^ny'y'^nx'$.

Remark.
The same is true with an initially nonempty stack.
Theorem

There exists a sequence $S$ such that

$$\rho_{LZ}(S) = 0 \text{ and } \rho_{PD}(S) = 1 \text{ (with endmarkers)}.$$
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There exists a sequence $S$ such that

$$\rho_{LZ}(S) = 0 \text{ and } \rho_{PD}(S) = 1 \ (\text{with endmarkers}).$$

Proof.

Let $w_i$ be a sufficiently big Kolmogorov-random word
→ cut it in three pieces $w_i = t_i u_i v_i$, with $u_i$ big enough (thus incompressible), according to the $i$-th pushdown transducers. Then

$$S = t_1 u_1^{n_1} t_2 u_2^{n_2} \ldots$$

(for sufficiently large integers $n_i$) is LZ-compressible but not PD-compressible. □
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Summary

- Introduction of *pushdown compression*.
- Strictly better than finite-state compression.
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- Better than Lempel-Ziv on some sequences (palindromes: compression ratio 1/2 instead of 1).
Summary

- Introduction of **pushdown compression**.

- Strictly better than finite-state compression.

- Better than Lempel-Ziv on some sequences (palindromes: compression ratio 1/2 instead of 1).

- Worse than Lempel-Ziv on some sequences (repetitions: compression ratio 1 instead of 0).
Future work

- Lower bound on the compression ratio of a PD compressor with $k$ states with endmarkers?
- Better separation for “PD beats LZ”?
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