BETA-EXPANSIONS OF RATIONAL NUMBERS IN QUADRATIC PISOT BASES

TOMÁŠ HEJDA AND WOLFGANG STEINER

Abstract. We study rational numbers with purely periodic Rényi β-expansions. For bases β satisfying $\beta^2 = a\beta + b$ with b dividing a, we give a necessary and sufficient condition for that all rational numbers $p/q \in [0,1)$ with $\gcd(q,b) = 1$ have a purely periodic β-expansion. A simple algorithm for determining the infimum of $p/q \in [0,1)$ with $\gcd(q,b) = 1$ and with not purely periodic β-expansion is described that works for all quadratic Pisot numbers β.

1. Introduction and main results

Rényi β-expansions [Rén57] provide a very natural generalization of standard positional numeration systems such as the decimal system. Let $\beta > 1$ denote the base. Expansions of numbers $x \in [0,1)$ are defined in terms of the β-transformation

$$T\colon [0,1) \to [0,1),\ x \mapsto \beta x - \lfloor \beta x \rfloor.$$

The expansion of x is the infinite string $x_1x_2x_3\cdots$ where $x_j := \lfloor \beta T^{j-1}x \rfloor$. For $\beta \in \mathbb{N}$, we recover the standard expansions in base β and the β-expansion of $x \in [0,1)$ is eventually periodic (i.e., there exist p, n such that $x_{k+p} = x_k$ for all $k \geq n$) if and only if $x \in \mathbb{Q}$. This result was generalized to all Pisot bases by Schmidt [Sch80], who proved that for a Pisot number β the expansion of $x \in [0,1)$ is eventually periodic if and only if x is an element of the number field $\mathbb{Q}(\beta)$. Moreover, he showed that when β satisfies $\beta^2 = a\beta + 1$, then each $x \in [0,1) \cap \mathbb{Q}$ has a purely periodic β-expansion.

Akiyama [Aki98] showed that if β is a Pisot unit satisfying a certain finiteness property then there exists $c > 0$ such that all rational numbers $x \in \mathbb{Q} \cap [0,c)$ have a purely periodic expansion. If β is not a unit, then a rational number $p/q \in [0,1)$ can have a purely periodic expansion only if q is co-prime to the norm $N(\beta)$. Many Pisot non-units satisfy that there exists $c > 0$ such that all rational numbers $\frac{p}{q} \in [0,c)$ with q co-prime to $N(\beta)$ have a purely periodic expansion. This stimulates for the following definition:

Definition 1.1. Let β be a Pisot number, and let $N(\beta)$ denote the norm of β. Then we define $\gamma(\beta) \in [0,1]$ as the maximal c such that all $\frac{p}{q} \in \mathbb{Q} \cap [0,c)$

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with $\gcd(q, N(\beta)) = 1$ have a purely periodic β-expansion. In other words,

$$\gamma(\beta) \coloneqq \inf\Big\{ \tfrac{p}{q} \in \mathbb{Q} \cap [0,1) : \gcd(q, N(\beta)) = 1, \\ \tfrac{p}{q} \text{ has a not purely periodic expansion} \Big\} \cup \{1\}.$$

The question is how to determine the value of $\gamma(\beta)$. As well, knowing when $\gamma(\beta) = 0$ or 1 is of big interest. Values of $\gamma(\beta)$ for whole classes of numbers as well as for particular numbers have been given [Aki98, ABBS08, AS05, MS14, Sch80]. Periodic greedy expansions in negative quadratic unit bases were studied in [**?**].

It is easy to observe that the expansion of x is purely periodic if and only if x is a periodic point of T, i.e., there exists $p \geq 1$ such that $T^p x = x$. The natural extension $(\mathcal{X}, \mathcal{T})$ of the dynamical system $([0,1), T)$ (w.r.t. its unique absolutely continuous invariant measure) can be defined in an algebraic way, cf. §2.3. Several authors contributed to proving the following result: A point $x \in [0,1)$ has a purely periodic β-expansion if and only if $x \in \mathbb{Q}(\beta)$ and its diagonal embedding lies in the natural extension domain $\mathcal{X}$. The quadratic unit case was solved by Hama and Imahashi [HI97], the confluent unit case by Ito and Sano [IS01, IS02]. Then Ito and Rao [IR05] resolved the unit case completely using an algebraic argument. For non-unit bases β, one has to consider finite (p-adic) places of the field $\mathbb{Q}(\beta)$. This consideration allowed Berthé and Siegel [BS07] to expand the result to all (non-unit) Pisot numbers.

The first values of $\gamma(\beta)$ for two particular quadratic non-units were provided by Akiyama et al. [ABBS08]. Recently, Minervino and the second author [MS14] described the boundary of $\mathcal{X}$ for quadratic non-unit Pisot bases. This allowed them to find the value of $\gamma(\beta)$ for an infinite class of quadratic numbers. Namely, let β be the positive root of $\beta^2 = a\beta + b$ for $a \geq b \geq 1$ two co-prime integers; then

$$\gamma(\beta) = \begin{cases} 1 - \frac{(b-1)b\beta}{\beta^2 - b^2} & \text{if } a > b(b-1), \\ 0 & \text{otherwise} \end{cases}$$

(note that this value is 1 if and only if $b = 1$).

The purpose of this article is to generalize this result to all quadratic Pisot numbers β with norm $N(\beta) < 0$. (Note that when $N(\beta) > 0$, then β has a positive Galois conjugate $\beta' > 0$ and $\gamma(\beta) = 0$ by [Aki98, Proposition 5].) To this end, we define β-adic expansions (not to be confused with the Rényi β-expansions) similarly to p-adic expansions with $p \in \mathbb{Z}$, see also §2.4.

Definition 1.2. Let β be an algebraic integer. The *β-adic expansion* of $x \in \mathbb{Z}[\beta]$ is the unique infinite word $\boldsymbol{h}(x) \coloneqq u_0 u_1 u_2 \cdots$ such that $u_n \in \{0, 1, \ldots, |N(\beta)| - 1\}$ and $x - \sum_{i=0}^{n-1} u_i \beta^i \in \beta^n \mathbb{Z}[\beta]$ for all $n \in \mathbb{N}$.

Theorem 1. *Let β be a quadratic Pisot number, root of $\beta^2 = a\beta + b$ with $a \ge b \ge 1$. Then*

$$\gamma(\beta) = \begin{cases} 0 & \text{if } \sup_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j-\beta)}(\beta') > \beta \text{ or } \inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta') < -1, \\ \beta - a & \text{if } \sup_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j-\beta)}(\beta') \in (2\beta - a - 1, \beta] \\ & \text{and } \inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta') \ge \beta - a - 1, \\ 1 + \inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta') & \text{otherwise,} \end{cases}$$

where $P_{u_0u_1u_2\cdots}(X) \coloneqq \sum_{n\ge 0} u_n X^n$.

In many cases, we obtain the following direct formula (which we conjecture to be true for all $a \ge b \ge 1$):

Theorem 2. *Let β be a quadratic Pisot number, root of $\beta^2 = a\beta + b$ for $a \ge b \ge 1$. Suppose $a > \frac{1+\sqrt{5}}{2} b$ or $a = b$ or $\gcd(a,b) = 1$. Then*

$$\gamma(\beta) = \max\Big\{0, 1 + \inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta')\Big\}. \tag{1.1}$$

The infimum in (1.1) can be computed easily with the help of Proposition 3.2 below. In the case $\frac{a}{b} \in \mathbb{Z}$, Proposition 4.1 provides an even faster algorithm, and we are able to prove a necessary and sufficient condition for $\gamma(\beta) = 1$:

Theorem 3. *Let β be a quadratic Pisot number, root of $\beta^2 = a\beta + b$ with $a \ge b \ge 1$ and such that b divides a.*

(i) *We have $\gamma(\beta) = 1$ if and only if $a \ge b^2$ or $(a,b) \in \{(24,6),(30,6)\}$.*
(ii) *If $a = b \ge 3$ then $\gamma(\beta) = 0$.*

This paper is organized as follows: In the next section, notions on words, representation spaces and β-tiles are recalled, and properties of β-adic expansions are studied. Section 3 connects tiles arising from the β-transformation and the value $\gamma(\beta)$ in order to prove Theorem 1. The proof of Theorem 2 is completed in Section 4, together with that of Theorem 3. Comments on the general case are in Section 5, along with a list of related open questions.

2. Preliminaries

2.1. **Words over a finite alphabet.** We consider both finite and infinite words over a finite alphabet $\mathcal{A}$. The set of finite words over $\mathcal{A}$ is denoted $\mathcal{A}^*$. The set of all (right) infinite words over $\mathcal{A}$ is denoted $\mathcal{A}^\omega$, and it is equipped with the Cantor topology. An infinite word is *(eventually) periodic* if it is of the form $vu^\omega \coloneqq vuuu\cdots$; a finite word v is the pre-period and a non-empty finite word u is the period; if the pre-period is empty, we speak about a *purely periodic word.* A prefix of a (finite or infinite) word w is any finite word v such that w can be written as $w = vu$ for some word u. We denote by $\boldsymbol{u}[\![n]\!]$ the prefix of length n of an infinite word $\boldsymbol{u}$.

To a finite word $w = w_0w_1\cdots w_{k-1}$ we assign the polynomial

$$P_w(X) \coloneqq \sum_{i=0}^{k-1} w_i X^i.$$

Similarly, $P_{\boldsymbol{u}}(X) := \sum_{i\ge 0} u_i X^i$ is a power series for an infinite word $\boldsymbol{u} = u_0u_1u_2\cdots$.

2.2. **Representation spaces.** The following notation will be used: For integers $a, b \in \mathbb{Z}$, we denote by $a \perp b$ the fact that a and b are co-prime, i.e., that $\gcd(a,b) = 1$. Moreover, for $b \ge 2$ we put $\mathbb{Z}_b := \{\, p/q : p, q \in \mathbb{Z},\ q \perp b \,\}$ (the ring of rational numbers with denominator co-prime to b).

We adopt the notation of [MS14], however, we restrict ourselves to β being a quadratic Pisot number. Let $K = \mathbb{Q}(\beta)$. Since β is quadratic, there are exactly two infinite places of K; they are given by the two Galois isomorphisms of $\mathbb{Q}(\beta)$: the identity and $x \mapsto x'$ that maps β to its Galois conjugate. Both these places have $\mathbb{R}$ as their completion.

If β is not a unit, then we have to consider finite places of K as well. We define the ring K_f as the direct product $K_\mathrm{f} := \prod_{\mathfrak{p}|(\beta)} K_\mathfrak{p}$, where $\mathfrak{p}$ runs through all prime ideals of $\mathbb{Q}(\beta)$ that divide the principal ideal (β) and $K_\mathfrak{p}$ is the associate completion of $\mathbb{K}$; for a precise definition, we refer to [MS14, §2.2]. The direct products $\mathbb{K} := K \times K' \times K_\mathrm{f}$ and $\mathbb{K}' := K' \times K_\mathrm{f}$ are called *representation spaces.* We define the diagonal embeddings

$$\delta\colon \mathbb{Q}(\beta) \to \mathbb{K},\ x \mapsto (x, x', x_\mathrm{f}) \quad \text{and} \quad \delta'\colon \mathbb{Q}(\beta) \to \mathbb{K}',\ x \mapsto (x', x_\mathrm{f}),$$

where x_f is the vector of the embeddings of x into the spaces $K_\mathfrak{p}$. We put

$$S_\mathrm{f} := \overline{\{\, x_\mathrm{f} : x \in S \,\}} \quad \text{for any } S \subseteq K.$$

In particular, we consider $\mathbb{Z}[\beta]_\mathrm{f}$, which is a compact subset of K_f. Since multiplication by β_f is a contraction on K_f, we have that $\beta_\mathrm{f}^n \mathbb{Z}[\beta]_\mathrm{f} \to \{0_\mathrm{f}\}$ as $n \to \infty$.

2.3. **Beta-tiles.** For $x \in [0,1)$, we define the (reflected and translated) *β-tile* of x as the Hausdorff limit

$$\mathcal{Q}(x) := \lim_{k\to\infty} \delta'(x - \beta^k T^{-k}(x)) \subseteq \mathbb{K}'.$$

Note that the standard definition of a β-tile for $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$ is $\mathcal{R}(x) := \delta'(x) - \mathcal{Q}(x)$, see e.g. [MS14]. For a quadratic Pisot number β, root of $\beta^2 = a\beta + b$ with $a \ge b \ge 1$, we have that $\mathcal{Q}(x) = \mathcal{Q}(0)$ for $x < \beta - a$ and $\mathcal{Q}(x) = \mathcal{Q}(\beta - a)$ otherwise. The dynamical system $([0,1), T)$ admits $(\mathcal{X}, \mathcal{T})$ as its natural extension, where

$$\mathcal{X} := \Big([0, \beta - a) \times \mathcal{Q}(0)\Big) \cup \Big([\beta - a, 1) \times \mathcal{Q}(\beta - a)\Big) \subset \mathbb{K}$$

is a union of two suspensions of β-tiles and $\mathcal{T}(x,y) := \delta(\beta)(x,y) - \delta(\lfloor \beta x \rfloor)$. The natural extension domain is often required to be a closed set, but here it is more convenient to work with the one above, since the following result holds:

Proposition 2.1 ([HI97, IR05, BS07]). *For a Pisot number β, we have that x has a purely periodic β-expansion if and only if $x \in \mathbb{Q}(\beta)$ and $\delta(x) \in \mathcal{X}$.*

2.4. **Beta-adic expansions.** In Definition 1.2, β-adic expansions are defined on $\mathbb{Z}[\beta]$. By Lemma 2.2 below, we extend this definition to the closure $\mathbb{Z}[\beta]_{\mathrm{f}}$ similarly to the p-adic case. To this end, let

$$D\colon \mathbb{Z}[\beta]_{\mathrm{f}} \to \mathbb{Z}[\beta]_{\mathrm{f}}, \quad x \mapsto \beta_{\mathrm{f}}^{-1}(z - d(z)_{\mathrm{f}}),$$

where $d(x)$ is the unique digit $d \in \mathcal{A} \coloneqq \{0, 1, \dots, |N(\beta)| - 1\}$ such that $\beta_{\mathrm{f}}^{-1}(x - d_{\mathrm{f}})$ is in $\mathbb{Z}[\beta]_{\mathrm{f}}$. Such d exists because $\mathbb{Z}[\beta] = \mathcal{A} + \beta\mathbb{Z}[\beta]$. It is unique because $(c + \beta\mathbb{Z}[\beta])_{\mathrm{f}} \cap (d + \beta\mathbb{Z}[\beta])_{\mathrm{f}} \neq \emptyset$ implies $(\beta^{-1}(c-d))_{\mathrm{f}} \in \mathbb{Z}[\beta]_{\mathrm{f}}$ and thus $c \equiv d \pmod{N(\beta)}$ by the following lemma:

Lemma 2.2 ([MS14, Lemma 5.2 and Eq. (5.1)])**.** *For each $x \in \mathbb{Z}[\beta^{-1}] \setminus \mathbb{Z}[\beta]$ we have $x_{\mathrm{f}} \notin \mathbb{Z}[\beta]_{\mathrm{f}}$. There exists $k \in \mathbb{N}$ such that $\mathbb{Z}[\beta^{-1}] \cap \beta^k \mathcal{O} \subseteq \mathbb{Z}[\beta]$, where $\mathcal{O}$ is the ring of integers in $\mathbb{Q}(\beta)$.*

Lemma 2.3. *The β-adic expansion map $\boldsymbol{h}_{\mathrm{f}} : \mathbb{Z}[\beta]_{\mathrm{f}} \to \mathcal{A}^\omega$ defined by*

$$\boldsymbol{h}_{\mathrm{f}}(z) \coloneqq u_0 u_1 u_2 \cdots, \quad \text{where} \quad u_i \coloneqq d(D^i(z)),$$

is a homeomorphism. It satisfies that $\boldsymbol{h}_{\mathrm{f}}(x_{\mathrm{f}}) = \boldsymbol{h}(x)$ for all $x \in \mathbb{Z}[\beta]$.

Proof. The map $\boldsymbol{h}_{\mathrm{f}}$ is surjective because $\boldsymbol{h}_{\mathrm{f}}(P_{\boldsymbol{u}}(\beta_{\mathrm{f}})) = \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathcal{A}^\omega$. It is injective because $\boldsymbol{h}_{\mathrm{f}}(z) = \boldsymbol{u} = u_0 u_1 u_2 \cdots$ implies that $z \in \sum_{i=0}^{n-1} u_i \beta_{\mathrm{f}}^i + \beta_{\mathrm{f}}^n \mathbb{Z}[\beta]_{\mathrm{f}}$ for all n, thus $z = P_{\boldsymbol{u}}(\beta_{\mathrm{f}})$.

Since $\mathcal{O}_{\mathrm{f}}$ is open and $\mathbb{Z}[\beta^{-1}]_{\mathrm{f}} = K_{\mathrm{f}}$, we get from Lemma 2.1 that $\mathbb{Z}[\beta]_{\mathrm{f}} = \bigcup_{x \in \mathbb{Z}[\beta]} x_{\mathrm{f}} + \beta_{\mathrm{f}}^k \mathcal{O}_{\mathrm{f}}$ for some $k \in \mathbb{N}$, and therefore it is an open set as well. Then the preimage $\boldsymbol{h}_{\mathrm{f}}^{-1}(v\mathcal{A}^\omega) = P_v(\beta_{\mathrm{f}}) + \beta_{\mathrm{f}}^n \mathbb{Z}[\beta]_{\mathrm{f}}$ is open for any $v \in \mathcal{A}^*$. As the cylinders $\{\, v\mathcal{A}^\omega : v \in \mathcal{A}^* \,\}$ form a base of the topology of $\mathcal{A}^\omega$, the map $\boldsymbol{h}_{\mathrm{f}}$ is continuous.

The inverse $\boldsymbol{h}_{\mathrm{f}}^{-1}$ is continuous because $\beta_{\mathrm{f}}^n \mathbb{Z}[\beta]_{\mathrm{f}} \to \{0_{\mathrm{f}}\}$ as $n \to \infty$.

For $x \in \mathbb{Z}[\beta]$, the equality $\boldsymbol{h}_{\mathrm{f}}(x_{\mathrm{f}}) = \boldsymbol{h}(x)$ follows from the fact that $\beta^{-1}(x - d(x_{\mathrm{f}})) \in \mathbb{Z}[\beta]$. □

Note that we can also identify the set $\mathbb{Z}[\beta]_{\mathrm{f}}$ with the inverse limit space $\varprojlim \mathbb{Z}[\beta]/\beta^n \mathbb{Z}[\beta]$. Indeed, the map

$$\kappa\colon u_0 u_1 u_2 \cdots \mapsto (\xi_1, \xi_2, \xi_3, \dots), \quad \text{where} \quad \xi_n = \sum_{i=0}^{n-1} u_i \beta^i$$

is an isomorphism $\mathcal{A}^\omega \to \varprojlim \mathbb{Z}[\beta]/\beta^n \mathbb{Z}[\beta]$, and the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{Z}[\beta]_{\mathrm{f}} & \xrightarrow{\quad D \quad} & \mathbb{Z}[\beta]_{\mathrm{f}} \\
{\scriptstyle \boldsymbol{h}}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \boldsymbol{h}}\big\downarrow{\scriptstyle \cong} \\
\mathcal{A}^\omega & \xrightarrow{\quad \text{(shift)} \quad} & \mathcal{A}^\omega \\
{\scriptstyle \kappa}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \kappa}\big\downarrow{\scriptstyle \cong} \\
\varprojlim \mathbb{Z}[\beta]/\beta^n \mathbb{Z}[\beta] & \longrightarrow & \varprojlim \mathbb{Z}[\beta]/\beta^n \mathbb{Z}[\beta]
\end{array}$$

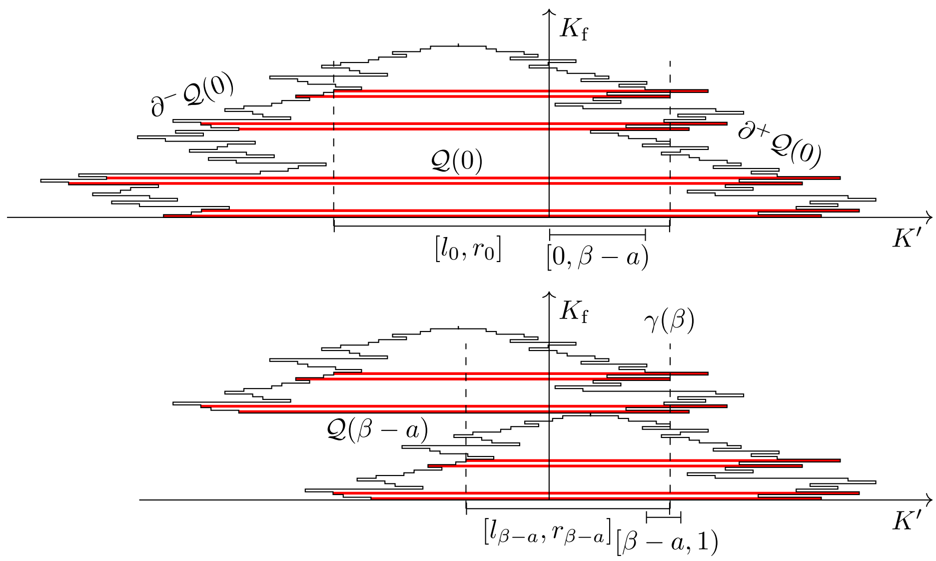

FIGURE 1. The tiles $\mathcal{Q}(0)$ and $\mathcal{Q}(\beta - a)$ for $\beta = 1 + \sqrt{3}$. The (red) stripes illustrate the intersection of $Y = K' \times (\mathbb{Z})_\mathrm{f}$ with the tiles.

3. Beta-tiles and the value $\gamma(\beta)$

The goal of this section is to prove Theorems 1 and 2, using the connection between β-tiles and the value of $\gamma(\beta)$. First we prove the following lemma about the closures of $\mathbb{Z}$ and $\mathbb{Z}_b$ in K_f:

Lemma 3.1. *We have that* $(\mathbb{Z})_\mathrm{f} = (\mathbb{Z}_b)_\mathrm{f} = (\mathbb{Z}_b \cap [c, d])_\mathrm{f}$ *for all* $c < d$.

Proof. We have that $(\mathbb{Z}_b)_\mathrm{f} = (\mathbb{Z}_b \cap [c, d])_\mathrm{f}$ by [ABBS08, Lemma 4.7]. Clearly $\mathbb{Z} \subseteq \mathbb{Z}_b$ whence $(\mathbb{Z})_\mathrm{f} \subseteq (\mathbb{Z}_b)_\mathrm{f}$. We will prove that $(\mathbb{Z}_b)_\mathrm{f} \subseteq (\mathbb{Z})_\mathrm{f}$, namely that every point $x/q \in \mathbb{Z}_b$ for $x, q \in \mathbb{Z}$ and $q \perp b$ can be approximated by integers. For each $n \in \mathbb{N}$, there exists $q_n \in \mathbb{Z}$ such that $q_n q \equiv 1 \pmod{b^n}$. Then $\frac{x}{q} - q_n x = (1 - q_n q)\frac{x}{q} \in \frac{1}{q} b^n \mathbb{Z} \subseteq \frac{1}{q}\beta^n \mathbb{Z}[\beta]$, therefore $(q_n x)_\mathrm{f} \to (x/q)_\mathrm{f}$. $\square$

Proof of Theorem 1. By Definition 1.1, Proposition **??** and since $\delta(1) \notin \mathcal{X}$, we have that

$$\gamma(\beta) = \inf\{\, x \in \mathbb{Z}_b : x \geq 0,\ \delta(x) \notin \mathcal{X} \,\}.$$

For $x \in \mathbb{Q} \cap [0, \beta - a)$, the condition $\delta(x) \in \mathcal{X}$ is equivalent to $\delta'(x) \in \mathcal{Q}(0)$; for $x \in \mathbb{Q} \cap [\beta - a, 1)$, it is equivalent to $\delta'(x) \in \mathcal{Q}(\beta - a)$.

We recall the results of [MS14, §9.3], where the shape of the tiles is described. The intersection of $\mathcal{Q}(x)$ with a line $K' \times \{z\}$ is a line segment for any $z \in \mathbb{Z}[\beta]_\mathrm{f}$ and it is empty for all $z \in K_\mathrm{f} \setminus \mathbb{Z}[\beta]_\mathrm{f}$, see Figure 1. Let $\partial^- \mathcal{Q}(x)$ denote the set of the segments' left end-points, and similarly $\partial^+ \mathcal{Q}(x)$ the set of the right end-points. For $x \in \{0, \beta - a\}$, put

$$l_x \coloneqq \sup \pi'(\delta^- \mathcal{Q}(x) \cap Y) \quad \text{and} \quad r_x \coloneqq \inf \pi'(\delta^+ \mathcal{Q}(x) \cap Y),$$

where $Y \coloneqq K' \times (\mathbb{Z}_b)_\mathrm{f}$ and π' denotes the projection $\pi' \colon K' \times K_\mathrm{f} \to K'$, $(y, z) \mapsto y$. Then all numbers $p/q \in \mathbb{Z}_b$ in $[l_0, r_0] \cap [0, \beta - a)$ have a purely periodic expansion, and so do all numbers $p/q \in \mathbb{Z}_b$ in $[l_{\beta-a}, r_{\beta-a}] \cap [\beta - a, 1)$.

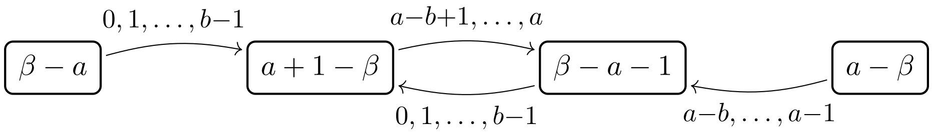

FIGURE 2. Boundary graph for quadratic β-tiles, cf. [MS14, Fig. 6]. Each arrow in the graph represents exactly b edges.

Outside these two sets, numbers $p/q \in \mathbb{Z}_b$ that do not have a purely periodic expansion are dense, since the points $\delta'(p/q)$ are dense in Y by Lemma 3.1. Therefore, the value of $\gamma(\beta)$ depends on the relative position of the above intervals (see Figure 1) in the following way:

(3.1)
$$\gamma(\beta) = \begin{cases} 0 & \text{if } l_0 > 0 \text{ or } r_0 < 0, \\ r_0 & \text{if } l_0 \le 0 \text{ and } r_0 \in [0, \beta - a), \\ \beta - a & \text{if } l_0 \le 0,\ r_0 \ge \beta - a \text{ and } \beta - a \notin [l_{\beta-a}, r_{\beta-a}], \\ \min\{r_{\beta-a}, 1\} & \text{if } l_0 \le 0,\ r_0 \ge \beta - a \text{ and } \beta - a \in [l_{\beta-a}, r_{\beta-a}]. \end{cases}$$

In the rest of the proof, we will show that

$$l_0 = l_{\beta-a} - 1 = -\beta + \sup_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j-\beta)}(\beta') \tag{3.2}$$

$$\text{and} \quad r_0 = r_{\beta-a} = 1 + \inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta'). \tag{3.3}$$

As $\inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta') \le P_{\boldsymbol{h}(0)}(\beta') = 0$, we see that (3.1) implies the statement of the theorem.

We use results of [MS14, §§8.3, 9.2 and 9.3], namely Equations (8.4) and (9.2), which read:

$$z \in \mathcal{R}(x) \cap \mathcal{R}(y) \quad \text{if and only if} \quad z = \delta'(x) + P_{\boldsymbol{u}}(\delta'(\beta)),$$

where $\boldsymbol{u} = u_0u_1u_2\cdots$ is an edge-labelling of a path in the boundary graph in Figure 2 that starts in the node $y - x$; and

$$\partial\mathcal{R}(x) = \Big(\mathcal{R}(x) \cap \mathcal{R}(x + \beta - \lfloor x + \beta \rfloor)\Big) \cup \Big(\mathcal{R}(x) \cap \mathcal{R}(x - \beta - \lfloor x - \beta \rfloor)\Big),$$

where the first part is the left boundary $\mathcal{R}^-(x)$ and the second part is the right boundary $\mathcal{R}^+(x)$. Therefore

$$\begin{aligned} \partial^-\mathcal{R}(0) = \partial^+\mathcal{R}(\beta - a) = \mathcal{R}(0) \cap \mathcal{R}(\beta - a) \\ = \{\, P_{\boldsymbol{u}}(\delta'(\beta)) : \boldsymbol{u} \in (\mathcal{AB})^\omega \,\}, \end{aligned}$$

$$\begin{aligned} \partial^+\mathcal{R}(0) = \mathcal{R}(a + 1 - \beta) \cap \mathcal{R}(0) \\ = \{\, \delta'(a + 1 - \beta) + P_{\boldsymbol{u}}(\delta'(\beta)) : \boldsymbol{u} \in (\mathcal{AB})^\omega \,\}, \end{aligned}$$

$$\begin{aligned} \partial^-\mathcal{R}(\beta - a) = \mathcal{R}(\beta - a) \cap \mathcal{R}(2\beta - \lfloor 2\beta \rfloor) \\ = \{\, \delta'(\beta - a) + P_{\boldsymbol{u}}(\delta'(\beta)) : \boldsymbol{u} \in (\mathcal{AB})^\omega \,\}, \end{aligned}$$

where we put $\mathcal{B} \coloneqq \{a-b+1, a-b+2, \dots, a\}$. We have that

$$\begin{aligned} \{\, P_{\boldsymbol{u}}(\delta'(\beta)) : \boldsymbol{u} \in (\mathcal{AB})^\omega \,\} = \{\, P_{((b-1)a)^\omega}(\delta'(\beta)) - P_{\boldsymbol{u}}(\delta'(\beta)) : \boldsymbol{u} \in \mathcal{A}^\omega \,\} \\ = -\delta'(1) - \{\, P_{\boldsymbol{u}}(\delta'(\beta)) : \boldsymbol{u} \in \mathcal{A}^\omega \,\}, \end{aligned}$$

since $\mathcal{A} = b-1-\mathcal{A}$ and $\mathcal{B} = a-\mathcal{A}$. Because $\mathcal{Q}(x) = \delta'(x) - \mathcal{R}(x)$, we have $\partial^\pm \mathcal{Q}(x) = \delta'(x) - \partial^\mp \mathcal{R}(x)$. We obtain

$$\begin{aligned}\partial^- \mathcal{Q}(0) &= \delta'(\beta-a) + \{\, P_{\boldsymbol{u}}(\delta'(\beta)) : \boldsymbol{u} \in \mathcal{A}^\omega \,\},\\ \partial^- \mathcal{Q}(\beta-a) &= \delta'(\beta-a+1) + \{\, P_{\boldsymbol{u}}(\delta'(\beta)) : \boldsymbol{u} \in \mathcal{A}^\omega \,\},\\ \partial^+ \mathcal{Q}(0) = \partial^+ \mathcal{Q}(\beta-a) &= \delta'(1) + \{\, P_{\boldsymbol{u}}(\delta'(\beta)) : \boldsymbol{u} \in \mathcal{A}^\omega \,\}.\end{aligned}$$

We have that

$$\begin{aligned}\delta'(1) + P_{\boldsymbol{u}}(\delta'(\beta)) \in Y \iff 1_{\mathrm{f}} + P_{\boldsymbol{u}}(\beta_{\mathrm{f}}) \in \mathbb{Z}_{\mathrm{f}}\\ \iff P_{\boldsymbol{u}}(\beta_{\mathrm{f}}) \in \mathbb{Z}_{\mathrm{f}} \iff \boldsymbol{u} \in \boldsymbol{h}_{\mathrm{f}}(\mathbb{Z}_{\mathrm{f}}),\end{aligned}$$

because $\boldsymbol{h}_{\mathrm{f}}(P_{\boldsymbol{u}}(\beta_{\mathrm{f}})) = \boldsymbol{u}$ and $\boldsymbol{h}_{\mathrm{f}}$ is a homeomorphism by Lemma 2.2. Then, since the map $\mathbb{Z}_{\mathrm{f}} \to K'$, $z \mapsto P_{\boldsymbol{h}_{\mathrm{f}}(z)}(\beta')$ is continuous, we get that

$$\inf \pi'(\partial^+ \mathcal{Q}(x) \cap Y) = 1 + \inf_{z \in \mathbb{Z}_{\mathrm{f}}} P_{\boldsymbol{h}_{\mathrm{f}}(z)}(\beta') = 1 + \inf_{j \in \mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta').$$

This justifies (**??**). Similarly, $\delta'(\beta-a) + P_{\boldsymbol{u}}(\delta'(\beta)) \in Y$ if and only if $\boldsymbol{u} \in \boldsymbol{h}_{\mathrm{f}}(\mathbb{Z}_{\mathrm{f}} - \beta_{\mathrm{f}})$, therefore

$$\sup \pi'(\partial^- \mathcal{Q}(\beta-a) \cap Y) - 1 = \sup \pi'(\partial^- \mathcal{Q}(0) \cap Y) = \beta' - a + \sup_{j \in \mathbb{Z}} P_{\boldsymbol{h}(j-\beta)}(\beta').$$

Since $\beta' - a = -\beta$, this justifies (**??**). □

Proof of Theorem 2, case $a > \frac{1+\sqrt{5}}{2} b$. Since $\beta' < 0$, we have that

$$\sup_{j \in \mathbb{Z}} P_{\boldsymbol{h}(j-\beta)}(\beta') \le \sup_{\boldsymbol{u} \in \mathcal{A}^\omega} P_{\boldsymbol{u}}(\beta') = P_{((b-1)0)^\omega}(\beta') = \frac{b-1}{1-(\beta')^2}.$$

We will show that this quantity is $< 2\beta - a - 1$. First, we derive, using $(\beta')^2 = a\beta' + b$, $\beta = a - \beta'$ and $1 - (\beta')^2 > 0$, that it is equivalent to

$$a + ab + \beta'(a^2 + a + 2b - 2) > 0. \tag{3.4}$$

We know that $\beta < a+1$, therefore $\beta = a + \frac{b}{\beta} > \frac{a(a+1)+b}{a+1}$ and $\beta' = -\frac{b}{\beta} > -\frac{(a+1)b}{a^2+a+b}$. As well, $a^2 + a + 2b - 2 > 0$, therefore we estimate

$$a + ab + \beta'(a^2 + a + 2b - 2) > \frac{ab^2((\frac{a}{b})^2 - \frac{a}{b} - 1) + b^2((\frac{a}{b})^2 + 2\frac{a}{b} - 2) + 2b}{a^2 + a + b}.$$

When $\frac{a}{b} > \frac{1+\sqrt{5}}{2}$, all three terms in the numerator are positive. Since the denominator is also positive, we get that $\sup_{j \in \mathbb{Z}} P_{\boldsymbol{h}(j-\beta)}(\beta') < 2\beta - a - 1$. Theorem 1 then implies (1.1). □

The proof of the case $a \perp b$ of Theorem 2 was given in [MS14, §9]. The proof of the case $a = b$ is given in the next section on page 9, because it falls under the case when b divides a.

The following proposition shows how to compute the infimum in Theorem 2 and thus the value of $\gamma(\beta)$ in a lot of (and possibly all) cases. Comments on the computation of $\gamma(\beta)$ by Theorem 1 are in Section 5. We recall that $\boldsymbol{u}[\![n]\!]$ denotes the prefix of $\boldsymbol{u}$ of length n.

$a/b=$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$b=1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	$\star$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	0	$\star$	1	1	1	1	1	1	1	1	1	1	1	1	1
4	0	$\star$	$\star$	1	1	1	1	1	1	1	1	1	1	1	1
5	0	$\star$	$\star$	$\star$	1	1	1	1	1	1	1	1	1	1	1
6	0	$\star$	$\star$	1	1	1	1	1	1	1	1	1	1	1	1
7	0	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1	1	1	1
8	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1	1	1
9	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1	1
10	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1	1
11	0	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1	1
12	0	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	1	1	1	1

TABLE 1. The values of $\gamma(\beta)$ for the case when b divides a. The star '$\star$' means that the value is strictly between 0 and 1.

Proposition 3.2. *Let $\beta^2 = a\beta + b$ with $a \geq b \geq 2$. Then for each $n \in \mathbb{N}$ we have*

$$\inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta') \in \min_{j\in\{0,1,\ldots,b^n-1\}} P_{\boldsymbol{h}(j)[\![n]\!]}(\beta') + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1]. \tag{3.5}$$

Lemma 3.3. *Let $x, y \in \mathbb{Z}[\beta]$ satisfy that $x - y \in b^n\mathbb{Z}[\beta]$. Then $\boldsymbol{h}(x)[\![n]\!] = \boldsymbol{h}(y)[\![n]\!]$.*

Proof. Since $b = \beta^2 - a\beta \in \beta\mathbb{Z}[\beta]$, we have that $x - y \in \beta^n\mathbb{Z}[\beta]$. Let $\boldsymbol{h}(x) = u_0u_1\cdots$. Then $x - \sum_{j=0}^{n-1} u_j\beta^j \in \beta^n\mathbb{Z}[\beta]$ and therefore $y - \sum_{j=0}^{n-1} u_j\beta^j \in \beta^n\mathbb{Z}[\beta]$, which means that $u_0\cdots u_{n-1}$ is a prefix of $\boldsymbol{h}(y)$. □

Proof of Proposition 3.2. Set $\mu_n := \min_{j\in\{0,1,\ldots,b^n-1\}} P_{\boldsymbol{h}(j)[\![n]\!]}(\beta')$. The statement actually consists of two inequalities, which will be proved separately. Let $j \in \mathbb{Z}$. Since $\boldsymbol{h}(j)[\![n]\!] = \boldsymbol{h}(j \bmod b^n)[\![n]\!]$ by Lemma 3.3 and since $\beta' < 0$, we have

$$P_{\boldsymbol{h}(j)}(\beta') \geq P_{\boldsymbol{h}(j)[\![n]\!](0(b-1)^\omega)}(\beta') \geq \mu_n + (\beta')^{n+1}\tfrac{b-1}{1-(\beta')^2} \quad \text{if } n \text{ is even,}$$
$$P_{\boldsymbol{h}(j)}(\beta') \geq P_{\boldsymbol{h}(j)[\![n]\!]((b-1)0)^\omega}(\beta') \geq \mu_n + (\beta')^{n}\tfrac{b-1}{1-(\beta')^2} \quad \text{if } n \text{ is odd.}$$

To prove the other inequality, let $k \in \{0, \ldots, b^n - 1\}$ be such that $\mu_n = P_{\boldsymbol{h}(k)[\![n]\!]}(\beta')$. Then

$$P_{\boldsymbol{h}(k)}(\beta') \leq P_{\boldsymbol{h}(k)[\![n]\!]((b-1)0)^\omega}(\beta') = \mu_n + (\beta')^{n}\tfrac{b-1}{1-(\beta')^2} \quad \text{if } n \text{ is even,}$$
$$P_{\boldsymbol{h}(k)}(\beta') \leq P_{\boldsymbol{h}(k)[\![n]\!](0(b-1)^\omega)}(\beta') = \mu_n + (\beta')^{n+1}\tfrac{b-1}{1-(\beta')^2} \quad \text{if } n \text{ is odd;}$$

this provides the upper bound on the infimum. □

4. THE CASE b DIVIDES a

In this section, we aim to prove Theorem 3, which deals with the particular case when b divides a. Table 1 shows whether $\gamma(\beta)$ is 0, 1 or strictly in between, for $b \leq 12$ and $a/b \leq 15$. The first non-trivial values are listed

a	b	$\gamma(\beta)$
2	2	$0.91480304419665\cdots$
6	3	$0.99296356010177\cdots$
8	4	$0.93354294467597\cdots$
12	4	$0.99989778900097\cdots$
10	5	$0.83415079417546\cdots$
15	5	$0.99530672367191\cdots$
20	5	$0.99999990711058\cdots$

a	b	$\gamma(\beta)$
12	6	$0.73611417827238\cdots$
18	6	$0.99389726639536\cdots$
14	7	$0.58490653345818\cdots$
21	7	$0.94452609461867\cdots$
28	7	$0.99798478808267\cdots$
35	7	$0.99998604176743\cdots$
42	7	$0.99999999999971\cdots$

TABLE 2. Numerical values of $\gamma(\beta)$, where $\beta^2 = a\beta + b$, that correspond to the first couple '$\star$' in Table 1.

in Table 2. The algorithm for obtaining these values is deduced from Theorem 2 (which covers all the cases when $\frac{a}{b} \in \mathbb{Z}$ since then either $a = b$ or $a \geq 2b > \frac{1+\sqrt{5}}{2}b$), and the following proposition, which improves the statement of Proposition 3.2.

Proposition 4.1. *Let $\beta^2 = a\beta + b$ with $a \geq b \geq 2$ and $\frac{a}{b} \in \mathbb{Z}$. Then for each $n \in \mathbb{N}$ we have*

$$\inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta') \in \min_{j\in\{0,1,\ldots,b^n-1\}} P_{\boldsymbol{h}(j)[\![2n]\!]}(\beta') + (\beta')^{2n} \tfrac{b-1}{1-(\beta')^2}[\beta', 0].$$

Lemma 4.2. *Let $\beta^2 = cb\beta + b$. Let $x, y \in \mathbb{Z}[\beta]$ satisfy that $x - y \in b^n\mathbb{Z}[\beta]$ for some $n \in \mathbb{N}$. Then $\boldsymbol{h}(x)[\![2n]\!] = \boldsymbol{h}(y)[\![2n]\!]$. Moreover, for all $x \in \mathbb{Z}[\beta]$ and $d \in \mathcal{A}$ there exists $y \in x + b^n\mathcal{A}$ such that $\boldsymbol{h}(y)[\![2n{+}1]\!] = \boldsymbol{h}(x)[\![2n]\!]d$.*

Proof. We have $\beta^2 = b(c\beta + 1) \in b\mathbb{Z}[\beta]$ and $b = \beta^2 - c(1 + c^2b)\beta^3 + c^2\beta^4 \in \beta^2 + \beta^3\mathbb{Z}[\beta] \subseteq \beta^2\mathbb{Z}[\beta]$, whence $\beta^2\mathbb{Z}[\beta] = b\mathbb{Z}[\beta]$ and $\beta^{2n}\mathbb{Z}[\beta] = b^n\mathbb{Z}[\beta]$ for all $n \in \mathbb{N}$. Following the lines of the proof of Lemma 3.3, we obtain that if $x - y \in b^n\mathbb{Z}[\beta]$ then $\boldsymbol{h}(x)$ and $\boldsymbol{h}(y)$ have a common prefix of length at least $2n$.

To prove the second statement, put $u_0u_1\cdots := \boldsymbol{h}(x)$. Since $b^n \in \beta^{2n} + \beta^{2n+1}\mathbb{Z}[\beta]$, we have that $u_0u_1\cdots u_{2n-1}d$ is a prefix of $\boldsymbol{h}(x + eb^n)$ for any $e \equiv d - u_{2n} \pmod{b}$. □

Proof of Proposition 4.1. We follow the lines of the proof of Proposition 3.2 for the case n even. The lower bound is the same in both statements, therefore we only need to prove that $\inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta') \leq P_{\boldsymbol{h}(k)[\![2n]\!]}(\beta')$, where $k := \arg\min_{j\in\{0,\ldots,b^n-1\}} P_{\boldsymbol{h}(j)[\![2n]\!]}(\beta')$. For each $m \in \mathbb{N}$, there exists $k_m \in \mathbb{Z}$ such that $\boldsymbol{h}(k_m)[\![2n{+}2m]\!] \in \boldsymbol{h}(k)[\![2n]\!](0\mathcal{A})^m$ by Lemma 4.2. Then

$$\inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta') \leq \inf_{m\in\mathbb{N}} P_{\boldsymbol{h}(k_m)}(\beta') \leq \inf_{m\in\mathbb{N}} P_{\boldsymbol{h}(k)[\![n]\!]0^{2m}((b-1)0)^\omega}(\beta') = P_{\boldsymbol{h}(k)[\![n]\!]}(\beta').$$

□

Remark 4.3. We have that

$$\mu_n := \min_{j\in\{0,1,\ldots,b^n-1\}} P_{\boldsymbol{h}(j)[\![2n]\!]}(\beta') = \min_{j\in J_{n-1}+b^{n-1}\mathcal{A}} P_{\boldsymbol{h}(j)[\![2n]\!]}(\beta'), \tag{4.1}$$

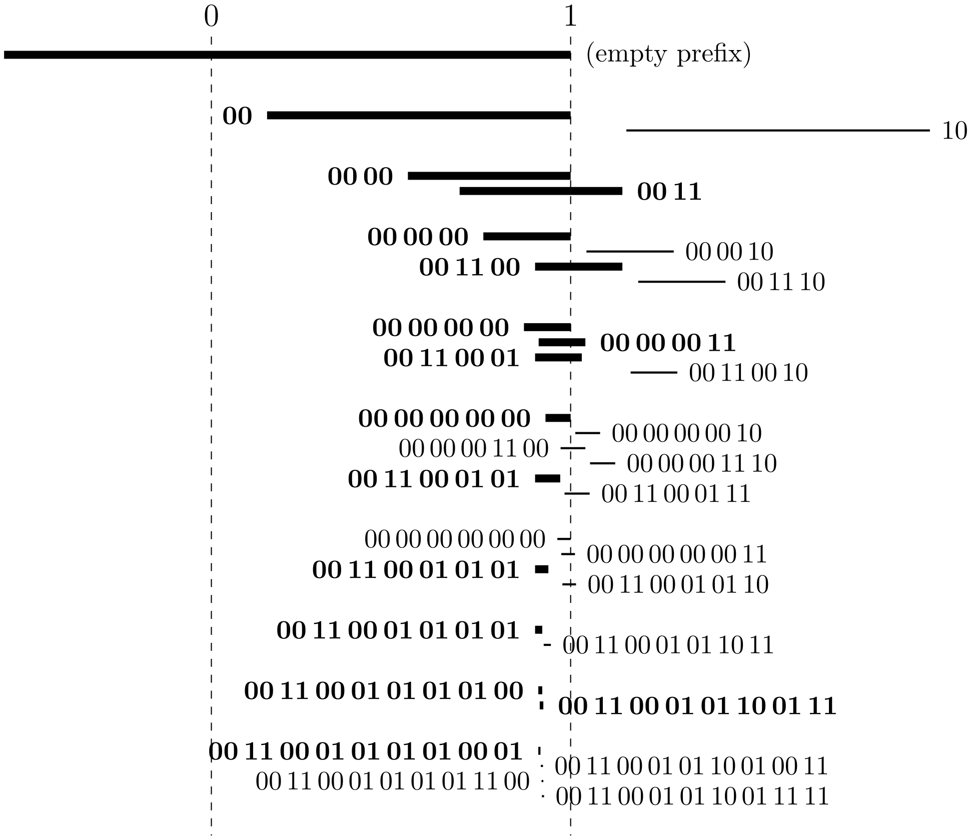

FIGURE 3. The computation of $\gamma(1+\sqrt{3})$. By a thick line with a bold label we denote the intervals that we 'keep' (these arise from numbers in J_n), by a thin line the ones that we 'forget'. The labels next to the intervals are the corresponding prefixes $\boldsymbol{h}(j)[\![2n]\!]$.

where

$$J_0 \coloneqq \{0\},$$
$$J_n \coloneqq \Big\{ j \in J_{n-1} + b^{n-1}\mathcal{A} : P_{\boldsymbol{h}(j)[\![2n]\!]}(\beta') < \mu_n + |\beta'|^{2n+1} \tfrac{b-1}{1-(\beta')^2} \Big\}.$$

To verify (4.1), we first show that the sequence $(\mu_n)_{n\in\mathbb{N}}$ is non-increasing. Let $j \in \{0,\dots,b^n-1\}$ be such that $\mu_n = P_{\boldsymbol{h}(j)[\![2n]\!]}(\beta')$. Then by Lemma 4.2 there exists $d \in \mathcal{A}$ such that $\boldsymbol{h}(j+db^n)[\![2n+1]\!] = \boldsymbol{h}(j)[\![2n]\!]0$, whence $\mu_{n+1} \le P_{\boldsymbol{h}(j+db^n)[\![2n+2]\!]}(\beta') \le \mu_n$.

Suppose now that $j \in \{0,\dots,b^n-1\} \setminus (J_{n-1} + b^{n-1}\mathcal{A})$. Then there exists $m < n$ such that $P_{\boldsymbol{h}(j)[\![2m]\!]}(\beta') \ge \mu_m + |\beta'|^{2m+1} \frac{b-1}{1-(\beta')^2}$, therefore $P_{\boldsymbol{h}(j)[\![2n]\!]}(\beta') > \mu_m \ge \mu_n$.

Example 4.4. As an example, the computation of $\gamma(\beta)$ for $\beta = 1+\sqrt{3}$, the Pisot root of $\beta^2 = 2\beta + 2$, is visualized in Figure 3. For each step of the algorithm, the value of $\gamma(\beta)$ lies in the left-most interval. Already in the 5th step we obtain that $\gamma(\beta) \in [0.900834, 0.970552]$, therefore it is strictly between 0 and 1. Note that in the 9th step we have that $\mu_9 = P_{t^{(9)}}(\beta')$ with $t^{(9)} = 001100010101010001$, and $\gamma(\beta) \in [0.910126652, 0.915876683]$. In the 40th step, we have that

$$t^{(40)} = 001100(01)^4 000100(0001)^4(00)^2(01)^5(00)^3(01)^6(00)^2 01$$

and $\gamma(\beta) \approx 0.914803044$.

Proof of Theorem 2, case $a = b$. Take $a = b \geq 4$. Then $b = \beta^2 + (b-1)\beta^3 + (2b+1)\beta^4$, therefore $\boldsymbol{h}(b)[\![4]\!] = 001(b-1)$. According to Proposition 4.1, we have that

$$A := \inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta') \leq P_{001(b-1)}(\beta') = (\beta')^2 + (b-1)(\beta')^3.$$

For $a = b \geq 5$, we use the estimate $-\beta' \in (\frac{b}{b+1}, 1)$ to obtain that $A < 1 - \frac{b^3(b-1)}{(b+1)^3} < -1$, therefore $\gamma(\beta) = 0$. For $a = b = 4$, we have $P_{001(b-1)}(\beta') \approx -1.0193$, thus $A < -1$.

When $a = b = 3$, we verify that $\boldsymbol{h}(21)[\![12]\!] = 001200020201$ and Proposition 4.1 yields $A \leq P_{001200020201}(\beta') \approx -1.0726 < -1$, therefore $\gamma(\beta) = 0$.

When $a = b = 2$, we can follow the lines of the proof of the case $a > \frac{1+\sqrt{5}}{2}b$, because we observe that (3.3) is satisfied, namely $6 + 8\beta' \approx 0.1436 > 0$. □

The proof of Theorem 3 is divided into several cases.

Proof of Theorem 3, case $a \geq b^2$. Any $j \in \mathbb{Z} \setminus \{0\}$ can be written as $j = b^n(j_0 + j_1 b)$, where $n \in \mathbb{N}$, $j_0 \in \mathcal{A}\setminus\{0\}$ and $j_1 \in \mathbb{Z}$. Then $\boldsymbol{h}(j)[\![2n+1]\!] = 0^{2n} j_0$ becase $b^n \in \beta^{2n} + \beta^{2n+1}\mathbb{Z}[\beta]$, whence

$$\begin{aligned} P_{\boldsymbol{h}(j)}(\beta') &\geq P_{\boldsymbol{h}(j)[\![2n+1]\!]((b-1)0)^\omega}(\beta') \geq P_{0^{2n}1((b-1)0)^\omega}(\beta') \\ &= (\beta')^{2n}\Big(1 + \frac{(b-1)\beta'}{1-(\beta')^2}\Big) = (\beta')^{2n}\Big(1 - \frac{(b-1)b\beta}{\beta^2 - b^2}\Big) > 0, \end{aligned}$$

where the last inequality was already proved in [MS14, Theorem 6]. As $\boldsymbol{h}(0) = 0^\omega$, we have $P_{\boldsymbol{h}(0)}(\beta') = 0$. From Theorem 2 we conclude that $\gamma(\beta) = 1 + \inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta') = 1$. □

The remaining cases of the proof of Theorem 3 make use of the following relations. Let $c := a/b \in \mathbb{Z}$. Then $\frac{b}{\beta^2} = \frac{1}{1+c\beta} \in 1 - c\beta + c^2\beta^2 - c^3\beta^3 + \beta^4\mathbb{Z}[\beta]$, and more generally,

$$\frac{b^n}{\beta^{2n}} \in 1 - nc\beta + \binom{n+1}{2}c^2\beta^2 - \binom{n+2}{3}c^3\beta^3 + \beta^4\mathbb{Z}[\beta] \quad \text{for any } n \in \mathbb{N}. \tag{4.2}$$

For $j = (j_0 + j_1 b)b^n$ with $n \in \mathbb{N}$, and $j_0, j_1 \in \mathbb{Z}$ we have that $\frac{j}{\beta^{2n}} = j_0\frac{b^n}{\beta^{2n}} + j_1\beta^2\frac{b^{n+1}}{\beta^{2n+2}}$, therefore

$$\begin{aligned} \frac{j}{\beta^{2n}} \in j_0 - j_0 nc\beta + \Big(j_0\binom{n+1}{2}c^2 + j_1\Big)\beta^2 \\ - \Big(j_0\binom{n+2}{3}c^3 + j_1(n+1)c\Big)\beta^3 + \beta^4\mathbb{Z}[\beta]. \end{aligned} \tag{4.3}$$

Proof of Theorem 3, case $\beta^2 = 30\beta + 6$. We have $b = 6$ and $c = 5$. As in the proof of the previous case, we will show that $P_{\boldsymbol{h}(j)}(\beta') \geq 0$ for all $j \in \mathbb{Z}$. Let $j \neq 0$ be written as $j = b^n(j_0 + j_1 b)$ with $j_0 \in \mathcal{A} \setminus \{0\}$ and $j_1 \in \mathbb{Z}$, then $\boldsymbol{h}(j) = 0^{2n} u_0 u_1 u_2 \cdots$ for some $u_0 u_1 \cdots \in \mathcal{A}^\omega$ with $u_0 = j_0$, and $P_{\boldsymbol{h}(j)}(\beta') = (\beta')^{2n} P_{u_0 u_1 \cdots}(\beta')$. We consider the following cases:

- If $u_0 \geq 2$, then $P_{u_0 u_1 \cdots}(\beta') \geq P_{2(50)^\omega}(\beta') > 0$.
- If $u_0 = 1$ and $u_1 \leq 4$, then $P_{u_0 u_1 \cdots}(\beta') \geq P_{14(05)^\omega}(\beta') > 0$.
- If $u_0 u_1 = 15$, then (4.3) yields that $j_0 = 1$ and $-j_0 nc \equiv 5 \pmod 6$, therefore $n \equiv -1 \pmod 6$ and $n = 6n_1 - 1$, i.e., $-j_0 nc\beta = 5\beta - 30n_1\beta \in 5\beta - 5n_1\beta^3 + \beta^4\mathbb{Z}[\beta]$. Therefore

$$\frac{j}{\beta^{2n}} \in 1 + 5\beta + \left(\binom{6n_1}{2} 5^2 + j_1\right)\beta^2 - \left(\tfrac{(6n_1+1)6n_1(6n_1-1)}{6} 5^3 + 30n_1 j_1 + 5n_1\right)\beta^3 + \beta^4\mathbb{Z}[\beta].$$

  The coefficient of β^3 is congruent to 0 modulo 6 regardless of the values of n_1 and j_1. This means that $u_3 = 0$. Then $P_{15u_2 0(05)^\omega}(\beta') \geq P_{1500(05)^\omega}(\beta') > 0$.

Therefore we have $P_{\boldsymbol{h}(j)}(\beta') \geq 0$ for all $j \in \mathbb{Z}$. □

Proof of Theorem 3, case $\beta^2 = 24\beta + 6$. We have $b = 6$ and $c = 4$. We use the same technique as in the case $\beta^2 = 30\beta + 6$.

- If $u_0 \geq 2$, then $P_{u_0 u_1 \cdots}(\beta') \geq P_{2(50)^\omega}(\beta') > 0$.
- If $u_0 = 1$ and $u_1 \leq 3$, then $P_{u_0 u_1 \cdots}(\beta') \geq P_{13(05)^\omega}(\beta') > 0$.
- Since c is even, we get that $u_1 \equiv -j_0 nc \pmod 6$ is even, therefore $u_0 u_1 \neq 15$.
- If $u_0 u_1 = 14$, then (4.3) gives $j_0 = 1$ and $-j_0 nc \equiv 4 \pmod 6$, i.e., $n \equiv -1 \pmod 3$ and $n = 3n_1 - 1$, whence $-j_0 nc\beta = 4\beta - 12n_1\beta \in 4\beta - 2n_1\beta^3 + \beta^4\mathbb{Z}[\beta]$. We derive that

$$\frac{j}{\beta^{2n}} \in 1 + 4\beta + (\text{some integer})\beta^2 - \left(144n_1^3 - 30n_1 + 12n_1 j_1\right)\beta^3 + \beta^4\mathbb{Z}[\beta].$$

  As above, we get that $u_3 = 0$ regardless of the values of n_1 and j_1, thus $P_{u_0 u_1 \cdots}(\beta') \geq P_{1400(05)^\omega}(\beta') > 0$. □

Proof of Theorem 3, case $c := a/b < b$ and $c \notin \{4, 5\}$ when $b = 6$. Let $n := \lceil \frac{c}{b-c} \rceil$. From (4.2), the β-adic expansion $\boldsymbol{h}(b^n)$ starts with $0^{2n}1(nb-nc)$. If $\frac{c}{b-c} \notin \mathbb{Z}$, then we have $nb - nc > c$ and thus $P_{1(nb-nc)}(\beta') \leq 1 + (c+1)\beta' < 0$, using that $\beta' = -\frac{b}{\beta} < -\frac{b}{cb+1} \leq -\frac{1}{c+1}$. By Proposition 4.1, this proves that $\gamma(\beta) < 1$ if c is not a multiple of $b - c$.

Assume now that $\frac{c}{b-c} \in \mathbb{Z}$, i.e., $n = \frac{c}{b-c}$. For $j := b^n - \binom{n+1}{2} c^2 b^{n+1}$, we have by (4.3) that

$$\frac{j}{\beta^{2n}} \in 1 - nc\beta - \left(\binom{n+2}{3} c^3 - \binom{n+1}{2} c^3 (n+1)\right)\beta^3 + \beta^4\mathbb{Z}[\beta].$$

Since $-nc = c - nb \in c - n\beta^2 + \beta^3\mathbb{Z}[\beta]$ and $(n+1)c = nb \in \beta\mathbb{Z}[\beta]$, we obtain that

$$\frac{j}{\beta^{2n}} \in 1 + c\beta - \left(\binom{n+2}{3}c^3 + n\right)\beta^3 + \beta^4\mathbb{Z}[\beta].$$

If $\binom{n+2}{3}c^3 + n \not\equiv 0 \pmod{b}$, then

$$P_{\boldsymbol{h}(j)[\![2n+4]\!]}(\beta') \le P_{0^{2n}1c01}(\beta') = \frac{(\beta')^{2n+2}}{b} + (\beta')^{2n+3} = (\beta')^{2n+2}\frac{\beta - b^2}{b\beta} < 0,$$

since $1 + c\beta' = \frac{(\beta')^2}{b}$ and $\beta < a + 1 \le b^2$, therefore $\gamma(\beta) < 1$ by Proposition 4.1.

It remains to consider the case that $\binom{n+2}{3}c^3 + n \equiv 0 \bmod b$, i.e.,

$$n \equiv -\frac{bn(n+2)}{6}c^2 n \bmod b,$$

because $(n+1)c = nb$. Multiplying by $b - c$ gives

$$c \equiv -\frac{bn(n+2)}{6}c^3 \bmod b.$$

Note that $\frac{bn(n+2)}{6} = (b-c)\binom{n+2}{3} \in \mathbb{Z}$. We distinguish four cases:

(i) If $6 \perp b$, then $c \equiv 0 \bmod b$, contradicting that $1 \le c < b$.

(ii) If $2 \mid b$ and $3 \nmid b$, then c is a multiple of $b/2$, i.e., $c = b/2$, $n = 1$. As n is also a multiple of $b/2$, we get that $b = 2$, thus $c = 1$. For $\beta^2 = 2\beta + 2$, we already know that $\gamma(\beta) < 1$, see Example 4.4.

(iii) If $3 \mid b$ and $2 \nmid b$, then c and n are multiples of $b/3$. For $c = b/3$ we have $n \notin \mathbb{Z}$. For $c = 2b/3$, we have $n = 2$, thus $b \in \{3, 6\}$. However, $b = 6$ contradicts $2 \nmid b$ and $b = 3$ (i.e., $c = 2$) contradicts $\binom{n+2}{3}c^3 + n \equiv 0 \bmod b$.

(iv) If $6 \mid b$, then c and n are multiples of $b/6$, thus $c \in \{b/2, 2b/3, 5b/6\}$, $n \in \{1, 2, 5\}$. If $n = 1$, then $b = 6$, thus $c = 3$, and $\binom{n+2}{3}c^3 + n \not\equiv 0 \bmod b$. If $n = 2$, then $b \in \{6, 12\}$; we have excluded that $b = 6$, $c = 4$; for $b = 12$, $c = 8$, we have $\binom{n+2}{3}c^3 + n \not\equiv 0 \bmod b$. If $n = 5$, then $b \in \{6, 30\}$; we have excluded that $b = 6$, $c = 5$; for $b = 30$, $c = 24$, we have $\binom{n+2}{3}c^3 + n \not\equiv 0 \bmod b$. □

5. The general case

In the general quadratic case where $1 < \gcd(a, b) < b$, the conditions of Theorem 2 need not be satisfied. This means that we have to rely on the more general Theorem 1, i.e., to compute the two values $\inf_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j)}(\beta')$ and $\sup_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j-\beta)}(\beta')$.

We can derive, in a similar manner to Proposition 3.2, that for all $n \in \mathbb{N}$,

$$\sup_{j\in\mathbb{Z}} P_{\boldsymbol{h}(j-\beta)}(\beta') \in \max_{j\in\{0,1,\dots,b^n-1\}} P_{\boldsymbol{h}(j-\beta)[\![n]\!]}(\beta') + (\beta')^n \tfrac{b-1}{1-(\beta')^2}[\beta', 1]. \tag{5.1}$$

Let now $s_n \ge 1$, for $n \in \mathbb{N}$, denote the smallest positive integer such that $s_n \in \beta^n\mathbb{Z}[\beta]$, and $r_n := \frac{s_n}{s_{n-1}}$. Then $x, y \in \mathbb{Z}$ have a common prefix of length n if and only if $y - x \in s_n\mathbb{Z}$. Therefore, in both (3.4) and (5.1) we can take $\{0, 1, \ldots, s_n - 1\}$ instead of $\{0, 1, \ldots, b^n - 1\}$. Moreover, following Remark 4.3, we can further restrict to the sets

$$J_0 \coloneqq \{0\}, \quad J'_0 \coloneqq \{-\beta\},$$
$$J_n \coloneqq \Big\{ j \in J_{n-1} + s_{n-1}\{0, \ldots, r_n - 1\} : P_{\boldsymbol{h}(j)[\![n]\!]}(\beta') \le \mu_n + |\beta'|^n \tfrac{b-1}{1+\beta'} \Big\},$$
$$J'_n \coloneqq \Big\{ j \in J_{n-1} + s_{n-1}\{0, \ldots, r_n - 1\} : P_{\boldsymbol{h}(j)[\![n]\!]}(\beta') \ge \nu_n - |\beta'|^n \tfrac{b-1}{1+\beta'} \Big\},$$

where we denote

$$\mu_n \coloneqq \min_{j \in \{0,1,\ldots,b^n-1\}} P_{\boldsymbol{h}(j)[\![n]\!]}(\beta') \quad \text{and} \quad \nu_n \coloneqq \max_{j \in \{0,1,\ldots,b^n-1\}} P_{\boldsymbol{h}(j-\beta)[\![n]\!]}(\beta').$$

We conclude by several open questions that arise in the study of rational numbers with purely periodic expansions:

(A) Prove or disprove that $\gamma(\beta) = 1$ for a quadratic Pisot number $\beta > 1$, root of $\beta^2 = a\beta + b$, if and only if $\frac{a}{b} \in \mathbb{Z}$ and $a \ge b^2$ or $(a, b) \in \{(24, 6), (30, 6)\}$.

(B) For which quadratic β we have that $\gamma(\beta) = 0$? Can we drop the restrictions on a and b in Theorem 2? More specifically, is it true that $a < \frac{1+\sqrt{5}}{2} b$ implies $\gamma(\beta) = 0$?

(C) What is the structure of the prefixes of β-adic expansions of integers for a general quadratic β?

(D) What about the cubic Pisot case? Akiyama and Scheicher [AS05] showed how to compute the value $\gamma(\beta)$ for $\beta \approx 1.325$ the minimal Pisot number (or Plastic number), root of $\beta^3 = \beta + 1$. Loridant et al. [LMST13] gave the contact graph of the β-tiles for cubic units, which could be used to determine $\gamma(\beta)$ for the units, in a similar way to what Akiyama and Scheicher did. The consideration of the β-adic spaces could then allow the results to be expanded to non-units as well.

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