Is this tiling pure discrete ?

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Self affine tilings and multi Delone sets

A **Delone set** in $\mathbb{R}^d$ $\Leftrightarrow$ relatively dense and uniformly discrete. A set Λ in $\mathbb{R}^d$ is a **Meyer set** if Λ and $\Lambda - \Lambda$ are Delone. $(\Lambda_i)_{i=1}^k$ is a substitutive Delone set if Λ_i is Delone and

$$\Lambda_i = \bigcup_{j=1}^{k} Q\Lambda_j + D_{ij}$$

for an expanding matrix Q. An **adjoint system** is

$$QA_j = \bigcup_{i=1}^{k} A_i + D_{ij},$$

which gives a unique non empty compact solution A_i. By these two equations, a tiling and a point set becomes **dual**.

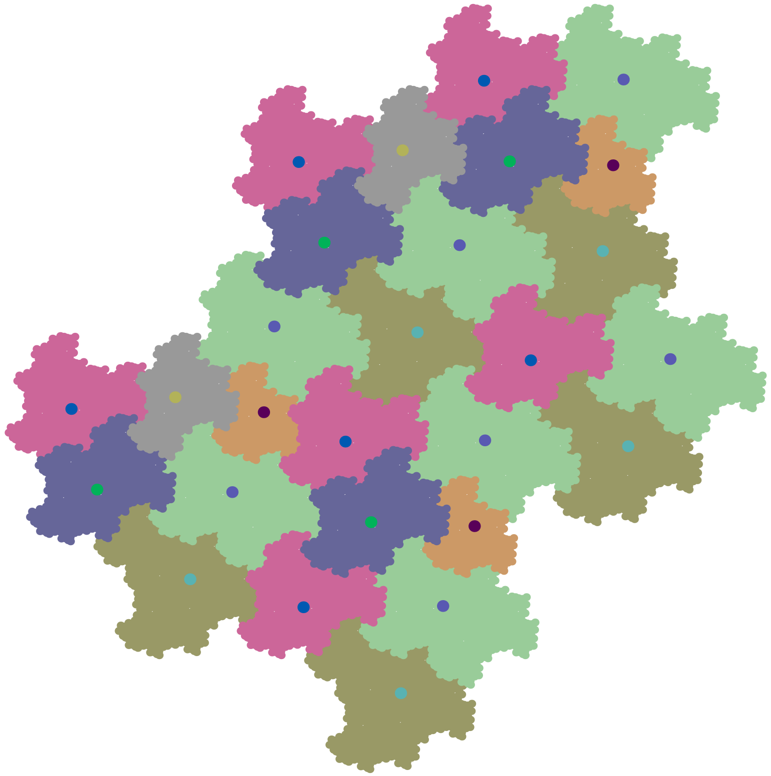

Figure 1: Tiling and multi color Delone set

By translation action we can define dynamical systems in both tilings and point sets. Dual systems are isomorphic. Assume that its substitution matrix $(\#D_{ij})$ is primitive. Then the system is minimal and uniquely ergodic. Therefore we can discuss their spectral properties.

We are interested in whether these system have **pure discrete spectrum** or not.

Given a self-affine tiling, Solomyak [7] gave a necessary and sufficient condition for pure discreteness, that is:

$$\text{density } \mathcal{T} \cap \ \mathcal{T} - Q^n v \to 1$$

for linearly independent d return vectors. This is a kind of "almost periodicity".

Overlap algorithm

Overlap is the triple (T, α, S) (up tp translation) that T and S are tiles in $\mathcal{T}$ and α is a return vector satisfying:

$$\left(\overset{\circ}{T} - \alpha\right) \cap \overset{\circ}{S} \neq \emptyset$$

The overlap is a **coincidence** if T, S are translationally equivalent and $\alpha = 0$. Among such overlaps, we can naturally create a graph by inflation: multiplication by Q. Then the above criterion is equivalent to show that for each overlap there is a path leading to coincidence. However in practice, this computation often becomes large and we require computer assistance to check. Moreover the overlap (T, α, S) is defined in topological terms:

$$(\mathring{T} - \alpha) \cap \mathring{S} \neq \emptyset,$$

it is not easy to tell this to our computer. Especially when the tile has fractal boundary, it is hard to tell even for us.

Therefore we introduce a weaker notation: (i, α, j) is a **potential overlap** if

$$|x_T - \alpha - x_S| \leq 2 \max_T \operatorname{diam}(T)$$

where $x_T \in \Lambda_i$ and $x_S \in \Lambda_j$. Taking control points of tiles T and S of an overlap, it must give a potential overlap.

For computational reason, we also need to introduce an **overlap graph with multiplicity**, in which the graph is made exactly by the same procedure as the overlap algorithm but we count multiplicities of occurrences of the same potential overlap in the inflated overlap.

In other words, a **potential overlap graph with multiplicity** $\mathcal{G}$ has s multiple edges from (i, α, j) to (i', α', j') iff $\Omega(i, \alpha, j)$ contains s copies of (i', α', j').

We say that (i, α, j) is a coincidence if $i = j$ and $\alpha = 0$. Let $\mathcal{G}_{coin}$ be the induced graph of $\mathcal{G}$ to the vertices which have a path leading to a coincidence. Also we define $\mathcal{G}_{res}$ by the induced graph generated by the complement of such vertices.

Theorem 1. $(X_{\mathcal{T}}, \mathbb{R}^d)$ *is pure discrete if and only if*

$$\rho(\mathcal{G}_{coin}) > \rho(\mathcal{G}_{res}).$$

Moreover the **modified** *Hausdorff dimension of the boundary of the tile is equal to* $d \log \rho(\mathcal{G}_{res}) / \log \rho(\mathcal{G}_{coin})$.

This algorithm is quite simple, easy to implement and applies to all self-affine tilings whenever we know the matrix Q and digits $D_{i,j}$ which defines the multi-color Delone set.

Our current bottle neck is the process to collect all potential overlaps for d return vectors.

Sufficiency of the inequality.

Consider the real overlap (T, α, S). Applying Ω^n we have

$$\mu_d(Q^n((T-\alpha)\cap S)) = |\det(Q)|^n \mu_d((T-\alpha)\cap S)$$

where μ_d is the d-dim Lebesgue measure. We must have $|\det(Q)| = \beta$ where β is the Perron Frobenius root of substitution matrix $(\#(D_{ij}))$ (Lagarias-Wang [2]). There are $r > 0$ and $R > 0$ such that each overlap $(\overset{\circ}{T} - \alpha) \cap \overset{\circ}{S}$ contains a ball of radius r and surrounded by a ball of radius R. After n-iteration of inflation, the number of overlaps K_n generated from (T, α, S) is estimated:

$$c_1 \beta^n \leq K_n \leq c_2 \beta^n$$

with positive constants c_1 and c_2. As each overlap gives this growth of overlaps, $\mathcal{G}_{res}$ can not contain an overlap (because we are taking into account the multiplicities of overlap growth).

Necessity of the inequality.

We prove that if all overlaps leads to a coincidence then $\mathcal{G}_{res}$ can not have a spectral radius β. Potential overlaps can be divided into three cases.

- Real overlap: $(\overset{\circ}{T} - \alpha) \cap \overset{\circ}{S} \neq \emptyset$,
- $T - \alpha$ and S are just touching at their boundaries,
- No intersection: $(T - \alpha) \cap S = \emptyset$.

If $(T - \alpha) \cap S$ is empty, then the distance becomes larger by the iteration of Ω. Therefore this potential overlap does not produce an infinite walk on the produced graph. This case does not contribute to the spectral radius. However when they are touching at their boundaries, by repeated inflation, this gives an infinite walk on this graph and may contribute to the spectral radius of $\mathcal{G}_{res}$. Our task is to prove that this contribution is small.

We recall a:

Conjecture 1 (Urbański (Solomyak [6])). *For d-dim self-affine tiling, each tile T satisfies* $d-1 \leq \dim_H(\partial(T)) < d$.

which is a folklore (every knows the 'fact' without proof).

We have to solve partly a version of this conjecture. We use the recent development to slightly modify the Hausdorff measure by He and Lau [1]. They introduced a new type of gauze function, called **pseudo norm** $w : \mathbb{R}^d \to \mathbb{R}_+$ corresponding to Q having key properties:

$$w(Qx) = |\det(Q)|^{1/d} w(x)$$

and

$$w(x+y) \leq c \max(w(x), w(y))$$

for some positive constant c. This w induces the same topology as Euclidean norm. By w, they modified the definition of

Hausdorff measure by:

$$\mathrm{diam}_w(U) = \sup_{x,y \in U} w(x-y),$$

$$\mathcal{H}^w_{s,\delta}(X) = \inf_{X \subset \bigcup_i U_i} \left\{ \sum_i \mathrm{diam}_w(U_i)^s \;\middle|\; \mathrm{diam}_w(U_i) < \delta \right\}$$

and

$$\mathcal{H}^w_s(X) = \lim_{\delta \downarrow 0} \mathcal{H}^w_{s,\delta}(X).$$

New Hausdorff dimension is $\dim^w_H(X) = \inf\{s | \mathcal{H}^w_s(X) = 0\}$. One can treat self-affine attractors almost as easy as self-similar ones.

Theorem 2. *For d-dim self-affine tiling, each tile T satisfies* $\dim_H^w(\partial T) < d$.

We prove Theorem 1 and 2 simultaneously.

Let us take a piece Y of graph directed set defined by a strongly connected component of $\mathcal{G}_{res}$ with maximal spectral radius and put $s = d \log \gamma / \log \beta$. Since all overlap leads to a coincidence, Y must be a part of the boundary of our tile.

GIFS satisfies OSC

This is done according to the method of [1, 3] using pseudo norm. Note that the uniform discreteness condition of inflated digits naturally follows from our setting. Then by mass distribution principle, we can show $\mathcal{H}_s^w(Y) > 0$ (c.f. [4, 3]). This shows

$$\dim_H^w(Y) = \frac{d \log(\rho(\mathcal{G}_{coin}))}{\log(\rho(\mathcal{G}_{res}))}.$$

Dimension can not be d.

Assume $s = d$, then this $\mathcal{H}_s^w$ must be a constant multiple of the d-dim Lebesgue measure, by the uniqueness of Haar measure. But Praggastis [5] showed that the d-dim Lebesgue measure of the boundary of self-affine tiles must be 0 which contradicts $\mathcal{H}_w^s(Y) > 0$. Thus we have $d \log \rho(\mathcal{G}_{coin}) / \log \rho(\mathcal{G}_{res}) < d$ which completes the proof of Theorem 1. Noting that potential overlap graph contain all boundary automata as a subgraph, $s = d \log \gamma / \log \beta < d$ gives Theorem 2.

Example by the endomorphism of free group (Dekking, Kenyon)

We consider a self similar tiling is generated a boundary substitution:

$$\begin{aligned}\theta(a) &= b\\ \theta(b) &= c\\ \theta(c) &= a^{-1}b^{-1}\end{aligned}$$

acting on the boundary word $aba^{-1}b^{-1}, aca^{-1}c^{-1}, bcb^{-1}c^{-1}$, representing three fundamental parallelogram. Associated the tile equation is

$$\begin{aligned} \alpha A_1 &= A_2 \\ \alpha A_2 &= (A_2 - 1 - \alpha) \cup (A_3 - 1) \\ \alpha A_3 &= A_1 - 1 \end{aligned}$$

with $\alpha \approx 0.341164 + i1.16154$ which is a root of the polynomial $x^3 + x + 1$.

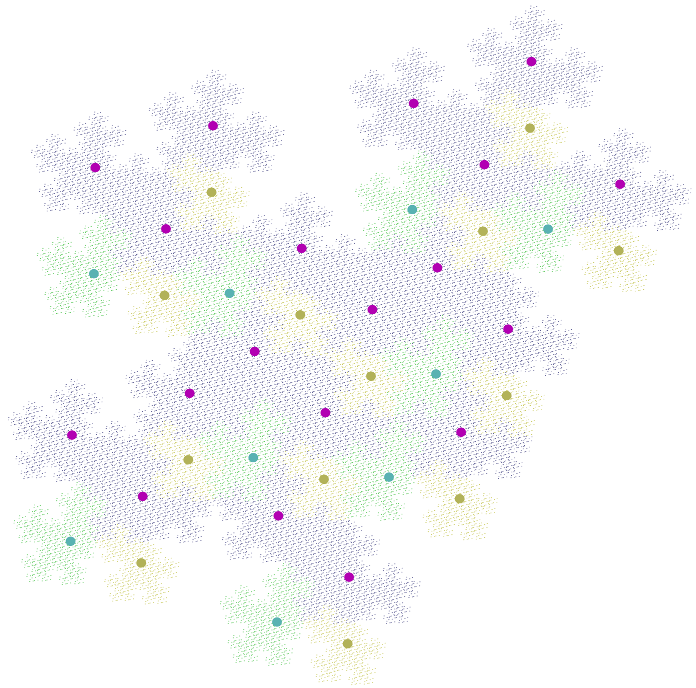

Figure 2: Tiling by boundary endomorphism

$\rho(\mathcal{G}_{coin}) \approx 1.46557$ and $\rho(\mathcal{G}_{res}) \approx 1.32472.$ This shows overlap coincidence and therefore the tiling dynamical system associated with this tiling has pure point spectrum. The Hausdorff dimension of the boundary is $2\log(\rho(\mathcal{G}_{coin}))/\log(\rho(\mathcal{G}_{res})) = 1.47131$.

Example: Arnoux-Furukado-Harriss-Ito tiling

Arnoux-Furukado-Harriss-Ito recently gave an explicit Markov partition of the toral automorphism for the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

which has two dim expanding and two dim contractive planes. They defined 2-dim substitution of 6 polygons. Let $\alpha = -0.518913 - 0.66661\sqrt{-1}$ a root of $x^4 - x^3 + 1$. The multi colour Delone set is given by 6×6 matrix:

$$\begin{pmatrix} \{\} & \{z/\alpha\} & \{z/\alpha\} & \{\} & \{\} & \{\} \\ \{\} & \{\} & \{\} & \{z/\alpha\} & \{z/\alpha\} & \{\} \\ \{\} & \{\} & \{\} & \{\} & \{\} & \{z/\alpha\} \\ \{z/\alpha\} & \{\} & \{\} & \{\} & \{\} & \{\} \\ \{\} & \{z/\alpha + 1 - \alpha\} & \{\} & \{\} & \{\} & \{\} \\ \{\} & \{\} & \{\} & \{(z-1)/\alpha + \alpha\} & \{\} & \{\} \end{pmatrix}$$

and the associated tiling for contractive plane is:

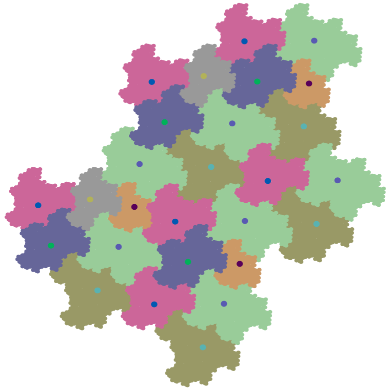

Figure 3: AFHI Tiling

Our program saids it is purely discrete. Number of potential overlap to make the graph is 264. he size of $\mathcal{G}_{coin}$ becomes 88 and $\mathcal{G}_{res}$ is 178.

$$\rho(\mathcal{G}_{coin}) = 1.40127, \ \rho(\mathcal{G}_{res}) = 1.22074$$

which shows **Overlap coincidence** and we have

$$\dim_H(\partial(T)) = 1.18239.$$

Possible Pisot Conjecture in Higher dimension (?)

Assume two conditions:

1. (Λ_i) has Meyer property,

2. Congruent tiles have the same color.

Then the tiling dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ may be pure discrete.

Lagarias-Wang condition and Pisot-family condition follows from Meyer property. If the substitution matrix $(\#D_{ij})$ is irreducible then the color condition holds.

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