The Thue-Morse sequence rarefied with respect to a prime difference.

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The Thue-Morse sequence with symbols 1 and -1 :

$$\tau(n) = (-1)^{\text{number of 1 digits in the binary expansion of } n}$$

It's also the fixed point of the morphism

$$\begin{cases} 1 & \rightarrow & 1\bar{1} \\ \bar{1} & \rightarrow & \bar{1}1 \end{cases}$$

First terms :

$$1 \;\; \bar{1} \;\; \bar{1}1 \;\; \bar{1}11\bar{1} \;\; \bar{1}11\bar{1}1\bar{1}\bar{1}1 \;\; \bar{1}11\bar{1}1\bar{1}\bar{1}11\bar{1}\bar{1}1\bar{1}11\bar{1} \;\ldots$$

$$\boxed{\text{Rarefied sums : } S_{p,0}(N) = \sum_{\substack{n < N \\ p \mid n}} \tau(n)}$$

The Thue-Morse sequence :

$$1\bar{1}\bar{1}1\bar{1}11\bar{1}\bar{1}11\bar{1}1\bar{1}\bar{1}1\bar{1}11\bar{1}1\bar{1}\bar{1}11\bar{1}\bar{1}1\bar{1}11\bar{1}\bar{1}11\bar{1}1\bar{1}\bar{1}1\ldots$$

The 3-rarefied Thue-Morse sequence :

$$1\,\bar{1}\,\bar{1}\,1\,\bar{1}\,1\,1\,\bar{1}\,\bar{1}\,1\,1\,\bar{1}\,1\,\bar{1}\,\bar{1}\,1\,\bar{1}\,1\,1\,\bar{1}\,1\,\bar{1}\,\bar{1}\,1\,1\,\bar{1}\,\bar{1}\,1\,\bar{1}\,1\,1\,\bar{1}\,\bar{1}\,1\,1\,\bar{1}\,1\,\bar{1}\,\bar{1}\,1\ldots$$

Proposition (Moser's conjecture, Newman's theorem (1979))

For all $N > 0$, we get : $S_{3,0}(N) = \sum_{\substack{0 \leqslant n < N \\ n \equiv 0 \pmod 3}} \tau(n) > 0.$

The 3-rarefied Thue-Morse sequence :

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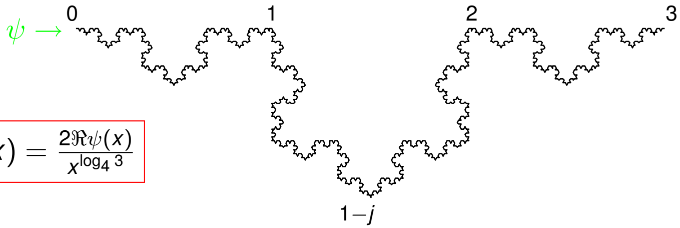
Sketch of the proof

Main idea : decompose $3\mathbb{1}_{3|n} = 1 + j + j^2$ and

pass from the base 2 to base 4.

$$\begin{aligned} S_{3,0}(N) = \quad & \tfrac{1}{3}\Big(\sum_{n<N} \tau(n) \quad + \quad 2\Re \sum_{n<N} \tau(n) j^n \Big) \\ = \quad & \eta \quad + \quad N^{\log_4 3} F(\{\log_4 N\}) \end{aligned}$$

where $\eta \in \{-\frac{1}{3}, 0, \frac{1}{3}\}$, $F > 0$ continuous.

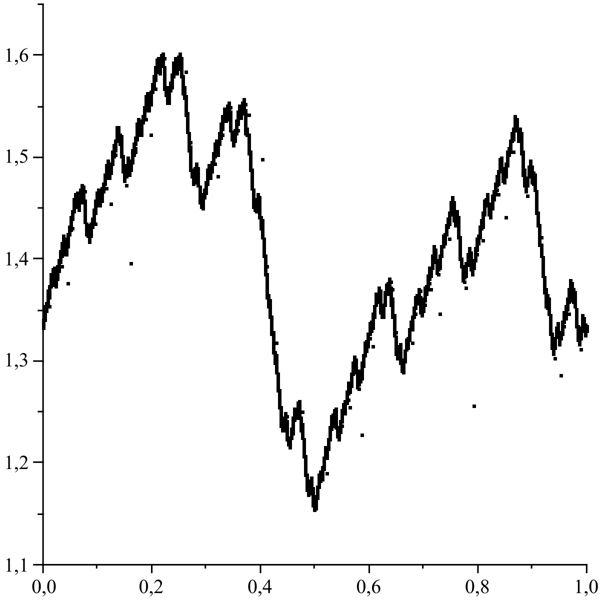


$$F(\log_4 x) = \frac{2\Re\psi(x)}{x^{\log_4 3}}$$

$$x = \{\log_4(N)\}$$

$$y = \frac{S_{3,0}(3N)}{N^{\log_4 3}}$$

$$N = 2 \ldots 10000$$



For a bigger prime number p :

Replace 4 by 2^s where $s = min\{s' | 2^{s'} \equiv 1 \bmod p\}$
and the Koch's curve by

$$\sum_{n=0}^{n=2^s-1} \zeta^n \tau(n) = \xi(\zeta), \qquad \zeta = \sqrt[p]{1}$$

$$\zeta_1 = \exp(\frac{2i\pi}{p})$$

$$\begin{aligned} S_{p,0}(N) = & \frac{1}{p} \sum_{n<N} (1 + \zeta_1 + \zeta_1^2 + \cdots + \zeta_1^{p-1}) \tau(n) \\ = & \eta + N^{\log_{2^s} \xi(\zeta_1)} F_1(\{\log_{2^s} N\}) + \ldots \\ & + N^{\log_{2^s} \xi(\zeta_1^{p-1})} F_{p-1}(\{\log_{2^s} N\}) \end{aligned}$$

Let's study the numbers ξ.

General facts about the sums ξ

If $s = p - 1$, $\xi(\zeta) = p$ for all $\zeta \neq 1$.
In general,

$$\xi(\zeta_1^j) = \begin{cases} 0 \text{ if } \zeta^j = 1 \\ \quad \text{otherwise} \\ j \in \mathbb{F}_p^\times / < 2 > \to \xi \in \mathbb{C}. \end{cases}$$

$$\xi = \sum_{n=0}^{n=2^s-1} \zeta^n \tau(n) = \prod_{j \in <2> \subset \mathbb{F}_p^\times} (1 - \zeta^j)$$

and the product of all the nonzero ξ is p.

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Algebra $\rightarrow$ we have to fix $k = \frac{p-1}{s}$ and define

$$\xi = \prod_{j \in \mathbb{F}_p^{\times k}} (1 - \zeta^j)$$

If $k = 2$, one can take any $p > 3$.

If $p \equiv 3 \bmod 4$, the two values are $i\sqrt{p}$ and $-i\sqrt{p}$.

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If $p \equiv 1 \bmod 4$, ξ's $\in \mathbb{R}$ and Galois-conjugate in $\mathbb{Q}(\sqrt{p})$.
Expression :

$$\begin{cases} \xi = \sqrt{p}\epsilon^{-h}, \\ \xi' = \sqrt{p}\epsilon^{h} \end{cases}$$

where ϵ is the regulator and

h is the class number of $\mathbb{Q}(\sqrt{p})$.

$\xi + \xi'$- ?

About the cubic case.
$p \equiv 1 \bmod 6$.
All three values are $\in \mathbb{R}_+^*$ and conjugate in a cubic field.

$$\sigma_1 = \xi^{I} + \xi^{II} + \xi^{III}$$
$$\sigma_2 = \xi^{I}\xi^{II} + \xi^{I}\xi^{III} + \xi^{II}\xi^{III}$$
$$\mathrm{Gal}(\mathrm{X}^3 - \sigma_1\mathrm{X}^2 + \sigma_2\mathrm{X} - \mathrm{p})$$

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Draw $\sum_{n=0}^{2^s-1} \tau(n)\zeta^n$ as sequence of 2^s line intervals.