ON THE STRUCTURE OF $(-\beta)$-INTEGERS

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Abstract. The $(-\beta)$-integers are natural generalisations of the β-integers, and thus of the integers, for negative real bases. When β is a $(-\beta)$-number, which is the analogue of a Parry number, we describe their structure by a fixed point of an anti-morphism.

1. Introduction

The aim of this paper is to study the structure of the set of real numbers having a digital expansion of the form

$$\sum_{k=0}^{n-1} a_k \, (-\beta)^k \,,$$

where $(-\beta)$ is a negative real base with $\beta > 1$, the digits $a_k \in \mathbb{Z}$ satisfy certain conditions specified below, and $n \geq 0$. These numbers are called *$(-\beta)$-integers*, and have been recently studied by Ambrož, Dombek, Masáková and Pelantová [1].

Before dealing with these numbers, we recall some facts about *β-integers*, which are the real numbers of the form

$$\pm \sum_{k=0}^{n-1} a_k \, \beta^k \quad \text{such that} \quad 0 \leq \sum_{k=0}^{m-1} a_k \, \beta^k < \beta^m \quad \text{for all } 1 \leq m \leq n \,,$$

i.e., $\sum_{k=0}^{n-1} a_k \, \beta^k$ is a greedy *β-expansion*. Equivalently, we can define the set of β-integers as

$$\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+) \quad \text{with} \quad \mathbb{Z}_\beta^+ = \bigcup_{n \geq 0} \beta^n \, T_\beta^{-n}(0) \,,$$

where T_β is the *β-transformation*, defined by

$$T_\beta : \, [0,1) \to [0,1) \,, \quad x \mapsto \beta x - \lfloor \beta x \rfloor \,.$$

This map and the corresponding β-expansions were first studied by Rényi [17].

The notion of β-integers was introduced in the domain of quasicrystallography, see for instance [5], and the structure of the β-integers is very well understood now. We have $\mathbb{Z}_\beta \subseteq \beta \, \mathbb{Z}_\beta$, the set of distances between consecutive elements of $\mathbb{Z}_\beta$ is

$$\Delta_\beta = \{T_\beta^n(1^-) \mid n \geq 0\} \,,$$

where $T_\beta(x^-) = \lim_{y \to x, \, y < x} T_\beta(y)$, and the sequence of distances between consecutive elements of $\mathbb{Z}_\beta^+$ is coded by the fixed point of a substition, see [8] for the case when Δ_β is a finite set, that is when β is a *Parry number*. We give short proofs of these facts in Section 2. More detailed properties of this sequence can be found e.g. in [2, 3, 4, 10, 14].

Closely related to $\mathbb{Z}_\beta^+$ are the sets

$$S_\beta(x) = \bigcup_{n\geq 0} \beta^n\, T_\beta^{-n}(x) \qquad (x \in [0,1)),$$

which were used by Thurston [18] to define (fractal) tilings of $\mathbb{R}^{d-1}$ when β is a Pisot number of degree d, i.e., a root of a polynomial $x^d + p_1 x^{d-1} + \cdots + p_d \in \mathbb{Z}[x]$ such that all other roots have modulus < 1, and an algebraic unit, i.e., $p_d = \pm 1$. These tilings allow e.g. to determine the k-th digit a_k of a number without knowing the other digits, see [13].

It is widely agreed that the greedy β-expansions are the natural representations of real numbers in a real base $\beta > 1$. For the case of negative bases, the situation is not so clear. Ito and Sadahiro [12] proposed recently to use the $(-\beta)$-*transformation* defined by

$$T_{-\beta}:\ \big[\tfrac{-\beta}{\beta+1}, \tfrac{1}{\beta+1}\big),\ x \mapsto -\beta x - \big\lfloor -\beta x + \tfrac{\beta}{\beta+1} \big\rfloor\,.$$

see also [9]. This transformation has the important property that $T_{-\beta}(-x/\beta) = x$ for all $x \in \big(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$. Some instances are depicted in Figures 1, 3 and 4.

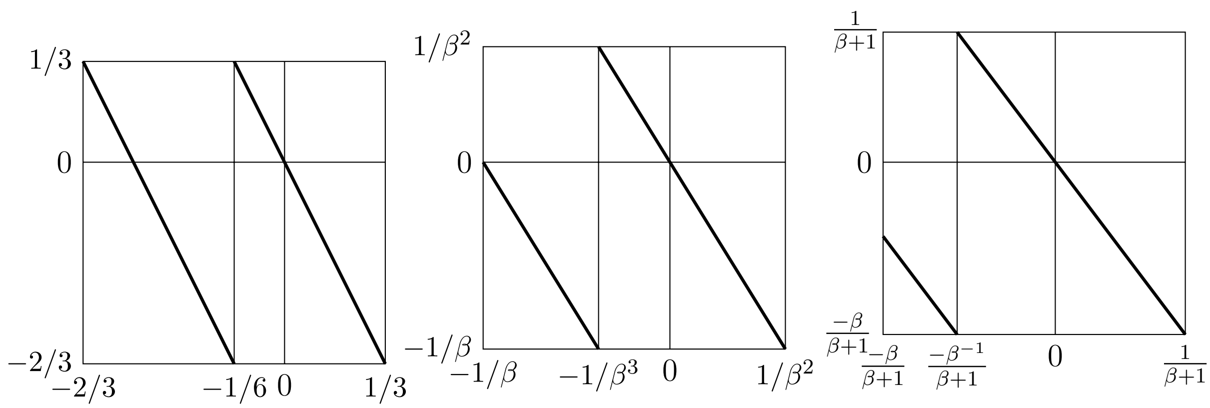

FIGURE 1. The $(-\beta)$-transformation for $\beta = 2$ (left), $\beta = \frac{1+\sqrt{5}}{2} \approx 1.618$ (middle), and $\beta = \frac{1}{\beta} + \frac{1}{\beta^2} \approx 1.325$ (right).

The set of $(-\beta)$-integers is therefore defined by

$$\mathbb{Z}_{-\beta} = \bigcup_{n\geq 0} (-\beta)^n\, T_{-\beta}^{-n}(0)\,.$$

These are the numbers

$$\sum_{k=0}^{n-1} a_k\,(-\beta)^k \quad \text{such that} \quad \frac{-\beta}{\beta+1} \leq \sum_{k=0}^{m-1} a_k\,(-\beta)^{k-m} < \frac{1}{\beta+1} \quad \text{for all } 1 \leq m \leq n\,.$$

Note that, in the case of β-integers, we have to add $-\mathbb{Z}_\beta^+$ to $\mathbb{Z}_\beta^+$ in order to obtain a set resembling $\mathbb{Z}$. In the case of $(-\beta)$-integers, this is not necessary because the $(-\beta)$-transformation allows to represent positive and negative numbers.

It is not difficult to see that $\mathbb{Z}_{-\beta} = \mathbb{Z} = \mathbb{Z}_\beta$ when $\beta \in \mathbb{Z}$, $\beta \geq 2$. Some other properties of $\mathbb{Z}_{-\beta}$ can be found in [1], mainly for the case when $T_{-\beta}^n\big(\frac{-\beta}{\beta+1}\big) \in \big[\frac{1-\lfloor\beta\rfloor}{\beta}, 0\big]$ for all $n \geq 1$. The set

$$V'_\beta = \big\{T_{-\beta}^n\big(\tfrac{-\beta}{\beta+1}\big) \mid n \geq 0\big\}$$

plays a similar role for $(-\beta)$-expansions as the set Δ_β for β-expansions. Consequently, we call $\beta > 1$ a $(-\beta)$-*number* if V'_β is a finite set, recalling that Parry numbers were originally called β-*numbers* in [16]. In Theorem 3, we describe, for any $(-\beta)$-number $\beta \geq (1+\sqrt{5})/2$, the sequence of $(-\beta)$-integers in terms of a two-sided infinite word on a finite alphabet which is a fixed point of an anti-morphism. Note that the orientation-reversing property of the map $x \mapsto -\beta x$ imposes the use of an anti-morphism instead of a morphism. Anti-morphisms were considered in a similar context by Enomoto [7].

For $1 < \beta < \frac{1+\sqrt{5}}{2}$, we have $\mathbb{Z}_{-\beta} = \{0\}$, as already proved in [1]. However, our study still makes sense for these bases $(-\beta)$ because we can also describe the sets

$$S_{-\beta}(x) = \lim_{n\to\infty} (-\beta)^n\, T_{-\beta}^{-n}(x) \qquad \big(x \in \big[\tfrac{-\beta}{\beta+1}, \tfrac{1}{\beta+1}\big)\big),$$

where the limit set consists of the numbers lying in all but finitely many sets $(-\beta)^n\, T_{-\beta}^{-n}(x)$, $n \geq 0$. The reason for taking the limit instead of the union over all $n \geq 0$ is that $T_{-\beta}^2\big(\frac{-\beta^{-1}}{\beta+1}\big) \neq \frac{-\beta}{\beta+1}$ when $\beta \notin \mathbb{Z}$, see Section 3. In [15], the $(-\beta)$-transformation is studied in detail for these values of β.

2. β-integers

In this section, we consider the structure of β-integers. The results are not new, but it is useful to state and prove them in order to understand the differences with $(-\beta)$-integers.

Recall that $\Delta_\beta = \{T_\beta^n(1^-) \mid n \geq 0\}$, and let Δ_β^* be the free monoid generated by Δ_β. Elements of Δ_β^* will be considered as words on the alphabet Δ_β, and the operation is the concatenation of words. The β-*substitution* is the morphism $\varphi_\beta : \Delta_\beta^* \to \Delta_\beta^*$, defined by

$$\varphi_\beta(x) = \underbrace{11\cdots1}_{\lceil\beta x\rceil - 1 \text{ times}}\, T_\beta(x^-) \qquad (x \in \Delta_\beta).$$

Here, 1 is an element of Δ_β and not the identity element of Δ_β^* (which is the empty word). Recall that, as φ_β is a morphism, we have $\varphi_\beta(uv) = \varphi_\beta(u)\varphi_\beta(v)$ for all $u, v \in \Delta_\beta^*$. Since $\varphi_\beta^{n+1}(1) = \varphi_\beta^n(\varphi_\beta(1))$ and $\varphi_\beta(1)$ starts with 1, $\varphi_\beta^n(1)$ is a prefix of $\varphi_\beta^{n+1}(1)$ for every $n \geq 0$.

Theorem 1. *For any $\beta > 1$, we have*

$$\mathbb{Z}_\beta^+ = \{z_k \mid k \geq 0\} \quad \textit{with} \quad z_k = \sum_{j=1}^k u_j\,,$$

where $u_1 u_2 \cdots$ is the infinite word with letters in Δ_β which has $\varphi_\beta^n(1)$ as prefix for all $n \geq 0$.

The set of differences between consecutive β-integers is Δ_β.

The main observation for the proof of the theorem is the following. We use the notation $|v| = k$ and $L(v) = \sum_{j=1}^k v_j$ for any $v = v_1 \cdots v_k \in \Delta_\beta^k$, $k \geq 0$.

Lemma 1. *For any $n \ge 0$, $1 \le k \le |\varphi_\beta^n(1)|$, we have*

$$T_\beta^n\Big(\Big[\frac{z_{k-1}}{\beta^n}, \frac{z_k}{\beta^n}\Big)\Big) = [0, u_k)\,,$$

and $z_{|\varphi_\beta^n(1)|} = L(\varphi_\beta^n(1)) = \beta^n$.

Proof. For $n = 0$, we have $|\varphi_\beta^0(1)| = 1$, $z_0 = 0$, $z_1 = 1$, $u_1 = 1$, thus the statements are true. Suppose that they hold for n, and consider

$$u_1 u_2 \cdots u_{|\varphi_\beta^{n+1}(1)|} = \varphi_\beta^{n+1}(1) = \varphi_\beta(\varphi_\beta^n(1)) = \varphi_\beta(u_1)\varphi_\beta(u_2)\cdots\varphi_\beta(u_{|\varphi_\beta^n(1)|})\,.$$

Let $1 \le k \le |\varphi_\beta^{n+1}(1)|$, and write $u_1 \cdots u_k = \varphi_\beta(u_1 \cdots u_{j-1})\, v_1 \cdots v_i$ with $1 \le j \le |\varphi_\beta^n(1)|$, $1 \le i \le |\varphi_\beta(u_j)|$, i.e., $v_1 \cdots v_i$ is a non-empty prefix of $\varphi_\beta(u_j)$.

For any $x \in (0,1]$, we have $T_\beta(x^-) = \beta x - \lceil \beta x \rceil + 1$, hence $L(\varphi_\beta(x)) = \beta x$ for $x \in \Delta_\beta$. This yields that

$$z_k = L(u_1 \cdots u_k) = \beta\, L(u_1 \cdots u_{j-1}) + L(v_1 \cdots v_i) = \beta z_{j-1} + i - 1 + v_i$$

and $z_{k-1} = \beta z_{j-1} + i - 1$, hence

$$\Big[\frac{z_{k-1}}{\beta}, \frac{z_k}{\beta}\Big) = \Big[z_{j-1} + \frac{i-1}{\beta}, z_{j-1} + \frac{i-1+v_i}{\beta}\Big) \subseteq [z_{j-1}, z_{j-1} + u_j) = [z_{j-1}, z_j)\,,$$

$$T_\beta^{n+1}\Big(\Big[\frac{z_{k-1}}{\beta^{n+1}}, \frac{z_k}{\beta^{n+1}}\Big)\Big) = T_\beta\Big(\Big[\frac{i-1}{\beta}, \frac{i-1+v_i}{\beta}\Big)\Big) = [0, v_i) = [0, u_k)\,.$$

Moreover, we have $L(\varphi_\beta^{n+1}(1)) = \beta\, L(\varphi_\beta^n(1)) = \beta^{n+1}$, thus the statements hold for $n+1$. □

Proof of Theorem 1. By Lemma 1, we have $z_{|\varphi_\beta^n(1)|} = \beta^n$ for all $n \ge 0$, thus $[0,1)$ is split into the intervals $[z_{k-1}/\beta^n, z_k/\beta^n)$, $1 \le k \le |\varphi_\beta^n(1)|$. Therefore, Lemma 1 yields that

$$T_\beta^{-n}(0) = \{z_{k-1}/\beta^n \mid 1 \le k \le |\varphi_\beta^n(1)|\}\,,$$

hence

$$\bigcup_{n \ge 0} \beta^n\, T_\beta^{-n}(0) = \{z_k \mid k \ge 0\}\,.$$

Since $u_k \in \Delta_\beta$ for all $k \ge 1$ and $u_{|\varphi^n(1)|} = T_\beta^n(1^-)$ for all $n \ge 0$, we have

$$\{z_k - z_{k-1} \mid k \ge 1\} = \{u_k \mid k \ge 1\} = \Delta_\beta\,.$$ □

For the sets $S_\beta(x)$, Lemma 1 gives the following corollary.

Corollary 1. *For any $x \in [0,1)$, we have*

$$S_\beta(x) = \{z_k + x \mid k \ge 0,\, u_{k+1} > x\} \subseteq x + S_\beta(0)\,.$$

Note that $S_\beta(x)$ is always the union of a sequence of nested sets because $y \in [0,1)$ implies $y/\beta \in [0,1)$ and $T_\beta(y/\beta) = y$, thus $\beta^n\, T_\beta^{-n}(x) \subseteq \beta^{n+1}\, T_\beta^{-n-1}(x)$ for all $x \in [0,1)$.

3. $(-\beta)$-integers

We now turn to the discussion of $(-\beta)$-integers and sets $S_{-\beta}(x)$, $x \in \big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$. The first technical problem comes from the fact $(-\beta)^n\, T_{-\beta}^{-n}(x) \subseteq (-\beta)^{n+1}\, T_{-\beta}^{-n-1}(x)$ is not always true because $\frac{-y}{\beta} \notin \big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$ for $y = \frac{-\beta}{\beta+1}$. However, we have the following lemma.

Lemma 2. *For any $\beta > 1$, $x \in \big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$, $n \geq 0$, we have*

$$T_{-\beta}^{-n}(x) \setminus \big\{\tfrac{-\beta}{\beta+1}\big\} \subseteq (-\beta)\, T_{-\beta}^{-n-1}(x)\,.$$

If $T_{-\beta}(x) = x$, in particular if $x = 0$, then

$$T_{-\beta}^{-n}(x) \subseteq \beta^2\, T_{-\beta}^{-n-2}(x)\,.$$

Proof. For any $y \in \big(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$, we have $T_{-\beta}^2(y/\beta^2) = T_{-\beta}(-y/\beta) = y$, which implies the first inclusion and $T_{-\beta}^{-n}(x) \setminus \big\{\frac{-\beta}{\beta+1}\big\} \subseteq \beta^2\, T_{-\beta}^{-n-2}(x)$. If $T_{-\beta}^n\big(\frac{-\beta}{\beta+1}\big) = x = T_{-\beta}(x)$, then

$$T_{-\beta}^{n+2}\big(\tfrac{-\beta^{-1}}{\beta+1}\big) = T_{-\beta}^{n+1}\big(\tfrac{-\beta}{\beta+1}\big) = T_{-\beta}(x) = x\,,$$

thus $\frac{-\beta}{\beta+1} \in \beta^2\, T_{-\beta}^{-n-2}(x)$ as well. □

The first two statements of the following proposition can also be found in [1].

Proposition 1. *For any $\beta > 1$, we have $\mathbb{Z}_{-\beta} \subseteq (-\beta)\,\mathbb{Z}_{-\beta}$.*

If $\beta < (1+\sqrt{5})/2$, then $\mathbb{Z}_{-\beta} = \{0\}$.

If $\beta \geq (1+\sqrt{5})/2$, then

$$\mathbb{Z}_{-\beta} \cap (-\beta)^n\, [-\beta, 1] = \big\{(-\beta)^n, (-\beta)^{n+1}\big\} \cup (-\beta)^{n+2}\, \big(T_{-\beta}^{-n-2}(0) \cap \big(\tfrac{-1}{\beta}, \tfrac{1}{\beta^2}\big)\big)$$

for all $n \geq 0$, in particular

$$\mathbb{Z}_{-\beta} \cap [-\beta, 1] = \begin{cases} \{-\beta, -\beta+1, \ldots, -\beta+\lfloor\beta\rfloor, 0, 1\} & \text{if } \beta^2 \geq \lfloor\beta\rfloor(\beta+1), \\ \{-\beta, -\beta+1, \ldots, -\beta+\lfloor\beta\rfloor-1, 0, 1\} & \text{if } \beta^2 < \lfloor\beta\rfloor(\beta+1). \end{cases}$$

Proof. For any $\beta > 1$, we have $T_{-\beta}(0) = 0$, thus $T_{-\beta}^{-n-1}(0) \subseteq T_{-\beta}^{-n}(0)$. This means that $(-\beta)^{n+1}\, T_{-\beta}^{-n-1}(0) \subseteq -\beta\big((-\beta)^n\, T_{-\beta}^{-n}(0)\big)$, hence $\mathbb{Z}_{-\beta} \subseteq (-\beta)\,\mathbb{Z}_{-\beta}$.

If $\beta < \frac{1+\sqrt{5}}{2}$, then $\frac{-1}{\beta} < \frac{-\beta}{\beta+1}$, hence $T_{-\beta}^{-1}(0) = \{0\}$ and $\mathbb{Z}_{-\beta} = \{0\}$, see Figure 1 (right).

If $\beta \geq \frac{1+\sqrt{5}}{2}$, then $\frac{-1}{\beta} \in T_{-\beta}^{-1}(0)$ implies $1 \in \mathbb{Z}_{-\beta}$, thus $(-\beta)^n \in \mathbb{Z}_{-\beta}$ for all $n \geq 0$. Clearly,

$$(-\beta)^{n+2}\, \big(T_{-\beta}^{-n-2}(0) \cap \big(\tfrac{-1}{\beta}, \tfrac{1}{\beta^2}\big)\big) \subseteq \mathbb{Z}_{-\beta} \cap (-\beta)^n\, (-\beta, 1)\,.$$

To show the other inclusion, let $z \in (-\beta)^m\, T_{-\beta}^{-m}(0) \cap (-\beta)^n\, (-\beta, 1)$ for some $m \geq 0$. If $m < n+2$, then Lemma 2 yields that $z \in (-\beta)^{n+2}\, T_{-\beta}^{-n-2}(0)$, since $(-1)^n\, z < \beta^n \leq \frac{\beta^{n+2}}{\beta+1}$ implies $z \neq \frac{(-\beta)^{n+2}}{\beta+1}$. If $m > n+2$, then $\big(\frac{-1}{\beta}, \frac{1}{\beta^2}\big) \subseteq \big(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$ implies that $T_{-\beta}^{n+2}\big(\frac{z}{(-\beta)^{n+2}}\big) = T_{-\beta}^m\big(\frac{z}{(-\beta)^m}\big) = 0$, thus we also have $z \in (-\beta)^{n+2}\, \big(T_{-\beta}^{-n-2}(0) \cap (-\beta)^n\, (-\beta, 1)\big)$.

Consider now $n = 0$, then

$$\mathbb{Z}_{-\beta} \cap [-\beta, 1] = \{-\beta, 1\} \cup \{z \in (-\beta, 1) \mid T_{-\beta}^2(z/\beta^2) = 0\}\,.$$

Since $\frac{-\lfloor\beta\rfloor}{\beta} \geq \frac{-\beta}{\beta+1}$ if and only if $\beta^2 \geq \lfloor\beta\rfloor(\beta+1)$, we obtain

$$(-\beta)\, T_{-\beta}^{-1}(0) = \begin{cases} \{0,1,\ldots,\lfloor\beta\rfloor\} & \text{if } \beta^2 \geq \lfloor\beta\rfloor(\beta+1), \\ \{0,1,\ldots,\lfloor\beta\rfloor-1\} & \text{if } \beta^2 < \lfloor\beta\rfloor(\beta+1). \end{cases}$$

If $T_{-\beta}^2(z/\beta^2) = 0$, then $z = -a_1\beta + a_0$ with $a_0 \in (-\beta)\, T_{-\beta}^{-1}(0)$, $a_1 \in \{0,1,\ldots,\lfloor\beta\rfloor\}$, and $\mathbb{Z}_{-\beta} \cap [-\beta, 1]$ consists of those numbers $-a_1\beta + a_0$ lying in $[-\beta, 1]$. □

This shows in particular that the maximal difference between consecutive $(-\beta)$-integers exceeds 1 whenever $\beta^2 < \lfloor\beta\rfloor(\beta+1)$, i.e., $T_{-\beta}\big(\frac{-\beta}{\beta+1}\big) < 0$. This was already shown for an example in [1]. Example 3 shows that the distance between two consecutive $(-\beta)$-integers can be even greater than 2, and the structure of $\mathbb{Z}_{-\beta}$ can be quite complicated. Therefore, we adapt a slightly different strategy as for $\mathbb{Z}_\beta$.

In the following, we always assume that the set

$$V_\beta = V'_\beta \cup \{0\} = \big\{T_{-\beta}^n\big(\tfrac{-\beta}{\beta+1}\big) \mid n \geq 0\big\} \cup \{0\}$$

is finite, i.e., β is a $(-\beta)$-number, and let β be fixed. For $x \in V_\beta$, let

$$r_x = \min\big\{y \in V_\beta \cup \big\{\tfrac{1}{\beta+1}\big\} \mid y > x\big\}, \quad \widehat{x} = \tfrac{x+r_x}{2}, \quad J_x = \{x\} \quad \text{and} \quad J_{\widehat{x}} = (x, r_x)\,.$$

Then $\{J_a \mid a \in A_\beta\}$ forms a partition of $\big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$, where

$$A_\beta = V_\beta \cup \widehat{V}_\beta\,, \quad \text{with} \quad \widehat{V}_\beta = \{\widehat{x} \mid x \in V_\beta\}\,.$$

We have $T_{-\beta}(J_x) = J_{T_{-\beta}(x)}$ for every $x \in V_\beta$, and the following lemma shows that the image of every $J_{\widehat{x}}$, $x \in V_\beta$, is a union of intervals J_a, $a \in A_\beta$.

Lemma 3. *Let $x \in V_\beta$ and*

$$J_{\widehat{x}} \cap T_{-\beta}^{-1}(V_\beta) = \{y_1, \ldots, y_m\}\,, \quad x = y_0 < y_1 < \cdots < y_m < y_{m+1} = r_x\,.$$

For any $0 \leq i \leq m$, we have

$$T_{-\beta}\big((y_i, y_{i+1})\big) = J_{\widehat{x_i}} \quad \text{with} \quad x_i = \lim\nolimits_{y\to y_{i+1},\, y<y_{i+1}} T_{-\beta}(y)\,, \ \text{i.e.,} \ \widehat{x_i} = T_{-\beta}\big(\tfrac{y_i+y_{i+1}}{2}\big)\,,$$

and $\beta(y_{i+1} - y_i) = \lambda(J_{\widehat{x_i}})$, where λ denotes the Lebesgue measure.

Proof. Since $T_{-\beta}$ maps no point in (y_i, y_{i+1}) to $\frac{-\beta}{\beta+1} \in V_\beta$, the map is continuous on this interval and $\lambda(T_{-\beta}((y_i, y_{i+1}))) = \beta(y_{i+1} - y_i)$. We have $x_i \in V_\beta$ since $x_i = T_{-\beta}(y_{i+1})$ in case $y_{i+1} < \frac{1}{\beta+1}$, and $x_i = \frac{-\beta}{\beta+1}$ in case $y_{i+1} = \frac{1}{\beta+1}$. Since $y_i = \max\{y \in T_{-\beta}^{-1}(V_\beta) \mid y < y_{i+1}\}$, we obtain that $r_{x_i} = \lim_{y\to y_i,\, y>y_i} T_{-\beta}(y)$, thus $T_{-\beta}((y_i, y_{i+1})) = (x_i, r_{x_i})$. □

Therefore, we define an anti-morphism $\psi_\beta:\, A_\beta^* \to A_\beta^*$ by

$$\psi_\beta(x) = T_{-\beta}(x) \quad \text{and} \quad \psi_\beta(\widehat{x}) = \widehat{x_m}\, T_{-\beta}(y_m) \cdots \widehat{x_1}\, T_{-\beta}(y_1)\, \widehat{x_0} \qquad (x \in V_\beta),$$

with m, x_i and y_i as in Lemma 3. Here, anti-morphism means that $\psi_\beta(uv) = \psi_\beta(v)\psi_\beta(u)$ for all $u, v \in A_\beta^*$. Now, the last letter of $\psi_\beta(\widehat{0})$ is $\widehat{t}$, $t = \max\{x \in V_\beta \mid x < 0\}$, and the first letter of $\psi_\beta(\widehat{t}\,)$ is $\widehat{0}$. Therefore, $\psi_\beta^{2n}(\widehat{0})$ is a prefix of $\psi_\beta^{2n+2}(\widehat{0}) = \psi_\beta^{2n}(\psi_\beta^2(\widehat{0}))$ and $\psi_\beta^{2n+1}(\widehat{0})$ is a suffix of $\psi_\beta^{2n+3}(\widehat{0})$ for every $n \geq 0$.

Theorem 2. *For any $(-\beta)$-number $\beta \geq (1+\sqrt{5})/2$, we have*

$$\mathbb{Z}_{-\beta} = \{z_k \mid k \in \mathbb{Z},\, u_k = 0\} \quad \text{with} \quad z_k = \begin{cases} \sum_{j=1}^{k} \lambda(J_{u_j}) & \text{if } k \geq 0\,, \\ -\sum_{j=1}^{|k|} \lambda(J_{u_{-j}}) & \text{if } k \leq 0\,, \end{cases}$$

where $\cdots u_{-1}u_0u_1\cdots$ is the two-sided infinite word on the finite alphabet A_β such that $u_0 = 0$, $\psi_\beta^{2n}(\widehat{0})$ is a prefix of $u_1u_2\cdots$ and $\psi_\beta^{2n+1}(\widehat{0})$ is a suffix of $\cdots u_{-2}u_{-1}$ for all $n \geq 0$.

Since $\psi_\beta(u_0) = u_0$, the word $\cdots u_{-1}u_0u_1\cdots$ can be seen as a fixed point of ψ_β.

The following lemma is the analogue of Lemma 1. Note that $u_{2k} \in V_\beta$ and $u_{2k+1} \in \widehat{V}_\beta$ for all $k \in \mathbb{Z}$, thus $z_{2k} = z_{2k-1}$ and $z_{-2k} = z_{1-2k}$ for all $k \geq 1$. We use the notation

$$L(v) = \sum_{j=1}^{k} \lambda(J_{v_j}) \quad \text{if } v = v_1\cdots v_k \in A_\beta^k\,.$$

Lemma 4. *For any $n \geq 0$, $0 \leq k < |\psi_\beta^n(\widehat{0})|/2$, we have*

$$T_{-\beta}^n\Big(\frac{z_{(-1)^n 2k}}{(-\beta)^n}\Big) = u_{(-1)^n 2k}\,, \quad T_{-\beta}^n\Big(\Big(\frac{z_{(-1)^n 2k}}{(-\beta)^n}, \frac{z_{(-1)^n(2k+1)}}{(-\beta)^n}\Big)\Big) = J_{u_{(-1)^n(2k+1)}}\,,$$

and $z_{(-1)^n|\psi_\beta^n(\widehat{0})|} = (-1)^n\, L\big(\psi_\beta^n(\widehat{0})\big) = r_0\,(-\beta)^n$.

Proof. The statements are true for $n = 0$ since $|\psi_\beta^0(\widehat{0})| = 1$, $z_0 = 0$, $z_1 = \lambda(J_{\widehat{0}}) = r_0$.

Suppose that they hold for even n, and consider

$$u_{-|\psi_\beta^{n+1}(\widehat{0})|}\cdots u_{-2}u_{-1} = \psi_\beta^{n+1}(\widehat{0}) = \psi_\beta\big(\psi_\beta^n(\widehat{0})\big) = \psi_\beta(u_{|\psi_\beta^n(\widehat{0})|})\cdots\psi_\beta(u_2)\psi_\beta(u_1)\,.$$

Let $0 \leq k < |\psi_\beta^{n+1}(\widehat{0})|/2$, and write

$$u_{-2k-1}\cdots u_{-1} = v_{-2i-1}\cdots v_{-1}\,\psi_\beta(u_1\cdots u_{2j})$$

with $0 \leq j < |\psi_\beta^n(\widehat{0})|/2$, $0 \leq i < |\psi_\beta(u_{2j+1})|/2$, i.e., $u_{-2i-1}\cdots u_{-1}$ is a suffix of $\psi_\beta(u_{2j+1})$.

By Lemma 3, we have $L(\psi_\beta(\widehat{x})) = \beta\,\lambda(J_{\widehat{x}})$ for any $x \in V_\beta$. This yields that

$$-z_{-2k-1} = \beta\, L(u_1\cdots u_{2j}) + L(v_{-2i-1}\cdots v_{-1}) = \beta\, z_{2j} + L(v_{-2i-1}\cdots v_{-1})$$

and $-z_{-2k} = \beta\, z_{2j} + L(v_{-2i}\cdots v_{-1})$. With $\widehat{x} = u_{2j+1}$ and y_i as in Lemma 3, we obtain

$$\begin{aligned} T_{-\beta}^{n+1}\Big(\frac{z_{-2k}}{(-\beta)^{n+1}}\Big) &= T_{-\beta}^{n+1}\Big(\frac{z_{2j}}{(-\beta)^n} - \frac{L(v_{-2i}\cdots v_{-1})}{(-\beta)^{n+1}}\Big) \\ &= \begin{cases} T_{-\beta}(u_{2j}) = \psi_\beta(u_{2j}) = u_{-2k} & \text{if } i = 0, \\ T_{-\beta}\big(x + L(v_{-2i}\cdots v_{-1})/\beta\big) = T_{-\beta}(y_i) = v_{-2i} = u_{-2k} & \text{if } i \geq 1, \end{cases} \end{aligned}$$

and

$$T_{-\beta}^{n+1}\Big(\Big(\frac{z_{-2k}}{(-\beta)^{n+1}}, \frac{z_{-2k-1}}{(-\beta)^{n+1}}\Big)\Big) = T_{-\beta}\big((y_i, y_{i+1})\big) = J_{v_{-2i-1}} = J_{u_{-2k-1}}\,.$$

Moreover, we have $L(\psi_\beta^{n+1}(\widehat{0})) = \beta\, L(\psi_\beta^n(\widehat{0})) = r_0\beta^{n+1}$, thus the statements hold for $n+1$.

The proof for odd n runs along the same lines and is therefore omitted. □

Proof of Theorem 2. By Lemma 4, we have $z_{(-1)^n|\psi_\beta^n(\widehat{0})|} = r_0\,(-\beta)^n$ for all $n \ge 0$, thus $[0, r_0)$ splits into the intervals $\big(z_{(-1)^n 2k}(-\beta)^{-n}, z_{(-1)^n(2k+1)}(-\beta)^{-n}\big)$ and points $z_{(-1)^n 2k}(-\beta)^{-n}$, $0 \le k < |\psi_\beta^n(\widehat{0})|/2$, hence

$$T_{-\beta}^{-n}(0) \cap [0, r_0) = \big\{z_{(-1)^n 2k}(-\beta)^{-n} \mid 0 \le k < |\psi_\beta^n(\widehat{0})|/2,\ u_{(-1)^n 2k} = 0\big\}\,.$$

Let $m \ge 1$ be such that $\beta^{2m} r_0 > \frac{1}{\beta+1}$. Then we have $\big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big) \subset (-\beta^{2m+1} r_0, \beta^{2m} r_0)$, and Lemma 2 yields that

$$T_{-\beta}^{-n}(0) \subseteq \beta^{2m}\big(T_{-\beta}^{-n-2m}(0) \cap [0, r_0)\big) \cup (-\beta)^{2m+1}\big(T_{-\beta}^{-n-2m-1}(0) \cap [0, r_0)\big)\,,$$

thus

$$\bigcup_{n\ge0}(-\beta)^n\, T_{-\beta}^{-n}(0) = \bigcup_{n\ge0}(-\beta)^n\, \big(T_{-\beta}^{-n}(0) \cap [0, r_0)\big) = \{z_{2k} \mid k \in \mathbb{Z},\ u_{2k} = 0\}\,. \qquad \square$$

As in the case of positive bases, the word $\cdots u_{-1}u_0u_1\cdots$ also describes the sets

$$S_{-\beta}(x) = \lim_{n\to\infty}(-\beta)^n\, T_{-\beta}^{-n}(x) = \bigcup_{n\ge0}(-\beta)^n\, \big(T_{-\beta}^{-n}(x) \setminus \big\{\tfrac{-\beta}{\beta+1}\big\}\big) \qquad \big(x \in \big[\tfrac{-\beta}{\beta+1}, \tfrac{1}{\beta+1}\big)\big),$$

where the second equality follows from Lemma 2. It is already indicated in the Introduction that $S_{-\beta}(x)$ can differ from $\bigcup_{n\ge0}(-\beta)^n\, T_{-\beta}^{-n}(x)$. Indeed, if $T_{-\beta}^n\big(\frac{-\beta}{\beta+1}\big) = x \ne T_{-\beta}(x)$, then

$$T_{-\beta}^{n+2}\big(\tfrac{-\beta^{-1}}{\beta+1}\big) = T_{-\beta}^{n+1}\big(\tfrac{-\beta}{\beta+1}\big) = T_{-\beta}(x) \ne x\,,$$

thus $\frac{(-\beta)^{n+1}}{\beta+1} \in (-\beta)^m\, T_{-\beta}^{-m}(x)$ if and only if $m = n$.

Theorem 2 and Lemma 4 give the following corollary.

Corollary 2. *For any $x \in V_\beta$, we have*

$$S_{-\beta}(x) = \{z_k \mid k \in \mathbb{Z},\ u_k = x\}$$

and, for any $y \in J_{\widehat{x}}$,

$$S_{-\beta}(y) = \{z_k + y - x \mid k \ge 0,\ u_{k+1} = \widehat{x}\} \cup \{z_k + y - x \mid k < 0,\ u_k = \widehat{x}\}\,.$$

Recall that our main goal is to understand the structure of $\mathbb{Z}_{-\beta}$ (in case $\beta \ge (1+\sqrt{5})/2$), i.e., to describe the occurrences of 0 in the word $\cdots u_{-1}u_0u_1\cdots$ defined in Theorem 2 and the words between two successive occurrences. Let

$$R_\beta = \{u_k u_{k+1}\cdots u_{s(k)-1} \mid k \in \mathbb{Z},\ u_k = 0\} \quad \text{with} \quad s(k) = \min\{j \in \mathbb{Z} \mid u_j = 0,\ j > k\}$$

be the set of return words of 0 in $\cdots u_{-1}u_0u_1\cdots$. Note that $s(k)$ is defined for all $k \in \mathbb{Z}$ since $(-\beta)^n \in \mathbb{Z}_{-\beta}$ for all $n \ge 0$ by Proposition 1.

For any $w \in R_\beta$, $\psi_\beta(w0)$ is a factor of $\cdots u_{-1}u_0u_1\cdots$ starting and ending with 0, thus we can write $\psi_\beta(w0) = w_1\cdots w_m0$ with $w_j \in R_\beta$, $1 \le j \le m$, and set

$$\varphi_{-\beta}(w) = w_1\cdots w_m\,.$$

This defines an anti-morphism $\varphi_{-\beta}:\, R_\beta^* \to R_\beta^*$, which plays the role of the β-substitution.

Since $\cdots u_{-1}u_0u_1\cdots$ is generated by $u_1=\widehat{0}$, we consider $w_\beta = u_0u_1\cdots u_{s(0)-1}$. We have

$$[0,1] = \big[0,\tfrac{1}{\beta+1}\big) \cup \big[\tfrac{1}{\beta+1},1\big]\,, \quad T_{-\beta}\big((-\beta)^{-1}\big[\tfrac{1}{\beta+1},1\big]\big) = \big[\tfrac{-\beta}{\beta+1},0\big]\,,$$

thus $L(w_\beta)=1$ and

$$w_\beta = 0\,\widehat{0}\,x_1\,\widehat{x_1}\,\cdots\,x_m\,\widehat{x_m}\,x_{-\ell}\,\widehat{x_{-\ell}}\,\cdots x_{-1}\,\widehat{x_{-1}}\,,$$

with $V_\beta = \{x_{-\ell},\dots,x_{-1},0,x_1,\dots,x_m\}$, $x_{-\ell}<\cdots<x_{-1}<0<x_1<\cdots<x_m$.

Theorem 3. *For any $(-\beta)$-number $\beta\ge(1+\sqrt{5})/2$, we have*

$$\mathbb{Z}_{-\beta} = \{z'_k \mid k\in\mathbb{Z}\} \quad \textit{with} \quad z'_k = \begin{cases} \sum_{j=1}^{k} L(u'_j) & \textit{if } k\ge 0\,,\\ -\sum_{j=1}^{|k|} L(u'_{-j}) & \textit{if } k\le 0\,,\end{cases}$$

where $\cdots u'_{-2}u'_{-1}u'_1u'_2\cdots$ is the two-sided infinite word on the finite alphabet R_β such that $\varphi_{-\beta}^{2n}(w_\beta)$ is a prefix of $u'_1u'_2\cdots$ and $\varphi_{-\beta}^{2n+1}(w_\beta)$ is a suffix of $\cdots u'_{-2}u'_{-1}$ for all $n\ge 0$.

The set of distances between consecutive $(-\beta)$-integers is

$$\Delta_{-\beta} = \{z'_{k+1}-z'_k \mid k\in\mathbb{Z}\} = \{L(w)\mid w\in R_\beta\}\,.$$

Note that the index 0 is omitted in $\cdots u'_{-2}u'_{-1}u'_1u'_2\cdots$ for reasons of symmetry.

Proof. The definitions of $\cdots u_{-1}u_0u_1\cdots$ in Theorem 2 and of $\varphi_{-\beta}$ imply that $\cdots u'_{-2}u'_{-1}$ $u'_1u'_2\cdots$ is the derived word of $\cdots u_{-1}u_0u_1\cdots$ with respect to R_β, that is

$$u'_k = u_{|u'_1\cdots u'_{k-1}|}\cdots u_{|u'_1\cdots u'_k|-1}\,, \quad u'_{-k} = u_{-|u'_{-k}\cdots u'_{-1}|}\cdots u_{-|u'_{1-k}\cdots u'_{-1}|-1} \quad (k\ge 1)$$

with

$$\{|u'_1\cdots u'_{k-1}| \mid k\ge 1\} \cup \{-|u'_{-k}\cdots u'_{-1}| \mid k\ge 1\} = \{k\in\mathbb{Z}\mid u_k=0\}\,.$$

Here, for any $v\in R_\beta^*$, $|v|$ denotes the length of v as a word in A_β^*, not in R_β^*. Since

$$z'_k = \sum_{j=1}^{k} L(u'_j) = \sum_{j=0}^{|u'_1\cdots u'_k|-1}\lambda(J_{u_j}) = \sum_{j=1}^{|u'_1\cdots u'_k|}\lambda(J_{u_j})\,, \quad z'_{-k} = -\sum_{j=1}^{k} L(u'_{-j}) = -\sum_{j=1}^{|u'_{-k}\cdots u'_{-1}|}\lambda(J_{u_{-j}})$$

for all $k\ge 0$, Theorem 2 yields that $\{z'_k\mid k\in\mathbb{Z}\}=\mathbb{Z}_{-\beta}$.

It follows from the definition of R_β that $\Delta_{-\beta}=\{L(w)\mid w\in R_\beta\}$.

It remains to show that R_β is a finite set. We first show that the restriction of ψ_β to $\widehat{V_\beta}$ is primitive, which means that there exists some $m\ge 1$ such that, for every $x\in V_\beta$, $\psi_\beta^m(\widehat{x})$ contains all elements of $\widehat{V_\beta}$. The proof is taken from [11, Proposition 8]. If $\beta>2$, then the largest connected pieces of images of $J_{\widehat{x}}$ under $T_{-\beta}$ grow until they cover two consecutive discontinuity points $\frac{1}{\beta+1}-\frac{a+1}{\beta}$, $\frac{1}{\beta+1}-\frac{a}{\beta}$ of $T_{-\beta}$, and the next image covers all intervals $J_{\widehat{y}}$, $y\in V_\beta$. If $\frac{1+\sqrt{5}}{2}<\beta\le 2$, then $\beta^2>2$ implies that the largest connected pieces of images of $J_{\widehat{x}}$ under $T_{-\beta}^2$ grow until they cover two consecutive discontinuity points of $T_{-\beta}^2$. Since

$$T_{-\beta}^2\big(\big(\tfrac{-\beta}{\beta+1},\tfrac{\beta^{-2}}{\beta+1}-\tfrac{1}{\beta}\big)\big) = \big(\tfrac{-\beta^3+\beta^2+\beta}{\beta+1},\tfrac{1}{\beta+1}\big)\,, \quad T_{-\beta}^2\big(\big(\tfrac{\beta^{-2}}{\beta+1}-\tfrac{1}{\beta},\tfrac{-\beta^{-1}}{\beta+1}\big)\big) = \big(\tfrac{-\beta}{\beta+1},\tfrac{\beta^2-\beta-1}{\beta+1}\big)\,,$$
$$T_{-\beta}^2\big(\big(\tfrac{-\beta^{-1}}{\beta+1},\tfrac{\beta^{-2}}{\beta+1}\big)\big) = \big(\tfrac{-\beta}{\beta+1},\tfrac{1}{\beta+1}\big)\,, \quad T_{-\beta}^2\big(\big(\tfrac{\beta^{-2}}{\beta+1},\tfrac{1}{\beta+1}\big)\big) = \big(\tfrac{-\beta}{\beta+1},\tfrac{\beta^2-\beta-1}{\beta+1}\big)\,,$$

the next image covers the fixed point 0, and further images grow until after a finite number of steps they cover all intervals $J_{\widehat{y}}$, $y \in V_\beta$. The case $\beta = \frac{1+\sqrt{5}}{2}$ is treated in Example 1.

If $T_{-\beta}^n\big(\frac{-\beta}{\beta+1}\big) \neq 0$ for all $n \geq 0$, then $u_k = 0$ is equivalent with $u_{k+1} = \widehat{0}$, see Proposition 2 below, thus we can consider the return words of $\widehat{0}$ in $\cdots u_{-1}u_0u_1\cdots$ instead of the return words of 0. Since $\psi_\beta^m(\widehat{x_0}\,x_1\,\widehat{x_2})$ has at least two occurrences of $\widehat{0}$ for all $x_0, x_1, x_2 \in V_\beta$, there are only finitely many such return words, cf. [6]. If $T_{-\beta}^n\big(\frac{-\beta}{\beta+1}\big) = 0$, then $\psi_\beta^n(x_0\,\widehat{x_1}\,x_2)$ starts and ends with 0 for all $x_0, x_1, x_2 \in V_\beta$, hence R_β is finite as well. $\square$

We remark that, for practical reasons, the set R_β can be obtained from the set $R = \{w_\beta\}$ by adding to R iteratively all return words of 0 which appear in $\varphi_{-\beta}(w)$ for some $w \in R$ until R stabilises. The final set R is equal to R_β.

Now, we apply the theorems in the case of two quadratic examples.

Example 1. Let $\beta = \frac{1+\sqrt{5}}{2}$, i.e., $\beta^2 = \beta + 1$, and $t = \frac{-\beta}{\beta+1} = \frac{-1}{\beta}$. We have $V_\beta = \{t, 0\}$. Since

$$J_{\widehat{t}} = (t,0) = \big(t, \tfrac{-1}{\beta^3}\big) \cup \big\{\tfrac{-1}{\beta^3}\big\} \cup \big(\tfrac{-1}{\beta^3}, 0\big)\,, \quad J_{\widehat{t}} = \big(0, \tfrac{1}{\beta^2}\big)\,,$$

see Figure 1 (middle), the anti-morphism ψ_β on A_β^* is defined by

$$\psi_\beta: \quad t \mapsto 0\,, \quad \widehat{t} \mapsto \widehat{0}\,t\,\widehat{t}\,, \quad 0 \mapsto 0\,, \quad \widehat{0} \mapsto \widehat{t}\,.$$

Its two-sided fixed point $\cdots u_{-1}u_0u_1\cdots$ is

$$\cdots \underbrace{0}_{\psi_\beta(0)}\underbrace{\widehat{0}\,t\,\widehat{t}}_{\psi_\beta(\widehat{t})}\underbrace{0}_{\psi_\beta(t)}\underbrace{\widehat{t}}_{\psi_\beta(\widehat{0})}\underbrace{0}_{\psi_\beta(0)}\underbrace{\widehat{0}\,t\,\widehat{t}}_{\psi_\beta(\widehat{t})}\underbrace{0}_{\psi_\beta(t)}\underbrace{\widehat{t}}_{\psi_\beta(\widehat{0})}\underbrace{\dot{0}}_{\psi_\beta(\dot{0})}\underbrace{\widehat{0}\,t\,\widehat{t}}_{\psi_\beta(\widehat{t})}\underbrace{0}_{\psi_\beta(0)}\underbrace{\widehat{0}\,t\,\widehat{t}}_{\psi_\beta(\widehat{t})}\underbrace{0}_{\psi_\beta(t)}\underbrace{\widehat{t}}_{\psi_\beta(\widehat{0})}\underbrace{0}_{\psi_\beta(0)}\underbrace{\widehat{t}}_{\psi_\beta(\widehat{0})}\underbrace{0}_{\psi_\beta(0)}\cdots\,,$$

where $\dot{0}$ marks the central letter u_0. The ψ_β-images of the complete return words of 0 are

$$\psi_\beta: \quad 0\,\widehat{0}\,t\,\widehat{t}\,0 \mapsto 0\,\widehat{0}\,t\,\widehat{t}\,0\,\widehat{t}\,0\,, \quad 0\,\widehat{t}\,0 \mapsto 0\,\widehat{0}\,t\,\widehat{t}\,0\,,$$

thus $R_\beta = \{A, B\}$ with $A = 0\,\widehat{0}\,t\,\widehat{t}$, $B = 0\,\widehat{t}$. The anti-morphism

$$\varphi_{-\beta}: \quad A \mapsto AB\,, \quad B \mapsto A\,,$$

has the two-sided fixed point

$$\cdots u'_{-2}u'_{-1}\,\dot{}\,u'_1u'_2\cdots = \cdots A\,AB\,A\,AB\,A\,AB\,AB\,\dot{}\,A\,AB\,A\,AB\,AB\,A\,AB\,AB\,A\,AB\,B\,\cdots\,.$$

We have $\lambda(J_{\widehat{0}}) = \frac{1}{\beta^2}$, $\lambda(J_{\widehat{t}}) = \frac{1}{\beta}$, thus $L(A) = 1$, $L(B) = \frac{1}{\beta} = \beta - 1$, and some $(-\beta)$-integers are shown in Figure 2. Note that $(-\beta)^n$ can also be represented as $(-\beta)^{n+2} + (-\beta)^{n+1}$.

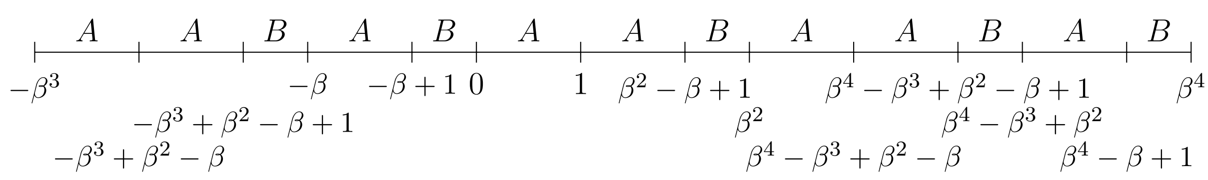

FIGURE 2. The $(-\beta)$-integers in $[-\beta^3, \beta^4]$, $\beta = (1+\sqrt{5})/2$.

Example 2. Let $\beta = \frac{3+\sqrt{5}}{2}$, i.e., $\beta^2 = 3\beta - 1$, then the $(-\beta)$-transformation is depicted in Figure 3, where $t_0 = \frac{-\beta}{\beta+1}$, $t_1 = T_{-\beta}(t_0) = \frac{\beta^2}{\beta+1} - 2 = \frac{-\beta^{-1}}{\beta+1}$, $T_{-\beta}(t_1) = \frac{1}{\beta+1} - 1 = t_0$. Therefore, $V_\beta = \{t_0, t_1, 0\}$ and the anti-morphism $\psi_\beta : A_\beta^* \to A_\beta^*$ is defined by

$$\psi_\beta : \quad t_0 \mapsto t_1\,, \quad \widehat{t_0} \mapsto \widehat{t_0}\, t_1 \widehat{t_1}\, 0\, \widehat{0}\, t_0\, \widehat{t_0}\,, \quad t_1 \mapsto t_0\,, \quad \widehat{t_1} \mapsto \widehat{0}\,, \quad 0 \mapsto 0\,, \quad \widehat{0} \mapsto \widehat{t_0}\, t_1\, \widehat{t_1}\,,$$

which has the two-sided fixed point

$$\cdots \underbrace{0}_{\psi_\beta(0)} \underbrace{\widehat{0}}_{\psi_\beta(\widehat{t_1})} \underbrace{t_0}_{\psi_\beta(t_1)} \underbrace{\widehat{t_0}\, t_1 \widehat{t_1}\, 0\, \widehat{0}\, t_0\, \widehat{t_0}}_{\psi_\beta(\widehat{t_0})} \underbrace{t_1}_{\psi_\beta(t_0)} \underbrace{\widehat{t_0}\, t_1\, \widehat{t_1}}_{\psi_\beta(\widehat{0})} \underbrace{\dot{0}}_{\psi_\beta(0)} \underbrace{\widehat{0}}_{\psi_\beta(\widehat{t_1})} \underbrace{t_0}_{\psi_\beta(t_1)} \underbrace{\widehat{t_0}\, t_1 \widehat{t_1}\, 0\, \widehat{0}\, t_0\, \widehat{t_0}}_{\psi_\beta(\widehat{t_0})} \cdots\,,$$

where $\dot{0}$ marks the central letter u_0. The ψ_β-images of the complete return words of 0 are

$$\begin{aligned}
\psi_\beta : \quad & 0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_1}\, 0 \mapsto 0\ \widehat{0}\ t_0\ \widehat{t_0}\, t_1\, \widehat{t_1}\, 0\, \widehat{0}\, t_0\, \widehat{t_0}\ t_1\ \widehat{t_0}\, t_1\, \widehat{t_1}\ 0\,,\\
& 0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_0}\, t_1\, \widehat{t_1}\, 0 \mapsto 0\ \widehat{0}\ t_0\ \widehat{t_0}\, t_1\, \widehat{t_1}\, 0\, \widehat{0}\, t_0\, \widehat{t_0}\ t_0\ \widehat{t_0}\, t_1\, \widehat{t_1}\, 0\, \widehat{0}\, t_0\, \widehat{t_0}\ t_1\ \widehat{t_0}\, t_1\, \widehat{t_1}\ 0\,,\\
& 0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_1}\, 0 \mapsto 0\ \widehat{0}\ t_0\ \widehat{t_0}\, t_1\, \widehat{t_1}\, 0\, \widehat{0}\, t_0\, \widehat{t_0}\ t_1\ \widehat{t_0}\, t_1\, \widehat{t_1}\, 0\, \widehat{0}\, t_0\, \widehat{t_0}\ t_1\ \widehat{t_0}\, t_1\, \widehat{t_1}\ 0\,.
\end{aligned}$$

Note that $0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_0}\, t_1\, \widehat{t_1}$ and $0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_1}$ differ only by a letter in V_β, and correspond therefore to intervals of same length. Since the letters t_0 and t_1 are never mapped to 0, we identify these two return words. Then we have $R_\beta = \{A, B\}$ with $A = 0\,\widehat{0}\, t_0\, \widehat{t_0}\, t_1\, \widehat{t_1}$, $B = 0\,\widehat{0}\, t_0\, \widehat{t_0}\, \{t_0, t_1\}\, \widehat{t_0}\, t_1\, \widehat{t_1}$. The anti-morphism

$$\varphi_{-\beta} : \quad A \mapsto AB\,, \quad B \mapsto ABB\,,$$

has the two-sided fixed point

$$\cdots ABB\, AB\, ABB\, ABB\, AB\, \dot{}\, ABB\, AB\, ABB\, ABB\, AB\, \cdots\,.$$

We have $L(A) = 1$, $L(B) = \beta - 1 > 1$, and some $(-\beta)$-integers are shown in Figure 3.

We remark that it is in general sufficient to consider the elements of $\widehat{V}_\beta$ when one is only interested in $\mathbb{Z}_{-\beta}$. This is made precise in the following proposition.

Proposition 2. *Let β and $\cdots u_{-1} u_0 u_1 \cdots$ be as in Theorem 2, $t = \max\{x \in V_\beta \mid x < 0\}$.*

If $0 \notin V'_\beta$ or the size of V'_β is odd, then $u_k = 0$ is equivalent with $u_{k+1} = \widehat{0}$ for all $k \in \mathbb{Z}$.

If $0 \notin V'_\beta$ or the size of V'_β is even, then $u_k = 0$ is equivalent with $u_{k-1} = \widehat{t}$ for all $k \in \mathbb{Z}$.

Proof. Let $k \in \mathbb{Z}$ and $m \geq 0$ such that $z_{2k}/\beta^{2m} \in \big(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$. Then we have

- $u_{2k} = 0$ if and only if $T_{-\beta}^{2m}(z_{2k}/\beta^{2m}) = 0$,
- $u_{2k+1} = \widehat{0}$ if and only if $\lim_{y \to z_{2k},\, y > z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$,
- $u_{2k-1} = \widehat{t}$ if and only if $\lim_{y \to z_{2k},\, y < z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m}) = 0$.

Recall that $u_{2k} \in V_\beta$ and $u_{2k+1} \in \widehat{V}_\beta$ for all $k \in \mathbb{Z}$. If z_{2k}/β^{2m} is a point of discontinuity of $T_{-\beta}^{2m}$, then we must have $T_{-\beta}^{\ell}(z_{2k}/\beta^{2m}) = \frac{-\beta}{\beta+1}$ for some $1 \leq \ell \leq 2m$.

If $0 \notin V'_\beta = \big\{T_{-\beta}^n\big(\frac{-\beta}{\beta+1}\big) \mid n \geq 0\big\}$, then $T_{-\beta}^{\ell}(z_{2k}/\beta^{2m}) = \frac{-\beta}{\beta+1}$ is not possible when $T_{-\beta}^{2m}(z_{2k}/\beta^{2m}) = 0$, thus $u_{2k-1} = \widehat{t}$, $u_{2k} = 0$ and $u_{2k+1} = \widehat{0}$ are equivalent.

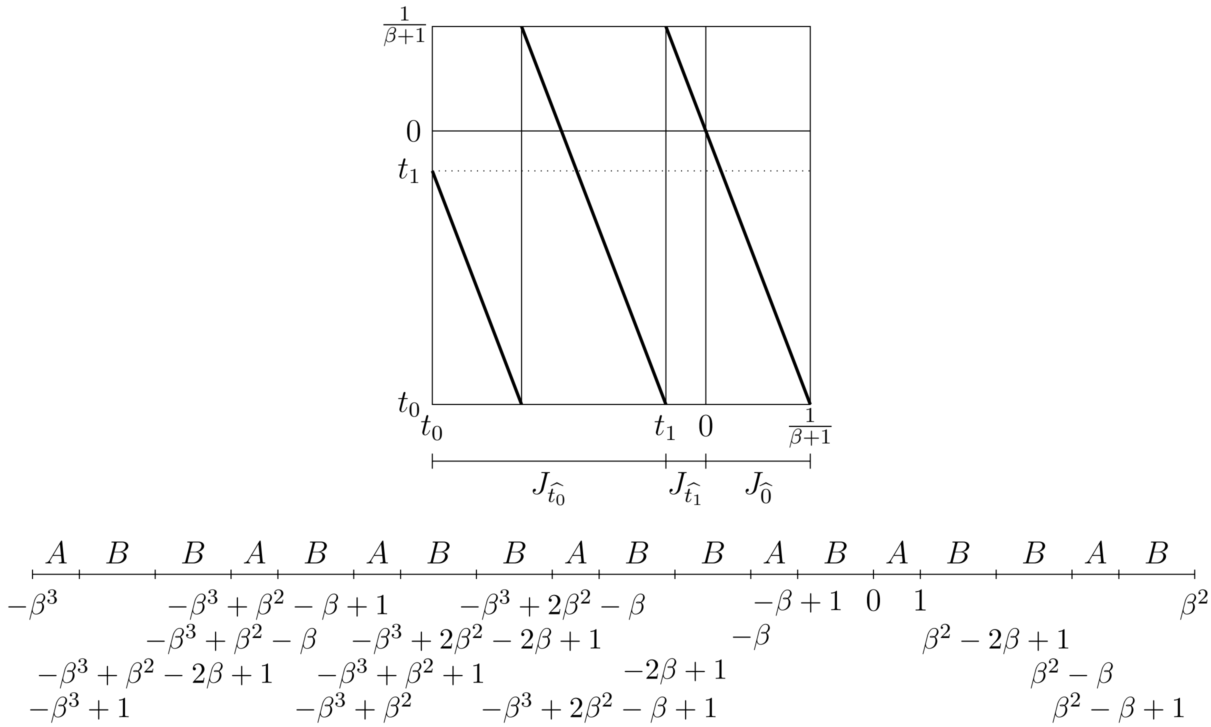

FIGURE 3. The $(-\beta)$-transformation and $\mathbb{Z}_{-\beta}\cap[-\beta^3,\beta^2]$, $\beta=(3+\sqrt{5})/2$.

Let now $T_{-\beta}^{\ell}(z_{2k}/\beta^{2m})=\frac{-\beta}{\beta+1}$ and $T_{-\beta}^{2m}(z_{2k}/\beta^{2m})=0$, thus $0\in V_\beta'$. Then the size of V_β' is the minimal $n\ge 2$ such that $T_{-\beta}^{n-1}\big(\frac{-\beta}{\beta+1}\big)=0$. Moreover, $T_{-\beta}^{j}(z_{2k}/\beta^{2m})\ne\frac{-\beta}{\beta+1}$ for all $j\ne\ell$.

If ℓ is even, then $\lim_{y\to z_{2k},\,y>z_{2k}} T_{-\beta}^{\ell}(y/\beta^{2m})=\frac{-\beta}{\beta+1}$, thus $\lim_{y\to z_{2k},\,y>z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m})=0$. From below, $\lim_{y\to z_{2k},\,y<z_{2k}} T_{-\beta}^{\ell}(y/\beta^{2m})=\frac{1}{\beta+1}$ and $\lim_{y\to z_{2k},\,y<z_{2k}} T_{-\beta}^{\ell+1}(y/\beta^{2m})=\frac{-\beta}{\beta+1}$, thus $\lim_{y\to z_{2k},\,y<z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m})=T_{-\beta}^{2m-\ell-1}\big(\frac{-\beta}{\beta+1}\big)$. By the definition of n, we have $2m-\ell\ge n-1$. If n is even, then we also have $2m-\ell-1\ge n-1$, thus $\lim_{y\to z_{2k},\,y<z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m})=0$.

If ℓ is odd, then the roles of $y>z_{2k}$ and $y<z_{2k}$ change, thus $\lim_{y\to z_{2k},\,y<z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m})=0$, $\lim_{y\to z_{2k},\,y>z_{2k}} T_{-\beta}^{\ell+1}(y/\beta^{2m})=\frac{-\beta}{\beta+1}$. Now, $\lim_{y\to z_{2k},\,y>z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m})=0$ if n is odd.

Therefore, $T_{-\beta}^{2m}(z_{2k}/\beta^{2m})=0$ is equivalent with $\lim_{y\to z_{2k},\,y>z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m})=0$ if the size of V_β' is odd, and $T_{-\beta}^{2m}(z_{2k}/\beta^{2m})=0$ is equivalent with $\lim_{y\to z_{2k},\,y<z_{2k}} T_{-\beta}^{2m}(y/\beta^{2m})=0$ if the size of V_β' is even. □

By Proposition 2, it suffices to consider the anti-morphism $\widehat{\psi}_\beta:\widehat{V}_\beta^*\to\widehat{V}_\beta^*$ defined by

$$\widehat{\psi}_\beta(\widehat{x})=\widehat{x_m}\cdots\widehat{x_1}\,\widehat{x_0}\quad\text{when}\quad \widehat{\psi}_\beta(\widehat{x})=\widehat{x_m}\,T_{-\beta}(y_m)\cdots\widehat{x_1}\,T_{-\beta}(y_1)\,\widehat{x_0}\quad(x\in V_\beta).$$

Then, $\Delta_{-\beta}$ is given by the set $\widehat{R}_\beta$ which consists of the return words of $\widehat{0}$ when $0\notin V_\beta'$ or the size of V_β' is odd. When $0\in V_\beta'$ and the size of V_β' is even, then $\widehat{R}_\beta$ consists of the words $w\,\widehat{t}$ such that $\widehat{t}\,w$ is a return word of $\widehat{t}$.

Example 3. Let $\beta > 1$ with $\beta^6 = 3\beta^5 + 2\beta^4 + 2\beta^3 + \beta^2 - 2\beta - 1$, i.e., $\beta \approx 3.695$, then the $(-\beta)$-transformation is depicted in Figure 4, where $t_n = T^n_{-\beta}\big(\frac{-\beta}{\beta+1}\big)$. We have $t_5 = \frac{-1}{\beta+1} = t_6$.

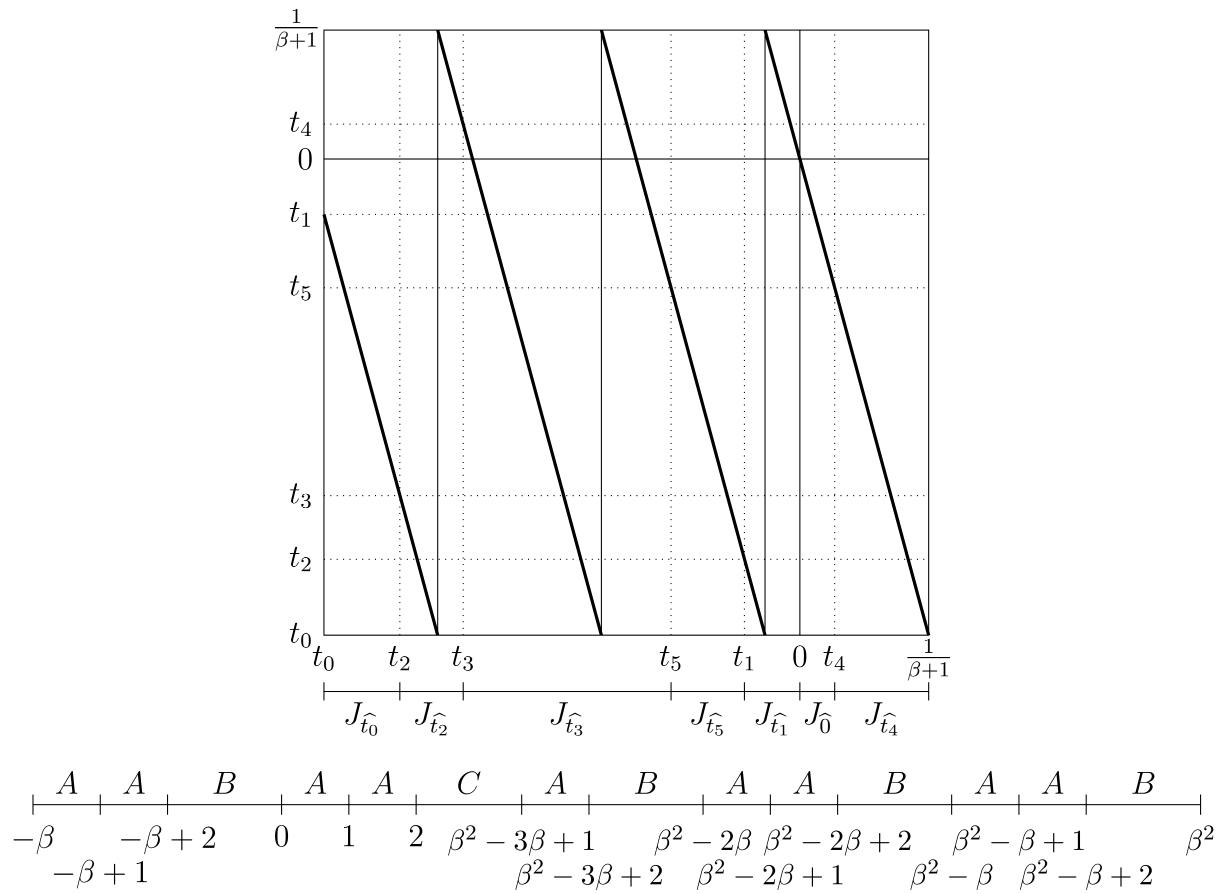

FIGURE 4. The $(-\beta)$-transformation and $\mathbb{Z}_{-\beta} \cap [-\beta, \beta^2]$ from Example 3.

The anti-morphism $\widehat{\psi}_\beta : \widehat{V}^*_\beta \to \widehat{V}^*_\beta$ is therefore defined by

$$\begin{array}{lllll} \widehat{\psi}_\beta : & \widehat{t_0} \mapsto \widehat{t_3}\,\widehat{t_5}\,, & \widehat{t_2} \mapsto \widehat{t_4}\,\widehat{t_0}\,\widehat{t_2}\,, & \widehat{t_3} \mapsto \widehat{t_5}\,\widehat{t_1}\,\widehat{0}\,\widehat{t_4}\,\widehat{t_0}\,\widehat{t_2}\,\widehat{t_3}\,\widehat{t_5}\,\widehat{t_1}\,\widehat{0}\,, & \\ & \widehat{t_5} \mapsto \widehat{t_2}\,\widehat{t_3}\,, & \widehat{t_1} \mapsto \widehat{0}\,\widehat{t_4}\,\widehat{t_0}\,, & \widehat{0} \mapsto \widehat{t_5}\,\widehat{t_1}\,, & \widehat{t_4} \mapsto \widehat{t_0}\,\widehat{t_2}\,\widehat{t_3}\,. \end{array}$$

It is convenient to group to letters forming the words

$$\begin{array}{llll} a = \widehat{0}\,\widehat{t_4}\,, & b = \widehat{t_0}\,\widehat{t_2}\,\widehat{t_3}\,\widehat{t_5}\,\widehat{t_1}\,, & c = \widehat{t_0}\,\widehat{t_2}\,\widehat{t_3}\,\widehat{t_5}\,, & d = \widehat{t_2}\,\widehat{t_3}\,\widehat{t_5}\,\widehat{t_1}\,, \\ e = \widehat{t_0}\,\widehat{t_2}\,, & f = \widehat{t_4}\,, & g = \widehat{t_0}\,\widehat{t_2}\,\widehat{t_3}\,, & h = \widehat{t_5}\,\widehat{t_1}\,, \end{array}$$

which correspond to the intervals

$$\begin{array}{llll} J_a = \big(0, \frac{1}{\beta+1}\big)\,, & J_b = (t_0, 0)\,, & J_c = (t_0, t_1)\,, & J_d = (t_2, 0)\,, \\ J_e = (t_0, t_3)\,, & J_f = \big(t_4, \frac{1}{\beta+1}\big)\,, & J_g = (t_0, t_5)\,, & J_h = (t_5, 0)\,. \end{array}$$

The anti-morphism $\widehat{\psi}_\beta$ acts on these words by

$$\widehat{\psi}_\beta: \quad a \mapsto b\,, \qquad b \mapsto ababac\,, \qquad c \mapsto dabac\,, \qquad d \mapsto ababae\,,$$
$$e \mapsto fc\,, \qquad f \mapsto g\,, \qquad g \mapsto habac\,, \qquad h \mapsto ag\,.$$

Since $\widehat{0}$ only occurs at the beginning of a, the return words of $\widehat{0}$ with their $\widehat{\psi}_\beta$-images are

$$\begin{aligned} ab &\mapsto ab\,ab\,acb\,, & aed &\mapsto ab\,ab\,aefcb\,,\\ acb &\mapsto ab\,ab\,acd\,ab\,acb\,, & aefcb &\mapsto ab\,ab\,acd\,ab\,acgfcb\,,\\ acd &\mapsto ab\,ab\,aed\,ab\,acb\,, & acgfcb &\mapsto ab\,ab\,acd\,ab\,\underbrace{acgh}_{=acb}\,ab\,acd\,ab\,acb\,. \end{aligned}$$

Then $\mathbb{Z}_{-\beta}$ is described by the anti-morphism $\widehat{\varphi}_{-\beta}: \widehat{R}_\beta^* \to \widehat{R}_\beta^*$ which is defined by

$$\begin{aligned} \widehat{\varphi}_{-\beta}: \quad A &\mapsto AAB\,, & L(A) &= 1\,,\\ B &\mapsto AACAB\,, & L(B) &= \beta - 2 \approx 1.695\,,\\ C &\mapsto AADAB\,, & L(C) &= \beta^2 - 3\beta - 1 \approx 1.569\,,\\ D &\mapsto AAE\,, & L(D) &= \beta^3 - 3\beta^2 - 2\beta - 1 \approx 1.104\,,\\ E &\mapsto AACAF\,, & L(E) &= \beta^4 - 3\beta^3 - 2\beta^2 - \beta - 2 \approx 2.081\,,\\ F &\mapsto AACABACAB\,, & L(F) &= \beta^5 - 3\beta^4 - 2\beta^3 - 2\beta^2 + \beta - 2 \approx 3.12\,. \end{aligned}$$

Some $(-\beta)$-integers are represented in Figure 4, and the two-sided fixed point is

$$\cdots\ AACAB\,AAB\,AADAB\,AAB\,AAB\,^{\cdot}AACAB\,AAB\,AAB\ \cdots\,.$$

Note that grouping the letters as in Example 3 is always possible. It is usually a good idea to start directly with the corresponding intervals, and this is even possible when β is not a $(-\beta)$-number. The drawback of this method is that the involved intervals can be a bit complicated to describe in the general case, e.g. $t_1 < \frac{1}{\beta+1} - \frac{\lfloor\beta\rfloor}{\beta}$ implies that (t_0, t_1) is mapped to (t_2, t_1), an interval which does not occur in Example 3. Determining the return words is also a bit more complicated since $J_{\widehat{0}}$ can be contained in several intervals, and it should be taken care of the fact that the union of two intervals can be another interval (minus one point), as for $J_g \cup J_h = J_b \setminus \{t_5\}$ in Example 3. Therefore, we do not give a general account of this method here.

4. Conclusions and open questions

With every $(-\beta)$-number $\beta \geq (1+\sqrt{5})/2$, we have associated an anti-morphism $\varphi_{-\beta}$ on a finite alphabet. The distances between consecutive $(-\beta)$-integers are described by a fixed point of $\varphi_{-\beta}$, and $\varphi_{-\beta}$ is given by a simple algorithm. In a forthcoming version of [1], the anti-morphism will be described explicitely for any $\beta > 1$ such that $T_{-\beta}^n\big(\frac{-\beta}{\beta+1}\big) \in \big[\frac{1-\lfloor\beta\rfloor}{\beta}, 0\big]$ for all $n \geq 1$. Example 3 shows that the situation is more complicated when this condition is not fulfilled. It would be interesting to have a reasonably simple description of $\varphi_{-\beta}$ in the general case as well.

It is well known that the maximal distance between consecutive β-integers is bounded by 1. We have seen that this is not true for $(-\beta)$-integers. Since the set $\Delta_{-\beta}$ is finite for any $(-\beta)$-number $\beta \geq (1+\sqrt{5})/2$, it is bounded. It is an open question whether there is a uniform bound on $\Delta_{-\beta}$. Another open question is whether $\Delta_{-\beta}$ is bounded when β is not a $(-\beta)$-number. It is possible that these questions can be answered only when the structure of $\mathbb{Z}_{-\beta}$ is well understood in general.

Another topic which is probably worth investigating is the structure of the sets $S_{-\beta}(x)$ for $x \neq 0$, and the corresponding tilings when β is a Pisot unit.

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