Finite beta-expansions with negative bases

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Abstract

The finiteness property is an important arithmetical property of beta-expansions. We exhibit classes of Pisot numbers β having the negative finiteness property, that is the set of finite $(-\beta)$-expansions is equal to $\mathbb{Z}[\beta^{-1}]$. For a class of numbers including the Tribonacci number, we compute the maximal length of the fractional parts arising in the addition and subtraction of $(-\beta)$-integers. We also give conditions excluding the negative finiteness property.

1 Introduction

Digital expansions in real bases $\beta > 1$ were introduced by Rényi [23]. Of particular interest are bases β satisfying the *finiteness property*, or Property (F), which means that each element of $\mathbb{Z}[\beta^{-1}] \cap [0, \infty)$ has a finite (greedy) β*-expansion.* We know from Frougny and Solomyak [13] that each base with Property (F) is a Pisot number, but the converse is not true. Partial characterizations are due to [13, 16, 1]. In [2], Akiyama et al. exhibited an intimate connection to *shift radix systems* (SRS), following ideas of Hollander [16]. For results on shift radix systems (with the finiteness property), we refer to the survey [18].

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Numeration systems with negative base $-\beta < -1$, or $(-\beta)$-*expansions*, received considerable attention since the paper [17] of Ito and Sadahiro in 2009. They are given by the $(-\beta)$-*transformation*

$$T_{-\beta} : [\ell_\beta, \ell_\beta + 1) \to [\ell_\beta, \ell_\beta + 1), \quad x \mapsto -\beta x - \lfloor -\beta x - \ell_\beta \rfloor, \quad \text{with } \ell_\beta = \tfrac{-\beta}{\beta+1};$$

see Section 2 for details. Certain arithmetic aspects seem to be analogous to those for positive base systems [12, 20], others are different, e.g., both negative and positive numbers have $(-\beta)$-expansions; for $\beta < \frac{1+\sqrt{5}}{2}$, the only number with finite $(-\beta)$-expansion is 0. We say that $\beta > 1$ has the *negative finiteness property*, or Property $(-\mathrm{F})$, if each element of $\mathbb{Z}[\beta^{-1}]$ has a finite $(-\beta)$-expansion. By Dammak and Hbaib [10], we know that β must be a Pisot number, as in the positive case. It was shown in [20] that the Pisot roots of $x^2 - mx + n$, with positive integers m, n, $m \geq n + 2$, satisfy the Property $(-\mathrm{F})$. This gives a complete characterization for quadratic numbers, as β does not possess Property $(-\mathrm{F})$ if β has a negative Galois conjugate, by [20].

First, we give other simple criteria when β does *not* satisfy Property $(-\mathrm{F})$. Surprisingly, this happens when ℓ_β has a finite $(-\beta)$-expansion, which is somewhat opposite to the positive case, where Property (F) implies that β is a simple Parry number.

Theorem 1. *If $T_{-\beta}^k(\ell_\beta) = 0$ for some $k \geq 1$, or if β is the root of a polynomial $p(x) \in \mathbb{Z}[x]$ with $|p(-1)| = 1$, then β does not possess Property (−F).*

The main tool we use is a generalization of shift radix systems. We show that the $(-\beta)$-transformation is conjugated to a certain α-SRS. Then we study properties of this dynamical system. We obtain a complete characterization for cubic Pisot units.

Theorem 2. *Let $\beta > 1$ be a cubic Pisot unit with minimal polynomial $x^3 - ax^2 + bx - c$. Then β has Property (−F) if and only if $c = 1$ and $-1 \leq b < a$, $|a| + |b| \geq 2$.*

Considering Pisot numbers of arbitrary degree, we have the following results.

Theorem 3. *Let $\beta > 1$ be a root of $x^d - mx^{d-1} - \cdots - mx - m$ for some positive integers d, m. Then β has Property (−F) if and only if $d \in \{1, 3, 5\}$.*

Theorem 4. *Let $\beta > 1$ be a root of $x^d - a_1x^{d-1} + a_2x^{d-2} + \cdots + (-1)^d a_d \in \mathbb{Z}[x]$ with $a_i \geq 0$ for $i = 1, \ldots, d$, and $a_1 \geq 2 + \sum_{i=2}^d a_i$. Then β has Property (−F).*

These theorems are proved in Section 3. In Section 4, we give a precise bound on the number of fractional digits arising from addition and subtraction of $(-\beta)$-integers in case $\beta > 1$ is a root of $x^3 - mx^2 - mx - m$ for $m \geq 1$. This is based on an extension of shift radix systems. The corresponding numbers for β-integers have not been calculated yet, although they can be determined in a similar way.

2 $(-\beta)$-expansions

For $\beta > 1$, any $x \in [\ell_\beta, \ell_\beta + 1)$ has an expansion of the form

$$x = \sum_{i=1}^{\infty} \frac{x_i}{(-\beta)^i} \quad \text{with} \quad x_i = \lfloor -\beta T_{-\beta}^{i-1}(x) - \ell_\beta \rfloor \text{ for all } i \geq 1.$$

This gives the infinite word $d_{-\beta}(x) = x_1 x_2 x_3 \cdots \in \mathcal{A}^{\mathbb{N}}$ with $\mathcal{A} = \{0, 1, \ldots, \lfloor \beta \rfloor\}$. Since the base is negative, we can represent any $x \in \mathbb{R}$ without the need of a minus sign. Indeed, let $k \in \mathbb{N}$ be minimal such that $\frac{x}{(-\beta)^k} \in (\ell_\beta, \ell_\beta + 1)$ and $d_{-\beta}(\frac{x}{(-\beta)^k}) = x_1 x_2 x_3 \cdots$. Then the $(-\beta)$-expansion of x is defined as

$$\langle x \rangle_{-\beta} = \begin{cases} x_1 \cdots x_{k-1} x_k \bullet x_{k+1} x_{k+2} \cdots & \text{if } k \geq 1, \\ 0 \bullet x_1 x_2 x_3 \cdots & \text{if } k = 0. \end{cases}$$

Similarly to positive base numeration systems, the set of $(-\beta)$-*integers* can be defined using the notion of $\langle x \rangle_{-\beta}$, by

$$\mathbb{Z}_{-\beta} = \{x \in \mathbb{R} : \langle x \rangle_{-\beta} = x_1 \cdots x_{k-1} x_k \bullet 0^\omega\} = \bigcup_{k \geq 0} (-\beta)^k T_{-\beta}^{-k}(0)\,,$$

where 0^ω is the infinite repetition of zeros. The set of numbers with finite $(-\beta)$-expansion is

$$\mathrm{Fin}(-\beta) = \{x \in \mathbb{R} : \langle x \rangle_{-\beta} = x_1 \cdots x_{k-1} x_k \bullet x_{k+1} \cdots x_{k+n} 0^\omega\} \;=\; \bigcup_{n \geq 0} \frac{\mathbb{Z}_{-\beta}}{(-\beta)^n}\,.$$

If $\langle x \rangle_{-\beta} = x_1 \cdots x_{k-1} x_k \bullet x_{k+1} \cdots x_{k+n} 0^\omega$ with $x_{k+n} \neq 0$, then $\mathrm{fr}(x) = n$ denotes the length of the *fractional part* of x; if $x \in \mathbb{Z}_{-\beta}$, then $\mathrm{fr}(x) = 0$.

3 Finiteness

In this section, we discuss the Property $(-\mathrm{F})$ for several classes of Pisot numbers β. Note that $\mathrm{Fin}(-\beta)$ is a subset of $\mathbb{Z}[\beta^{-1}]$ since β is an algebraic integer, hence Property $(-\mathrm{F})$ means that $\mathrm{Fin}(-\beta) = \mathbb{Z}[\beta^{-1}]$, i.e., $\mathrm{Fin}(-\beta)$ is a ring. We start by showing that bases β satisfying $d_{-\beta}(\ell_\beta) = d_1 d_2 \ldots d_k 0^\omega$, which can be considered as analogs to simple Parry numbers, do not possess Property $(-\mathrm{F})$. This was conjectured in [19] and supported by the fact that $d_{-\beta}(\ell_\beta) = d_1 d_2 \ldots d_k 0^\omega$ with $d_1 \geq d_j + 2$ for all $2 \leq j \leq k$ implies that $d_{-\beta}(\beta - 1 - d_1) = (d_2+1)(d_3+1) \cdots (d_k+1) 1^\omega$. However, the assumption $d_1 \geq d_j + 2$ is not necessary for showing that Property $(-\mathrm{F})$ does not hold.

We also prove that a base with Property $(-\mathrm{F})$ cannot be the root of a polynomial of the form $a_0 x^d + a_1 x^{d-1} + \cdots + a_d$ with $|\sum_{i=0}^{d} (-1)^i a_i| = 1$.

Proof of Theorem 1. If $T_{-\beta}^k(\ell_\beta) = 0$, i.e., $d_{-\beta}(\ell_\beta) = d_1 d_2 \dots d_k 0^\omega$, then we have

$$\frac{-\beta}{\beta+1} = \frac{d_1}{-\beta} + \frac{d_2}{(-\beta)^2} + \cdots + \frac{d_k}{(-\beta)^k}$$

and thus $\frac{-1}{\beta+1} \in \mathbb{Z}[\beta^{-1}]$. However, we have $\frac{-1}{\beta+1} \notin \mathrm{Fin}(-\beta)$ since $T_{-\beta}(\frac{-1}{\beta+1}) = \frac{-1}{\beta+1}$, i.e., $d_{-\beta}(\frac{-1}{\beta+1}) = 1^\omega$. Hence β does not possess Property (−F).

If $p(\beta) = 0$ with $|p(-1)| = 1$, then write

$$p(x-1) = xf(x) + p(-1),$$

with $f(x) \in \mathbb{Z}[x]$. Then we have $\frac{1}{\beta+1} = |f(\beta+1)| \in \mathbb{Z}[\beta]$ and thus $\frac{-(-\beta)^{-j}}{\beta+1} \in \mathbb{Z}[\beta^{-1}]$ for some $j \geq 0$. Now, $d_{-\beta}(\frac{-(\beta)^{-j}}{\beta+1}) = 0^j 1^\omega$ implies that β does not have the Property (−F). $\square$

The main tool we will be using in the rest of the paper are α-shift radix systems. An α-SRS is a dynamical system acting on $\mathbb{Z}^d$ in the following way. For $\alpha \in \mathbb{R}$, $\mathbf{r} = (r_0, r_1, \dots, r_{d-1}) \in \mathbb{R}^d$, and $\mathbf{z} = (z_0, z_1, \dots, z_{d-1}) \in \mathbb{Z}^d$, let $\tau_{\mathbf{r},\alpha}$ be defined as

$$\tau_{\mathbf{r},\alpha}(z_0, z_1, \dots, z_{d-1}) = (z_1, \dots, z_{d-1}, z_d),$$

where z_d is the unique integer satisfying

$$0 \leq r_0 z_0 + r_1 z_1 + \cdots + r_{d-1} z_{d-1} + z_d + \alpha < 1. \tag{1}$$

Alternatively, we can say that

$$\tau_{\mathbf{r},\alpha}(z_0, z_1, \dots, z_{d-1}) = (z_1, \dots, z_{d-1}, -\lfloor \mathbf{rz} + \alpha \rfloor),$$

where $\mathbf{rz}$ stands for the scalar product.

The usefulness of α-SRS with $\alpha = 0$ for the study of finiteness of β-expansions was first shown by Hollander in his thesis [16]. His approach was later formalized in [2] where the case $\alpha = 0$ was extensively studied. The symmetric case with $\alpha = \frac{1}{2}$ was then studied in [4]. Finally, general α-SRS were considered by Surer [24].

We say that $\tau_{\mathbf{r},\alpha}$ has the finiteness property if for each $\mathbf{z} \in \mathbb{Z}^d$ there exists $k \in \mathbb{N}$ such that $\tau_{\mathbf{r},\alpha}^k(\mathbf{z}) = \mathbf{0}$. The finiteness property of $\tau_{\mathbf{r},\alpha}$ is closely related to the Property (−F), thus it is desirable to study the set

$$\mathcal{D}_{d,\alpha}^0 = \{\mathbf{r} \in \mathbb{R}^d : \forall \mathbf{z} \in \mathbb{Z}^d, \exists k, \tau_{\mathbf{r},\alpha}^k(\mathbf{z}) = \mathbf{0}\}.$$

The following proposition shows the link between $(-\beta)$-expansions and α-SRS.

Proposition 5. *Let $\beta > 1$ be an algebraic integer with minimal polynomial $x^d + a_1 x^{d-1} + \cdots + a_{d-1}x + a_d$. Set $\alpha = \frac{\beta}{\beta+1}$ and let $(r_0, r_1, \dots, r_{d-2}) \in \mathbb{R}^{d-1}$ be such that*

$$x^d + (-1)a_1 x^{d-1} + \cdots + (-1)^d a_d = (x+\beta)(x^{d-1} + r_{d-2}x^{d-2} + \cdots + r_1 x + r_0),$$

$$\text{i.e.,} \quad r_i = (-1)^{d-i}\left(\frac{a_{d-i}}{\beta} + \cdots + \frac{a_d}{\beta^{i+1}}\right) \quad \text{for } i = 0, 1, \dots, d-2.$$

Then β has Property (−F) if and only if $(r_0, r_1, \dots, r_{d-2}) \in \mathcal{D}_{d-1,\alpha}^0$.

Proof. Let $\mathbf{r} = (r_0, r_1, \dots, r_{d-2})$. First we show that for $\phi : \mathbf{z} \mapsto \mathbf{rz} - \lfloor \mathbf{rz} + \alpha \rfloor$ the following commutation diagram holds, i.e., the systems $(\tau_{\mathbf{r},\alpha}, \mathbb{Z}^{d-1})$ and $(T_{-\beta}, \mathbb{Z}[\beta] \cap [\ell_\beta, \ell_\beta + 1))$ are conjugated.

$$\begin{tikzcd} \mathbb{Z}^{d-1} \arrow[r, "\tau_{r,\alpha}"] \arrow[d, "\phi"] & \mathbb{Z}^{d-1} \arrow[d, "\phi"] \\ \mathbb{Z}[\beta] \cap [\ell_\beta, \ell_\beta + 1) \arrow[r, "T_{-\beta}"] & \mathbb{Z}[\beta] \cap [\ell_\beta, \ell_\beta + 1) \end{tikzcd}$$

Since $r_i = (-1)^{d-i-1}(\beta^{d-i-1} + a_1\beta^{d-i-2} + \cdots + a_{d-i-1})$ for $0 \le i \le d-2$, the set $\{r_i : 0 \le i < d\}$ with $r_{d-1} = 1$ forms a basis of $\mathbb{Z}[\beta]$, hence ϕ is a bijection. Moreover, we have $-\beta r_i = r_{i-1} + c_i$ with $c_i \in \mathbb{Z}$ and $r_{-1} = 0$. For $\mathbf{z} = (z_0, z_1, \dots, z_{d-2})$, we have $\phi(\mathbf{z}) = \sum_{i=0}^{d-1} r_i z_i$ with $z_{d-1} = -\lfloor \mathbf{rz} + \alpha \rfloor$, thus

$$T_{-\beta}(\phi(\mathbf{z})) = -\beta\phi(\mathbf{z}) + n = \sum_{i=1}^{d-1} r_{i-1} z_i + n' = \phi(z_1, \dots, z_{d-2}, z_{d-1}) = \phi(\tau_{\mathbf{r},\alpha}(\mathbf{z})),$$

where n and n' are integers; for the third equality, we have used that $T_{-\beta}(\phi(\mathbf{z})) \in [\ell_\beta, \ell_\beta+1)$.

Therefore, we have $\mathbf{r} \in \mathcal{D}^0_{d-1,\alpha}$ if and only if for each $x \in \mathbb{Z}[\beta] \cap [\ell_\beta, \ell_\beta + 1)$ there exists $k \ge 0$ such that $T^k_{-\beta}(x) = 0$. Since for each $x \in \mathbb{Z}[\beta^{-1}] \cap [\ell_\beta, \ell_\beta + 1)$ we have $T^n_{-\beta}(x) \in \mathbb{Z}[\beta]$ for some $n \in \mathbb{N}$, Property $(-\mathrm{F})$ is equivalent to $\mathbf{r} \in \mathcal{D}^0_{d-1,\alpha}$. $\square$

Thus the problem of finiteness of $(-\beta)$-expansions can be interpreted as the problem of finiteness of the corresponding α-SRS. This problem is often decidable by checking the finiteness of α-SRS expansions of a certain subset of $\mathbb{Z}^d$. A *set of witnesses* of $\mathbf{r} \in \mathbb{R}^d$ is a set $\mathcal{V} \subset \mathbb{Z}^d$ that satisfies

1. $\pm\mathbf{e}_i \in \mathcal{V}$ where $\mathbf{e}_i$ denotes the standard basis of $\mathbb{R}^d$,
2. if $\mathbf{z} \in \mathcal{V}$, then $\tau_{\mathbf{r},0}(\mathbf{z}), -\tau_{\mathbf{r},0}(-\mathbf{z}) \in \mathcal{V}$.

The following proposition is due to Surer [24] and Brunotte [7].

Proposition 6. *Let $\alpha \in [0, 1)$ and $\mathbf{r} \in \mathbb{R}^d$. Then $\mathbf{r} \in \mathcal{D}^0_{d,\alpha}$ if and only if there exists a set of witnesses that does not contain nonzero periodic elements of $\tau_{\mathbf{r},\alpha}$.*

Sets of witnesses for several classes of $\mathbf{r} \in \mathbb{R}^d$ were derived in [3]. Exploiting their explicit form, several regions of finiteness can be determined; see in particular [3, Theorems 3.3–3.5]. An α-SRS analogy of some of those regions was given by Brunotte [7]. Brunotte's result, however, is unsuitable for our purposes. The next proposition gives several regions of finiteness of α-SRS.

Proposition 7. *Let $\mathbf{r} = (r_0, r_1, \dots, r_{d-1}) \in \mathbb{R}^d$ and $\alpha \in [0, 1)$.*

1. *If $\sum_{i=0}^{d-1} |r_i| \le \alpha$ and $\sum_{r_i<0} r_i > \alpha - 1$, then $\mathbf{r} \in \mathcal{D}^0_{d,\alpha}$.*
2. *If $0 \le r_0 \le r_1 \le \cdots \le r_{d-1} \le \alpha$, then $\mathbf{r} \in \mathcal{D}^0_{d,\alpha}$.*

3. *If* $\sum_{i=0}^{d-1} |r_i| \le \alpha$ *and* $r_i < 0$ *for exactly one index* $i = d-k$*, then* $\mathbf{r} \in \mathcal{D}_{d,\alpha}^0$ *if and only if*
$$\sum_{1\le j\le d/k} r_{d-jk} > \alpha - 1. \tag{2}$$

Proof. 1. The set $\mathcal{V} = \{-1,0,1\}^d$ is closed under $\tau_{\mathbf{r},0}(\mathbf{z})$ and $-\tau_{\mathbf{r},0}(-\mathbf{z})$, hence it is a set of witnesses. For any $\mathbf{z} \in \mathcal{V}$ we have $|\mathbf{rz}| \le \alpha$, thus $\lfloor \mathbf{rz} + \alpha \rfloor \in \{0,1\}$. Hence any periodic point of $\tau_{\mathbf{r},\alpha}$ is in $\{0,-1\}^d$. For $\mathbf{z} \in \{0,-1\}^d$ we have $\mathbf{rz} + \alpha \le -\sum_{r_j<0} r_j + \alpha < 1$. Therefore $\lfloor \mathbf{rz} + \alpha \rfloor = 0$, so the only period is the trivial one.

2. In this case we take as a set of witnesses the elements of $\{-1,0,1\}^d$ with alternating signs, i.e., $z_i z_j \le 0$ for any pair of indices $i < j$ such that $z_k = 0$ for each $i < k < j$. For any $\mathbf{z} \in \mathcal{V}$ we have again $|\mathbf{rz}| \le \alpha$, thus $\lfloor \mathbf{rz} + \alpha \rfloor \in \{0,1\}$ and $\tau_{\mathbf{r},\alpha}(\mathbf{z}) \in \mathcal{V}$. Therefore, we have $\tau_{\mathbf{r},\alpha}^n(\mathbf{z}) = (-1,0,\dots,0)$ for some $n \ge 0$, hence $\tau_{\mathbf{r},\alpha}^{n+1}(\mathbf{z}) = \mathbf{0}$.

3. In this case we have $\mathcal{V} = \{-1,0,1\}^d$. As above, all periodic points of $\tau_{\mathbf{r},\alpha}$ are in $\{0,-1\}^d$. If $\mathbf{z} = (z_0, z_1, \dots, z_{d-1})$ is a periodic point with $z_d = -\lfloor \mathbf{rz} + \alpha \rfloor = -1$, then we must have $z_{d-k} = -1$, and consequently $z_{d-jk} = -1$ for all $1 \le j \le d/k$. Then $z_d = -1$ also implies that $-\sum_{1\le j\le d/k} r_{d-jk} + \alpha \ge 1$, i.e., (2) does not hold. On the other hand, if (2) holds, then the vector $(z_0, z_1, \dots, z_{d-1})$ with $z_{d-jk} = -1$ for $1 \le j \le d/k$, $z_i = 0$ otherwise, is a periodic point of $\tau_{\mathbf{r},\alpha}$. $\square$

Next we prove Property (−F) when β is a root of a polynomial with alternating coefficients, where the second highest coefficient is dominant.

Proof of Theorem 4. Let $\beta > 1$ be a root of $p(x) = x^d - a_1 x^{d-1} + a_2 x^{d-2} + \cdots + (-1)^d a_d \in \mathbb{Z}[x]$ with $a_i \ge 0$ for $i = 1,\dots,d$, and $a_1 \ge 2 + \sum_{i=2}^d a_i$. As $\frac{\mathrm{d}}{\mathrm{d}x}(p(x)x^{-d}) \ge \frac{a_1}{x^2} - \frac{a_1-2}{x^3} > 0$ for $x > 1$, the polynomial $p(x)$ has a unique root $\beta > 1$, and we have $\beta > a_1 - 1$ since $p(a_1-1) \le -(a_1-1)^{d-1} + (a_1-2)(a_1-1)^{d-2} < 0$. By Proposition 5, Property (−F) holds if and only if $(r_0, r_1, \dots, r_{d-2}) \in \mathcal{D}_{d-1,\alpha}^0$, with $r_i = a_{d-i}\beta^{-1} - a_{d-i+1}\beta^{-2} + a_{d-i+2}\beta^{-3} - \cdots + (-1)^{d-i} a_d \beta^{-d+i-1}$. We have
$$-\sum_{r_i<0} r_i \le \frac{a_1-2}{\beta^2} + \frac{a_1-2}{\beta^4} + \cdots + \frac{a_1-2}{\beta^{2\lceil d/2\rceil-2}} \le \frac{a_1-2}{\beta^2-1} < \frac{1}{\beta+1}$$

and

$$
\begin{aligned}
&\frac{\beta+1}{\beta}\sum_{i=0}^{d-1}|r_i| \le \frac{\beta+1}{\beta}\left(\frac{a_2+\cdots+a_d}{\beta}+\frac{a_3+\cdots+a_d}{\beta^2}+\cdots+\frac{a_d}{\beta^{d-1}}\right)\\
&= \frac{a_2+\cdots+a_d}{\beta}+\frac{a_2+2a_3+\cdots+2a_d}{\beta^2}+\frac{a_3+2a_4+\cdots+2a_d}{\beta^3}+\cdots+\frac{a_{d-1}+2a_d}{\beta^{d-1}}+\frac{a_d}{\beta^d}\\
&\le \frac{a_1-2}{\beta}+\frac{2(a_1-2)-a_2}{\beta^2}+\frac{2(a_1-a_2-2)-a_3}{\beta^3}+\cdots+\frac{2(a_1-a_2-\cdots-a_{d-1}-2)-a_d}{\beta^d}\\
&\le 1-2\left(\frac{1}{\beta}-\frac{a_1-2}{\beta^2}-\frac{a_1-a_2-2}{\beta^3}-\cdots-\frac{a_1-a_2-\cdots-a_{d-1}-2}{\beta^d}\right)\\
&\le 1-\frac{2}{\beta}\left(1-\frac{a_1-2}{\beta-1}\right)<1.
\end{aligned}
$$

Therefore, item 1 of Proposition 7 gives that Property (−F) holds. □

Now we can classify the cubic Pisot units with Property (−F). The following description of cubic Pisot numbers in terms of the coefficients of the minimal polynomial is due to Akiyama [1, Lemma 1].

Lemma 8. *A number $\beta > 1$ with minimal polynomial $x^3 - ax^2 + bx - c$ is Pisot if and only if*

$$|b+1| < a+c \quad \textit{and} \quad b+c^2 < \operatorname{sgn}(c)(1+ac).$$

Proof of Theorem 2. Let $\beta > 1$ be a cubic Pisot unit with minimal polynomial $x^3 - ax^2 + bx - c$. If $c = -1$, then β has a negative conjugate, which contradicts Property (−F) by [20]. Therefore, we assume in the following that $c = 1$. Then from Lemma 8 we have that $-a-1 \le b < a$. By Proposition 5, Property (−F) holds if and only if $(r_0, r_1) \in \mathcal{D}^0_{2,\alpha}$, with $(r_0, r_1) = (\frac{1}{\beta}, \frac{b}{\beta} - \frac{1}{\beta^2})$ and $\alpha = \frac{\beta}{\beta+1}$. We distinguish five cases for the value of b.

1. $b = 0$: If $a \ge 2$, then we have $|r_0| + |r_1| = \frac{1}{\beta} + \frac{1}{\beta^2} < \alpha$ and $r_0 + r_1 > 0 > \alpha - 1$, so we apply item 3 of Proposition 7. If $a = 1$, then we have $T_{-\beta}^{-1}(0) = \{0\}$ as $\beta < \frac{1+\sqrt{5}}{2}$, thus $\mathrm{Fin}(-\beta) = \{0\}$.

2. $b = -1$: If $a \ge 1$, then $r_0 + r_1 = -\frac{1}{\beta^2} > \frac{-1}{\beta+1} = \alpha - 1$. If $a \ge 3$, then we also have $|r_0| + |r_1| < \alpha$ and use item 3 of Proposition 7. If $a = 2$, then $r_0 \approx 0.39$, $r_1 \approx -0.55$, $\alpha \approx 0.72$, $\{-1, 0, 1\}^2$ is a set of witnesses, and Property (−F) holds because $\tau_{\mathbf{r},\alpha}$ acts on this set in the following way:

$$(-1,1)\mapsto(1,1)\mapsto(1,0)\mapsto(0,-1)\mapsto(-1,-1)\mapsto(-1,0)\mapsto(0,0),$$
$$(0,1)\mapsto(1,0),\ (1,-1)\mapsto(-1,-1).$$

   For $a = 1$, we refer to Theorem 3, which is proved below. If $a = 0$, then $\beta < \frac{1+\sqrt{5}}{2}$ and thus $\mathrm{Fin}(-\beta) = \{0\}$.

3. $1 \le b \le a-2$: For $b \ge 2$, we have $0 < r_0 < r_1 < \alpha$ and thus $(r_0, r_1) \in \mathcal{D}^0_{2,\alpha}$ by item 2 of Proposition 7. If $b = 1$, then we can use item 1 of Proposition 7 because $r_0, r_1 > 0$ and $r_0 + r_1 < \alpha$.

4. $1 \le b = a-1$: We have $\beta = b + \frac{1}{\beta(\beta-1)}$. For $b \ge 3$, we have $0 < r_0 < \alpha < r_1 < 1$, the set $\{-1,0,1\}^2 \setminus \{(1,1),(-1,-1)\}$ is a set of witnesses, and $\tau_{\mathbf{r},\alpha}$ acts on this set by
$$(1,0) \mapsto (0,-1) \mapsto (-1,1) \mapsto (1,-1) \mapsto (-1,0) \mapsto (0,0), \quad (0,1) \mapsto (1,-1),$$
thus Property $(-\mathrm{F})$ holds. If $b = 2$, then $0 < r_0 < r_1 < \alpha$ and we can use item 2 of Proposition 7. If $b = 1$, then $r_0 \approx 0.57$, $r_1 \approx 0.25$, $\alpha \approx 0.64$, thus $\{-1,0,1\}^2$ is a set of witnesses, with
$$(-1,-1) \mapsto (-1,1) \mapsto (1,0) \mapsto (0,-1) \mapsto (-1,0) \mapsto (0,0),$$
$$(0,1) \mapsto (1,0), \ (1,1) \mapsto (1,-1) \mapsto (-1,0).$$

5. $-a-1 \le b \le -2$: We have $-r_0 - r_1 + \alpha = \frac{-b-1}{\beta} + \frac{1}{\beta^2} + \frac{\beta}{\beta+1} > 1$, thus $\tau_{\mathbf{r},\alpha}(-1,-1) = (-1,-1)$, hence $(r_0, r_1) \notin \mathcal{D}^0_{2,\alpha}$.

Therefore, β has Property $(-\mathrm{F})$ if and only if $-1 \le b < a$, $|a| + |b| \ge 2$. $\square$

Finally, we study generalized d-bonacci numbers.

Proof of Theorem 3. Let $\beta > 1$ be a root of $x^d - mx^{d-1} - \cdots - mx - m$ with $d, m \in \mathbb{N}$.

If $d = 1$ (and $m \ge 2$), then β is an integer, and Property $(-\mathrm{F})$ follows from $\mathbb{Z}_{-\beta} = \mathbb{Z}$; see e.g. [20].

If $d = 3$, then $\mathbf{r} = (\frac{m}{\beta}, -\frac{m}{\beta} - \frac{m}{\beta^2})$, $0 < r_0 < \alpha < -r_1 < 1$, with $\alpha = \frac{\beta}{\beta+1}$, and $\tau_{\mathbf{r},\alpha}$ satisfies
$$(0,1) \mapsto (1,1) \mapsto (1,0) \mapsto (0,-1) \mapsto (-1,-1) \mapsto (-1,0) \mapsto (0,0),$$
with $\{-1,0,1\}^2 \setminus \{(1,-1),(-1,1)\}$ being a set of witnesses.

If $d = 5$, then $\mathbf{r} = (\frac{m}{\beta}, -\frac{m}{\beta} - \frac{m}{\beta^2}, \frac{m}{\beta} + \frac{m}{\beta^2} + \frac{m}{\beta^3}, -\frac{m}{\beta} - \frac{m}{\beta^2} - \frac{m}{\beta^3} - \frac{m}{\beta^4})$, which gives $0 < r_0 < \alpha < -r_1 < r_2 < -r_3 < 1$ and the $\tau_{\mathbf{r},\alpha}$-transitions
$$(0,1,0,0) \mapsto (1,0,0,1) \mapsto (0,0,1,0) \mapsto (0,1,0,-1) \mapsto (1,0,-1,0) \mapsto (0,-1,0,0) \mapsto$$
$$(-1,0,0,-1) \mapsto (0,0,-1,-1) \mapsto (0,-1,-1,0) \mapsto (-1,-1,0,0) \mapsto (-1,0,0,0) \mapsto (0,0,0,0),$$
$$(0,0,-1,0) \mapsto (0,-1,0,1) \mapsto (-1,0,1,0) \mapsto (0,1,0,-1),$$
$$(0,1,1,1) \mapsto (1,1,1,1) \mapsto (1,1,1,0) \mapsto (1,1,0,-1) \mapsto (1,0,-1,-1) \mapsto$$
$$(0,-1,-1,-1) \mapsto (-1,-1,-1,-1) \mapsto (-1,-1,-1,0) \mapsto (-1,-1,0,0),$$
$$(0,0,0,1) \mapsto (0,0,1,1) \mapsto (0,1,1,0) \mapsto (1,1,0,0) \mapsto (1,0,0,0) \mapsto (0,0,0,-1) \mapsto (0,0,-1,-1),$$
$$(-1,-1,0,1) \mapsto (-1,0,1,1) \mapsto (0,1,1,0).$$

Let $\mathcal{V}$ be the set of these states. We have $\pm\mathbf{e}_i \in \mathcal{V}$, $\mathbf{z} \in \mathcal{V}$ if and only if $-\mathbf{z} \in \mathcal{V}$ and $\tau_{\mathbf{r},0}(\mathbf{z}) \in \mathcal{V}$ for all $\mathbf{z} \in \mathcal{V}$, thus $\mathcal{V}$ is a set of witnesses. As $\tau_{\mathbf{r},\alpha}^{11}(\mathbf{z}) = (0,0,0,0)$ for all $\mathbf{z} \in \mathcal{V}$, β has Property $(-\mathrm{F})$.

For odd $d \geq 7$, Property $(-\mathrm{F})$ does not hold since $T_{-\beta}^{d-1}(\frac{m}{\beta^2}+\frac{m}{\beta^3}+\frac{m}{\beta^4}-1) = \frac{m}{\beta^2}+\frac{m}{\beta^3}+\frac{m}{\beta^4}-1$, i.e., $\tau_{\mathbf{r},\alpha}^{d-1}(-1,0,0,-1,0,0,\ldots,0) = (-1,0,0,-1,0,0,\ldots,0)$. For even $d \geq 2$, we use the second condition of Theorem 1, or that $\tau_{\mathbf{r},\alpha}(-1,\ldots,-1) = (-1,\ldots,-1)$.

Therefore, β has Property $(-\mathrm{F})$ if and only if $d \in \{1,3,5\}$. □

4 Addition and subtraction

In this section, we consider the lengths of fractional parts arising in the addition and subtraction of $(-\beta)$-integers; we prove the following theorem.

Theorem 9. *Let $\beta > 1$ be a root of $x^3 - m\beta^2 - m\beta - m$, $m \geq 1$. We have*

$$\max\{\mathrm{fr}(x \pm y) : x, y \in \mathbb{Z}_{-\beta}\} = 3m + \begin{cases} 3 & \text{if } m = 1 \text{ or } m \text{ is even,} \\ 4 & \text{if } m \geq 3 \text{ is odd.} \end{cases}$$

Throughout the section, let β be as in Theorem 9, $\mathbf{r} = (r_0, r_1) = (\frac{m}{\beta}, -\frac{m}{\beta} - \frac{m}{\beta^2})$ and $\alpha = \frac{\beta}{\beta+1}$. Recall that $x, y \in \mathbb{Z}_{-\beta}$ means that $T_{-\beta}^k(\frac{x}{(-\beta)^k}) = 0 = T_{-\beta}^k(\frac{y}{(-\beta)^k})$, and $\mathrm{fr}(x \pm y) = n$ is the minimal $n \geq 0$ such that $T_{-\beta}^{k+n}(\frac{x \pm y}{(-\beta)^k}) = 0$, with $k \geq 0$ such that $\frac{x}{(-\beta)^k}, \frac{y}{(-\beta)^k}, \frac{x\pm y}{(-\beta)^k} \in (\ell_\beta, \ell_\beta + 1)$. To determine $\mathrm{fr}(x - y)$, set

$$s_j = T_{-\beta}^j(\tfrac{x-y}{(-\beta)^k}) + T_{-\beta}^j(\tfrac{y}{(-\beta)^k}) - T_{-\beta}^j(\tfrac{x}{(-\beta)^k})$$

for $j \geq 0$. Then we have $s_j = T_{-\beta}^j(\frac{x-y}{(-\beta)^k})$ for $j \geq k$, and, for all $j \geq 0$,

$$s_{j+1} \in -\beta s_j + \mathcal{B} \quad \text{with} \quad \mathcal{B} = -\mathcal{A} - \mathcal{A} + \mathcal{A} = \{-2m, -2m+1, \ldots, m\},$$
$$s_j \in [\ell_\beta, \ell_\beta + 1) + [\ell_\beta, \ell_\beta + 1) - [\ell_\beta, \ell_\beta + 1) = (\ell_\beta - 1, \ell_\beta + 2).$$

As $s_0 = 0$, we have $s_j \in \mathbb{Z}[\beta]$ for $j \geq 0$. Therefore, we extend the bijection $\phi : \mathbb{Z}^2 \to \mathbb{Z}[\beta] \cap [\ell_\beta, \ell_\beta + 1)$ to

$$\Phi : \mathbb{Z}^2 \times \{-1,0,1\} \to \mathbb{Z}[\beta] \cap [\ell_\beta - 1, \ell_\beta + 2), \quad (\mathbf{z}, h) \mapsto \mathbf{r}\mathbf{z} - \lfloor \mathbf{r}\mathbf{z} + \alpha \rfloor + h.$$

Note that $\Phi(\mathbf{z}, 0) = \phi(\mathbf{z})$.

Lemma 10. *Let $\mathbf{z} = (z_0, z_1) \in \mathbb{Z}^2$, $h \in \{-1,0,1\}$ and $b \in \mathcal{B}$. Then*

$$-\beta\,\Phi(\mathbf{z},h)+b = \Phi\big(z_1, h - \lfloor \mathbf{r}\mathbf{z}+\alpha \rfloor, (z_1 - z_0 - h + \lfloor \mathbf{r}\mathbf{z}+\alpha \rfloor)\, m + \lfloor r_0 z_1 + r_1 h - r_1 \lfloor \mathbf{r}\mathbf{z}+\alpha \rfloor + \alpha \rfloor + b\big).$$

Proof. We have

$$\begin{aligned} -\beta\,\Phi(\mathbf{z},h) + b &= -z_0 m + z_1 m + \frac{z_1 m}{\beta} + \lfloor \mathbf{r}\mathbf{z} + \alpha \rfloor \beta - h\beta + b \\ &= r_0 z_1 + r_1 \big(h - \lfloor \mathbf{r}\mathbf{z} + \alpha \rfloor\big) + \big(z_1 - z_0 - h + \lfloor \mathbf{r}\mathbf{z} + \alpha \rfloor\big)\, m + b. \end{aligned}$$

□

Hence, we have $s_j \in \Phi(\tilde{\tau}_{\mathbf{r},\alpha}^j(\mathbf{0},0))$, where $\tilde{\tau}_{\mathbf{r},\alpha}$ extends $\tau_{\mathbf{r},\alpha}$ to a set-valued function by

$$\begin{aligned}\tilde{\tau}_{\mathbf{r},\alpha} : \mathbb{Z}^2 \times \{-1,0,1\} \to \mathcal{P}(\mathbb{Z}^2 \times \{-1,0,1\}), \quad (\mathbf{z},h) \mapsto \big\{(z_1, h - \lfloor \mathbf{r}\mathbf{z}+\alpha\rfloor, h') :\\ h' \in \{-1,0,1\} \cap \big(\big(z_1 - z_0 - h + \lfloor \mathbf{r}\mathbf{z}+\alpha\rfloor\big)\, m + \big\lfloor r_0 z_1 + r_1 h - r_1\lfloor \mathbf{r}\mathbf{z}+\alpha\rfloor + \alpha\big\rfloor + \mathcal{B}\big)\big\}.\end{aligned}$$

To give a bound for the sets $\tilde{\tau}_{\mathbf{r},\alpha}^j(\mathbf{0},0)$, let

$$\begin{gathered}A_k = \{(j,k) : -1 \le j < k\},\ B_k = \{(k,j) : 1 \le j \le k\},\ C_k = \{(j, j-k) : 1 \le j \le k\},\\ D_k = \{(-j,-k) : 0 \le j < k\},\ E_k = \{(-k,-j) : 2 \le j \le k\},\\ F_k = \{(-j, k-j) : 2 \le j \le k+1\}.\end{gathered}$$

Then $\bigcup_{k\ge 0}\{A_k, B_k, C_k, D_k, E_k, F_k\}$ forms a partition of $\mathbb{Z}^2 \setminus \{(0,0),(-1,-1)\}$, with the sets B_0, C_0, D_0, E_0, F_0, and E_1 being empty, see Figure 1. If $m \ge 2$, then let

$$\begin{aligned}V = \Bigg(&\bigcup_{0\le k\le m} \big(A_k \cup B_k \cup C_k \cup D_k \cup E_k \cup F_k\big) \times \{-1,0,1\}\Bigg) \setminus \{(-1,m,1),(0,m,1)\}\\ &\cup \Big(\big(C_{m+1} \setminus \{(m+1,0)\}\big) \times \{1\}\Big) \cup \Big(D_{m+1} \times \{1\}\Big) \cup \Big(\big(D_{m+1} \setminus \{(0,-m-1)\}\big) \times \{0\}\Big)\\ &\cup \big(D_{m+1} \setminus \{(0,-m-1),(-1,-m-1),(-2,-m-1)\}\big) \times \{-1\}\\ &\cup \Big(\big(\{(0,0),(-1,-1)\} \cup E_{m+1}\big) \setminus \{(-m-1,-m-1)\}\Big) \times \{-1,0,1\}\\ &\cup \Big(\big(F_{m+1} \setminus \{(-m-2,-1),(-m-1,0)\}\big) \times \{-1,0\}\Big).\end{aligned}$$

If $m = 1$, then we add the point $(-2,0,-1)$ to this set, i.e.,

$$\begin{aligned}V = \big(\big\{(0,0),(1,1),(1,0),(0,-1),(-1,-1),(-1,0),(-2,-1)\big\} \times \{-1,0,1\}\big)\\ \cup\big(\big\{(-1,1),(0,1)\big\}\times\{-1,0\}\big)\cup\big(\{(-1,-2)\}\times\{0,1\}\big)\cup\big\{(1,-1,1),(0,-2,1),(-2,0,-1)\big\}.\end{aligned}$$

We call a point $\mathbf{z} \in \mathbb{Z}^2$ *full* if $\{\mathbf{z}\} \times \{-1,0,1\} \subset V$.

The following result is the key lemma of this section.

Lemma 11. *Let $x, y \in [\ell_\beta, \ell_\beta+1)$ such that $x - y \in [\ell_\beta, \ell_\beta+1)$. Then $T_{-\beta}^j(x-y) + T_{-\beta}^j(y) - T_{-\beta}^j(x) \in \Phi(V)$ for all $j \ge 0$.*

To prove Lemma 11, we first determine the value of $\lfloor \mathbf{r}\mathbf{z}+\alpha\rfloor$ for $(\mathbf{z},h) \in V$.

Lemma 12. *Let $\mathbf{z} = (z_0,z_1) \in \mathbb{Z}^2$ with $-m-1 \le z_0 \le m$, $|z_1| \le m+1$ and $|z_0-z_1| \le m+1$. Then*

$$\lfloor \mathbf{r}\mathbf{z}+\alpha\rfloor = z_0 - z_1 + \begin{cases}0 & \text{if } z_0 \ge 0 \text{ or } z_1 \le z_0 = -1,\\ 1 & \text{if } z_0 \le -2 \text{ or } z_1 > z_0 = -1.\end{cases}$$

Proof. We have $z_0 r_0 + z_1 r_1 = z_0 - z_1 - z_0\frac{m}{\beta^2} + (z_1 - z_0)\frac{m}{\beta^3}$ and

$$\frac{-\beta}{\beta+1} < -\frac{m^2}{\beta^2} - \frac{(m+1)m}{\beta^3} \le -\frac{z_0 m}{\beta^2} + \frac{(z_1-z_0)m}{\beta^3} \le \frac{(m+1)m}{\beta^2} + \frac{(m+1)m}{\beta^3} < 1 + \frac{1}{\beta+1},$$

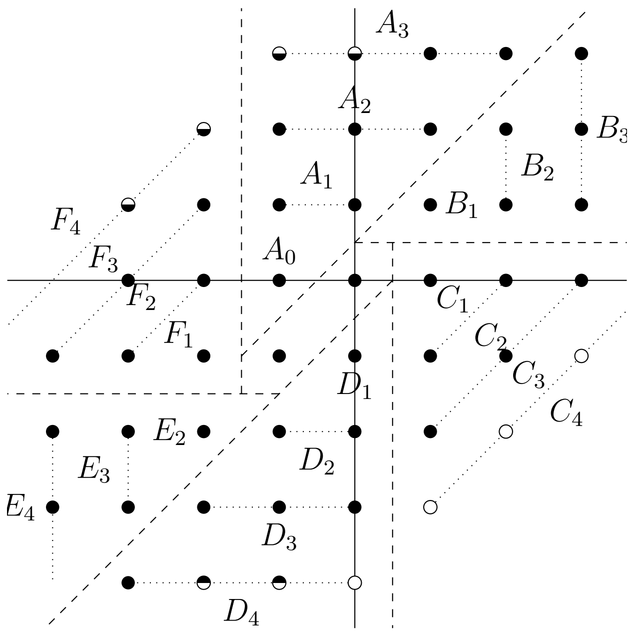

Figure 1: The set V for $m = 3$. Full points are represented by disks, points $\mathbf{z}$ with $\{\mathbf{z}\} \times \{0,1\} \subset V$, $\{\mathbf{z}\} \times \{-1,0\} \subset V$ and $\{\mathbf{z}\} \times \{1\} \subset V$ by upper half-disks, lower half-disks and circles respectively.

thus $\lfloor \mathbf{rz} + \alpha \rfloor \in z_0 - z_1 + \{0,1\}$.

If $z_0 \geq 0$, then we have $-z_0 \frac{m}{\beta^2} + (z_1 - z_0)\frac{m}{\beta^3} \leq (m+1)\frac{m}{\beta^3} < \frac{1}{\beta+1}$. If $z_1 \leq z_0 = -1$, then $-z_0 \frac{m}{\beta^2} + (z_1 - z_0)\frac{m}{\beta^3} \leq \frac{m}{\beta^2} < \frac{1}{\beta+1}$. This shows that $\lfloor \mathbf{rz} + \alpha \rfloor = z_0 - z_1$ in these two cases.

If $z_1 > z_0 = -1$, then we have $-z_0 \frac{m}{\beta^2} + (z_1 - z_0)\frac{m}{\beta^3} \geq \frac{m}{\beta^2} + \frac{m}{\beta^3} > \frac{1}{\beta+1}$. Finally, if $z_0 \leq -2$, then $-z_0 \frac{m}{\beta^2} + (z_1 - z_0)\frac{m}{\beta^3} \geq 2\frac{m}{\beta^2} - (m-1)\frac{m}{\beta^3} > \frac{1}{\beta+1}$, thus $\lfloor \mathbf{rz} + \alpha \rfloor = z_0 - z_1 + 1$ in the latter cases. $\square$

Proof of Lemma 11. We have already seen above that $T_{-\beta}^j(x-y) + T_{-\beta}^j(y) - T_{-\beta}^j(x) \in \Phi(\tilde{\tau}_{\mathbf{r},\alpha}^j(\mathbf{0},0))$. As $(\mathbf{0},0) \in V$, it suffices to show that $\tilde{\tau}_{\mathbf{r},\alpha}(V) \subseteq V$.

Let $(\mathbf{z},h) \in V$ and $b \in B$ such that

$$h' = \big(z_1 - z_0 + \lfloor \mathbf{rz} + \alpha \rfloor - h\big)\, m + \big\lfloor r_0 z_1 + r_1 h - r_1 \lfloor \mathbf{rz} + \alpha \rfloor + \alpha \big\rfloor + b \in \{-1,0,1\},$$

i.e., $(z_1, h - \lfloor \mathbf{rz} + \alpha \rfloor, h') \in \tilde{\tau}_{\mathbf{r},\alpha}(\mathbf{z},h)$. If $(z_1, h - \lfloor \mathbf{rz} + \alpha \rfloor)$ is full, then we clearly have $\tilde{\tau}_{\mathbf{r},\alpha}(\mathbf{z},h) \subset V$. Otherwise, we have to consider the possible values of h'. We distinguish seven cases.

1. $\mathbf{z} \in \{(0,0),(-1,-1)\}$: We have $\lfloor \mathbf{rz} + \alpha \rfloor = 0$.

   If $m \geq 2$, then $(0,h)$ and $(-1,h)$ are full since $(0,-1) \in D_1$, $(0,1),(-1,1) \in A_1$, $(-1,0) \in A_0$, and $(0,0)$, $(-1,-1)$ are also full.

   If $m = 1$, then $(0,-1)$, $(0,0)$, $(-1,-1)$, $(-1,0)$ are full. For $h = 1$, we have $h' = -1 + \lfloor r_0 z_1 + r_1 + \alpha \rfloor + b = b - 2$, thus $h' = 1$. The points $(z_1, 1, -1)$ for $z_1 \in \{-1,0\}$ are in V.

2. $\mathbf{z} = (j,k) \in A_k$: We have $\lfloor \mathbf{rz} + \alpha \rfloor = -k$ for $j = -1$ and $\lfloor \mathbf{rz} + \alpha \rfloor = j - k$ for $0 \le j < k$.

   If $k = 0$, then $\mathbf{z} = (-1, 0)$, and $(0, h)$ is full for $m \ge 2$. If $m = 1$, then $(0,0)$ and $(0,-1)$ are full, and $h = 1$ gives that $h' = \lfloor r_1 + \alpha \rfloor + b = b - 1 \in \{-1, 0\}$, thus $\tilde{\tau}_{\mathbf{r},\alpha}(\mathbf{z}, 1) \subset V$.

   If $1 \le k < m$, then $(k, h+k)$, $(k, h-j+k)$ lie in $B_k \cup C_k \cup \{(k, k+1)\}$ and are full.

   If $k = m$, then we have either $h \in \{-1, 0\}$, thus $(m, h+m)$ and $(m, h-j+m)$ lie in the set of full points $B_m \cup C_m$, or $h = 1$ and $1 \le j < m$, in which case $(m, 1-j+m) \in B_m$ is also full. (Note that $(-1, m, 1), (0, m, 1) \notin V$.)

3. $\mathbf{z} = (k,j) \in B_k$: We have $\lfloor \mathbf{rz} + \alpha \rfloor = k - j$, $1 \le j \le k$.

   For $h \in \{0, 1\}$, the point $(j, h+j-k)$ is in C_k and $C_{k-1} \cup B_k$ respectively, hence full. The point $(j, j-k-1) \in C_{k+1}$ is full if $k < m$. Finally, if $k = m$ and $h = -1$, then $h' = m + \lfloor r_0 j + r_1(j-m-1) + \alpha \rfloor + b = 2m + 1 + b = 1$, and $(j, j-m-1, 1) \in V$.

4. $\mathbf{z} = (j, j-k) \in C_k$: We have $\lfloor \mathbf{rz} + \alpha \rfloor = k$, $1 \le j \le k$.

   The point $(j-k, h-k)$, $h \in \{-1, 0, 1\}$, is in D_{k+1}, D_k, and $D_{k-1} \cup E_{k-1} \cup \{(0,0), (-1,-1)\}$ respectively, hence full for all $k < m$, $k \le m$, and $k \le m+1$ respectively. It remains to consider $h = -1$, $k = m$. For $1 \le j \le m-3$, the point $(j-m, -m-1)$ is full; we have

$$h' = m + \lfloor r_0(j-m) - r_1(m+1) + \alpha \rfloor + b = \begin{cases} 2m+1+b = 1 & \text{if } j = m, \\ 2m + b \in \{0,1\} & \text{if } j \in \{m-2, m-1\}, \end{cases}$$

   $(0, -m-1, 1) \in V$, and $\{(j-m, -m-1)\} \times \{0,1\} \subset V$ for $\max(1, m-2) \le j < m$.

5. $\mathbf{z} = (-j, -k) \in D_k$: We have $\lfloor \mathbf{rz} + \alpha \rfloor = k - j$ if $j \in \{0, 1\}$, $\lfloor \mathbf{rz} + \alpha \rfloor = k - j + 1$ if $2 \le j < k$.

   Let first $k = 1$, i.e., $\mathbf{z} = (0, -1)$. The point $(-1, h-1)$ lies in $D_2 \cup \{(-1,-1)\} \cup A_0$ and is full, except if $m = 1$, $h = -1$; in the latter case, we have $h' = 1 + \lfloor -r_0 - 2r_1 + \alpha \rfloor + b = b + 2 \in \{0, 1\}$, and $\{(-1, -2)\} \times \{0, 1\} \in V$.

   For $2 \le k \le m$, the points $(-k, h+j-k)$, $j \in \{0,1\}$, and $(-k, h+j-k-1)$, $2 \le j < k$, lie in $\{(-k, -i) : 0 \le i \le k+1\}$, and are full, except for $k = m = 2$, $h = -1$, $j = 0$; in the latter case, we have $h' = 2 + \lfloor -2r_0 - 3r_1 + \alpha \rfloor + b = b + 4 \in \{0, 1\}$, and $\{(-2, -3)\} \times \{0, 1\} \in V$.

   Finally, for $k = m+1$, we have $j = 0$, $h = 1$, or $1 \le j \le \min(m, 2)$, $h \in \{0, 1\}$, or $3 \le j \le m$, $h \in \{-1, 0, 1\}$, thus the points $(-m-1, h+j-m-1)$, $j \in \{0, 1\}$, and $(-m-1, h+j-m-2)$, $2 \le j \le m$, lie in $\{(-m-1, -i) : \min(m-1, 1) \le i \le m\}$ and are full, except for $m = j = h = 1$; in the latter case, we have $h' = -1 + \lfloor -2r_0 + \alpha \rfloor + b = b - 2 = -1$, and $(-2, 0, -1) \in V$.

6. $\mathbf{z} = (-k, -j) \in E_k$: We have $\lfloor \mathbf{r}\mathbf{z} + \alpha \rfloor = j - k + 1$, $2 \le j \le k$.

   The point $(-j, h-j+k-1) \in F_{k-2} \cup F_{k-1} \cup F_k \cup \{(-k, -2)\}$ is full, except for $k = m+1$, $h = 1$; in the latter case, we have $2 \le j \le m$, $h' = \lfloor -r_0 j + r_1(m-j+1) + \alpha \rfloor + b = b - m \in \{-1, 0\}$, and $\{(-j, m-j+1)\} \times \{-1, 0\} \subset V$.

7. $\mathbf{z} = (-j, k-j) \in F_k$: We have $\lfloor \mathbf{r}\mathbf{z} + \alpha \rfloor = 1 - k$, $2 \le j \le k+1$.

   If $1 \le k \le m$, then the point $(k-j, h+k-1) \in A_{k-2} \cup A_{k-1} \cup A_k \cup \{(k-2, k-2)\}$ is full, except for $k = m$, $j \in \{m, m+1\}$, $h = 1$; in the latter case, we have $h' = \lfloor r_0(m-j) + r_1 m + \alpha \rfloor + b = b - m \in \{-1, 0\}$, and $\{(m-j, m)\} \times \{-1, 0\} \subset V$.

   If $k = m+1$, then $2 \le j \le m$, $h \in \{-1, 0\}$, or $m = 1$, $j = 2$, $h = -1$, and $(m+1-j, h+m) \in A_{m-1} \cup A_m \cup \{(m-1, m-1)\}$ is full. $\square$

Lemma 13. *For the following chains of sets, $\tau_{\mathbf{r},\alpha}$ maps elements of a set into its successor:*

$$C_k \setminus \{(m+1, 0)\} \to D_k \to E_k \to F_{k-1} \quad (3 \le k \le m+1),$$
$$F_{k+1} \to A_k \to B_k \to C_k \to D_k \quad (1 \le k \le m).$$

On the remaining $\mathbf{z} = (z_0, z_1) \in \mathbb{Z}^2$ with $-m-1 \le z_0 \le m$, $-m-1 \le z_1 \le m$ and $|z_0 - z_1| \le m+1$, $\tau_{\mathbf{r},\alpha}$ acts by

$$(0, -2) \mapsto (-2, -2) \mapsto (-2, -1) \mapsto (-1, 0) \mapsto (0, 0),$$
$$(-1, -2) \mapsto (-2, -1), \quad (0, -1) \mapsto (-1, -1) \mapsto (-1, 0).$$

Proof. This is a direct consequence of Lemma 12, except for $(-m-2, -1) \in F_{m+1}$; see also the proof of Lemma 11. As $\frac{1}{\beta+1} < \frac{(m+2)m}{\beta^2} + \frac{(m+1)m}{\beta^3} < 1 + \frac{m}{\beta^2} < 1 + \frac{1}{\beta+1}$, the proof of Lemma 12 shows that $\tau_{\mathbf{r},\alpha}(-m-2, -1) = (-1, m) \in A_m$. $\square$

Proposition 14. *We have*

$$\max\{\mathrm{fr}(x-y) : x, y \in \mathbb{Z}_{-\beta}\} = 3m + \begin{cases} 3 & \text{if } m = 1 \text{ or } m \text{ is even}, \\ 4 & \text{if } m \ge 3 \text{ is odd}. \end{cases}$$

Proof. Let $k \ge 0$ be such that $\frac{x}{(-\beta)^k}, \frac{y}{(-\beta)^k}, \frac{x-y}{(-\beta)^k} \in (\ell_\beta, \ell_\beta + 1)$. Then $\mathrm{fr}(x-y) = n$ is the minimal $n \ge 0$ such that $T_{-\beta}^{k+n}(\frac{x-y}{(-\beta)^k}) = 0$. Let $\mathbf{z} \in \mathbb{Z}^2$ be such that $T_{-\beta}^k(\frac{x-y}{(-\beta)^k}) = \phi(\mathbf{z})$. Then $\mathrm{fr}(x-y)$ is the minimal $n \ge 0$ such that $\tau_{\mathbf{r},\alpha}^n(\mathbf{z}) = 0$, and we have $(\mathbf{z}, 0) \in V$, i.e.,

$$\mathbf{z} \in \{(0,0), (-1,-1), (0,-1)\} \cup \bigcup_{0 \le k \le m} \big(A_k \cup B_k \cup C_k \cup D_{k+1} \cup E_{k+1} \cup F_{k+1}\big)$$
$$\setminus \{(0, -m-1), (-m-1, -m-1), (-m-2, -1), (-m-1, 0)\}.$$

Therefore, $\mathrm{fr}(x-y)$ is bounded by the maximal length of the path from $\mathbf{z}$ to $(0,0)$ given by Lemma 13.

For $1 \le k \le m/2$, the sets F_{2k+1}, A_{2k}, B_{2k}, and C_{2k} are mapped to D_2 in $6k-2$, $6k-3$, $6k-4$, and $6k-5$ steps respectively. For $2 \le k \le (m+1)/2$, the sets D_{2k} and E_{2k} are mapped to D_2 in $6k-6$ and $6k-7$ steps respectively. The points $(0,-2)$ and $(-1,-2)$ in D_2 are mapped to $(0,0)$ in 4 and 3 steps respectively.

Similarly, for $1 \le k \le (m+1)/2$, the sets F_{2k}, A_{2k-1}, B_{2k-1}, and C_{2k-1} are mapped to $D_1 = \{(0,-1)\}$ in $6k-2$, $6k-3$, $6k-4$, and $6k-5$ steps respectively. For $1 \le k \le m/2$, the sets D_{2k+1} and E_{2k+1} are mapped to D_1 in $6k$ and $6k-1$ steps respectively. Finally, the point $(0,-1) \in D_1$ is mapped to $(0,0)$ in 3 steps.

For even m, the longest path comes thus from D_{m+1} and has length $3m+3$. For odd $m \ge 3$, the longest path comes from F_{m+1} and has length $3m+4$. For $m=1$, the longest path comes from A_1 (since $F_2 \times \{0\} \cap V = \emptyset$ in this case) and has length 6. This proves the upper bound for $\mathrm{fr}(x-y)$.

For $m=1$, this bound is attained by $x = 1-\beta$, $y = \beta^4 - \beta^3$, since $\mathrm{fr}(x-y) = \mathrm{fr}(\frac{1}{\beta^3} - \beta^3) = \mathrm{fr}(\frac{1}{\beta^3}) = 6$. Assume in the following that $m \ge 2$. Then the points $(-m,-m-1,0) \in D_{m+1} \times \{0\}$ and $(-2,m-1,0) \in F_{m+1} \times \{0\}$ are in $\tilde{\tau}_{\mathbf{r},\alpha}^j(0,0,0)$ for sufficiently large j because they can be attained from $(0,0,0)$ by transitions

$$(\mathbf{z},h) \xrightarrow{b} \big(z_1, h - \lfloor \mathbf{rz} + \alpha \rfloor, (z_1 - z_0 + \lfloor \mathbf{rz} + \alpha \rfloor - h)\, m + \lfloor r_0 z_1 + r_1 h - r_1 \lfloor \mathbf{rz} + \alpha \rfloor + \alpha \rfloor + b\big)$$

with $b \in B$ by the following paths (cf. the proof of Lemma 11):

$$\begin{gathered}
(0,k,0) \xrightarrow{-1} (k,k,-1) \xrightarrow{-m-k-2} (k,-1,-1) \xrightarrow{-m-k-2} (-1,-k-2,-1), \quad (0 \le k \le m-2) \\
(-1,-k,-1) \xrightarrow{-m} (-k,-k,1) \xrightarrow{k} (-k,0,1) \xrightarrow{k} (0,k,0), \qquad (2 \le k \le m) \\
(0,m-1,0) \xrightarrow{-1} (m-1,m-1,-1) \xrightarrow{-2m} (m-1,-1,0) \xrightarrow{-m} (-1,-m,-1), \\
(0,m,0) \xrightarrow{0} (m,m,0) \xrightarrow{-m} (m,0,0) \xrightarrow{-m-1} (0,-m,-1), \\
(0,-m,-1) \xrightarrow{-m-2} (-m,-m-1,0) \xrightarrow{-1} (-m-1,-2,1) \xrightarrow{m} (-2,m-1,0).
\end{gathered}$$

For even $m \ge 2$, these paths correspond to

$$\begin{aligned}
&\mathrm{fr}(000022\,000044 \cdots 0000mm\,0000 \bullet 0^\omega - 022000\,044000 \cdots 0mm000\,0012 \bullet 0^\omega) \\
&\quad = \mathrm{fr}((1mmm00)^{m/2}\,0mmm \bullet d_{-\beta}(\phi(-m,-m-1))) = \mathrm{fr}(\phi(-m,-m-1)) = 3m+3;
\end{aligned}$$

for the second equality, we have used that $(1mmm00)^{m/2}\,0mmm \bullet d_{-\beta}(\phi(-m,-m-1))$ is a $(-\beta)$-expansion. Indeed, this follows from the lexicographic conditions given in [17] since $d_{-\beta}(\ell_\beta) = m0m^\omega$ and $d_{-\beta}(\phi(-m,-m-1))$ starts with 2 (as $-\beta\phi(-m,-m-1)-2 = \phi(-m-1,-2)$). For odd $m \ge 3$, we have

$$\begin{aligned}
&\mathrm{fr}(000022\,000044 \cdots 0000(m{-}1)(m{-}1)\,0000mm\,00000m \bullet 0^\omega \\
&\qquad\qquad - 022000\,044000 \cdots 0(m{-}1)(m{-}1)000\,0m0000\,001200 \bullet 0^\omega) \\
&\quad = \mathrm{fr}((1mmm00)^{(m+1)/2}\,0mmm10 \bullet d_{-\beta}(\phi(-2,m-1))) = \mathrm{fr}(\phi(-2,m-1)) = 3m+4.
\end{aligned}$$

This concludes the proof of the proposition. □

Proposition 15. *We have*

$$\max\{\mathrm{fr}(x+y) : x,y \in \mathbb{Z}_{-\beta}\} \le \max\{\mathrm{fr}(x-y) : x,y \in \mathbb{Z}_{-\beta}\}.$$

Proof. Let $\mu = \max\{\mathrm{fr}(x-y) : x,y \in \mathbb{Z}_{-\beta}\}$. For $x,y \in \mathbb{Z}_{-\beta}$, $\mathrm{fr}(x+y)$ is the minimal $n \ge 0$ such that $T_{-\beta}^{k+n}(\frac{x+y}{(-\beta)^k}) = 0$, with $k \ge 0$ such that $\frac{x}{(-\beta)^k}, \frac{y}{(-\beta)^k}, \frac{x+y}{(-\beta)^k} \in (\ell_\beta, \ell_\beta+1)$. By Lemma 11, we have

$$T_{-\beta}^j(\tfrac{x}{(-\beta)^k}) + T_{-\beta}^j(\tfrac{y}{(-\beta)^k}) - T_{-\beta}^j(\tfrac{x+y}{(-\beta)^k}) \in \Phi(V)$$

for all $j \ge 0$, thus $T_{-\beta}^k(\frac{x+y}{(-\beta)^k}) \in -\Phi(V)$. Therefore, we have $T_{-\beta}^k(\frac{x+y}{(-\beta)^k}) = \phi(\mathbf{z}) = -\Phi(-\mathbf{z},h)$ for some $\mathbf{z} = (z_0,z_1) \in \mathbb{Z}^2$ and $h \in \{0,1\}$ with $(-\mathbf{z},h) \in V$.

If $(\mathbf{z},0) \in V$, then the proof of Proposition 14 shows that $\tau_{\mathbf{r},\alpha}^\mu(\mathbf{z}) = \mathbf{0}$, thus $\mathrm{fr}(x+y) \le \mu$.

Assume now that $(\mathbf{z},0) \notin V$. Then

$$-\mathbf{z} \in D_{m+1} \cup \big\{(-m-1,-j) : 1 \le j \le m\big\} \cup \big\{(-j,m-j+1) : 1 \le j \le m\big\}.$$

We can exclude $-\mathbf{z} = (-j,m-j+1)$, $1 \le j \le m$, because this would imply $h=0$ and

$$-\Phi(-\mathbf{z},h) = 1 - \frac{jm}{\beta^2} - \frac{(m+1)m}{\beta^3} \ge \frac{m}{\beta} - \frac{m^2}{\beta^3} > \frac{1}{\beta+1}.$$

This means that $\mathbf{z} \in (A_{m+1} \cup B_{m+1}) \setminus \{(-1,m+1),(m+1,m+1)\}$. With the notation of Lemma 13, we have

$$A_{m+1} \setminus \{(-1,m+1)\} \to B_{m+1} \setminus \{(m+1,m+1)\} \to C_m,$$

where we have used Lemma 12 and that $\lfloor r_0(m+1) + r_1 j + \alpha \rfloor = m-j$ for $1 \le j \le m$, as

$$-\frac{\beta}{\beta+1} < \frac{m}{\beta} - \frac{m^2}{\beta^2} + \frac{m-m^2}{\beta^3} \le \frac{m}{\beta} - \frac{m^2}{\beta^2} + \frac{(j-m)m}{\beta^3} \le \frac{m}{\beta} - \frac{m^2}{\beta^2} < \frac{1}{\beta+1}.$$

Hence, the points in $A_{m+1} \setminus \{(-1,m+1)\}$ and $B_{m+1} \setminus \{(m+1,m+1)\}$ are mapped to $(0,0)$ in the same number of steps as those in A_m and B_m respectively, thus $\mathrm{fr}(x+y) \le \mu$. □

Now, Theorem 9 is an immediate consequence of Propositions 14 and 15.

Remark 16. *It is also possible to determine the exact value of* $\max\{\mathrm{fr}(x+y) : x,y \in \mathbb{Z}_{-\beta}\}$ *in the same fashion as in the proof of Proposition 14; we have*

$$\max\{\mathrm{fr}(x+y) : x,y \in \mathbb{Z}_{-\beta}\} = 3m + \begin{cases} 1 & \text{if } m = 2, \\ 2 & \text{if } m \ge 4 \text{ is even}, \\ 3 & \text{if } m = 1, \\ 4 & \text{if } m \ge 3 \text{ is odd}. \end{cases}$$

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