Regularity and Optimization , Part 2

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INRIA

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Sturmian Words: 3 equivalent definitions

Consider an infinite word:

$$001010010010100101 00100\ldots$$

- minimal complexity : $n+1$ factors of length n.
  example: 4 factors of length 3: 001, 010, 100 and 101.
- balanced : number of 1 only differ by 1 in factors of same length.
  - length 3: 1 or 2.
  - length 4: 1 or 2.
  - ...
- mechanical:
  - for all i: $w_i = \lfloor \alpha * (i+1) + \theta \rfloor - \lfloor \alpha * i + \theta \rfloor$
    or for all i: $w_i = \lceil \alpha * (i+1) + \theta \rceil - \lceil \alpha * i + \theta \rceil$

Problem

Can we extend theses notions to trees?

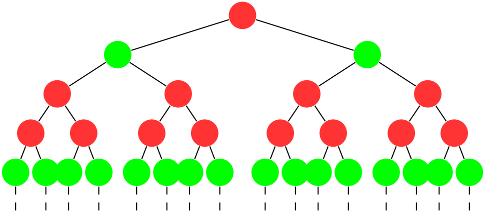
- sturmian
- balanced
- mechanical

Previous Work

Definition (Berstel, Boasson, Carton and Fagnot, 2007)

A Sturmian tree is a tree with $n+1$ subtrees of size n.

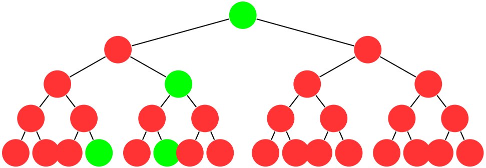
Simple example:



Example: The uniform tree corresponding to 0100101 . . .

Properties

- There is a natural bijection between binary trees and binary languages.
- This provides interesting examples, like the Dyck tree.



Example: Dyck Trees

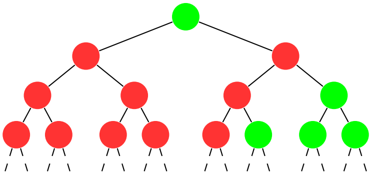
But

- the *balanced* property is lost (important in optimization problems),
- no simple equivalent characterization.

Infinite Labeled Binary Trees

Our trees are:

- rooted
- labeled by 0 or 1
- infinite
- Non-planar
  ($\neq$ original definition of Sturmian Trees)



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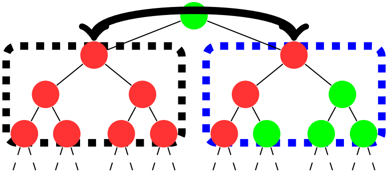

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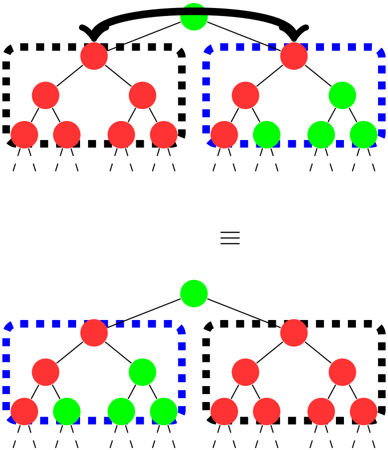
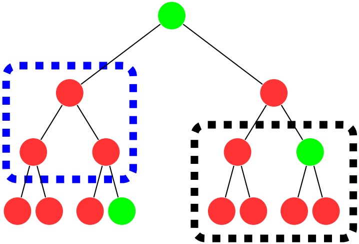


Subtrees and Density

We define:

- Subtree of height n.
- Difftree of width k and height n.
- Density of a subtree = average number of 1.
- If d_n is the density of the rooted subtree of height n:
  - density $= \lim_n d_n$
  - average density $= \lim_n \frac{1}{n} \sum_{k=1}^{n} d_k$



First simple case

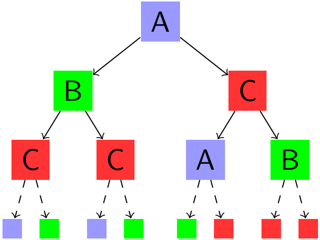
What is a non-planar Rational Tree?

Rational Trees: Definition

We call $P(n)$ = number of subtrees of size n.

Rational Trees: 3 equivalent definitions:

- $P(n)$ bounded.
- $\exists n / P(n) = P(n+1)$
- $\exists n / P(n) \leq n$.



Rational Trees: average Density

Theorem

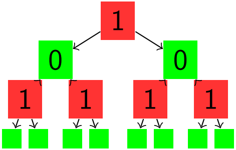
- *A rational Tree has an average density* α *which is rational.*

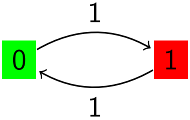
  α *is not necessarily a density but:*

- *If the associated Markov chain is aperiodic then there exists a density.*

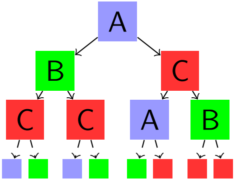
Example of density

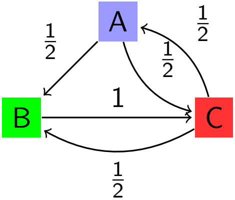
- Periodic = average density $d_{\text{average}} = \frac{1}{2}$





- Aperiodic : density $d = \frac{2}{9}\ell_A + \frac{1}{3}\ell_B + \frac{4}{9}\ell_C$





Second case

Balanced and Mechanical Trees

Balanced Trees and Strongly Balanced Trees

- Balanced tree: number of 1 in subtrees of height n only differ by 1.
- Strongly balanced tree: same property with difftrees of height n and width k.

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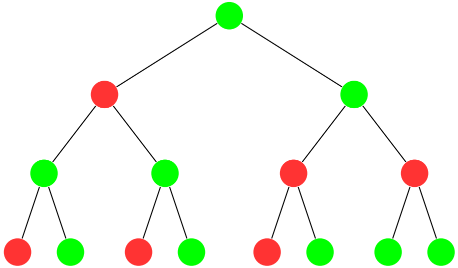

Example: Balanced tree not strongly balanced

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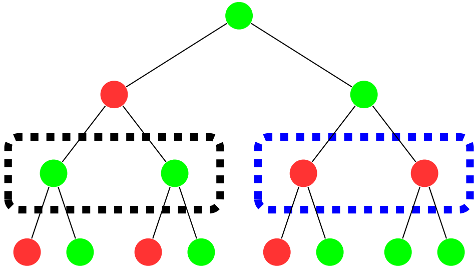

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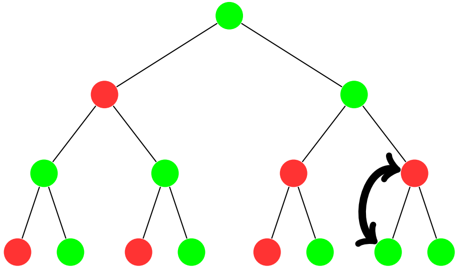

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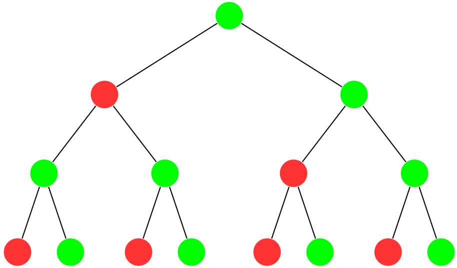


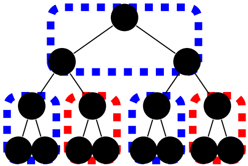
Example: Strongly balanced tree

Density of a Balanced Tree

Theorem

- *A balanced tree has a density.*

Sketch of the proof

1. A tree of size n has a density α_n or $\alpha_n + \frac{1}{2^n - 1}$
2. 

   If blue has a density α_2 and red $\alpha_2 + \frac{1}{3}$ then $\alpha_2 \leq \alpha_4 \leq \alpha_2 + \frac{1}{3}$
3. Take limits.

Mechanical Trees

- Subtree of size n has $2^n - 1$ nodes.
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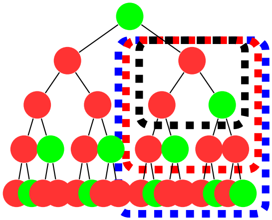

$\alpha = 0.3$, $\phi = 0.55$.

n	$(2^n - 1)\alpha + \phi$
1	0.85
2	1.45
3	2.65
4	5.05

Mechanical tree of density α:

- For all node i, there is a phase $\phi_i \in [0; 1)$ such that the number of 1 in a subtree of height n and root i is $\lfloor (2^n - 1)\alpha + \phi_i \rfloor$ (resp. for all i: $\lceil (2^n - 1)\alpha + \phi_i \rceil$)

Uniqueness of a mechanical Tree

Theorem

- *There exists a unique mechanical tree if* (α, ϕ_0) *is fixed.*

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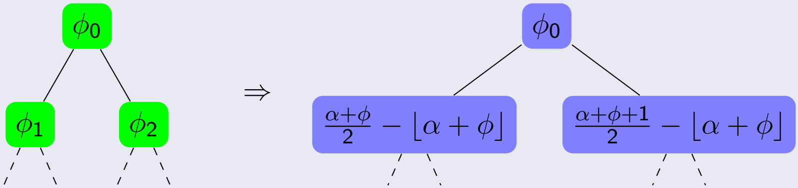


- *The phase* ϕ_0 *of the root is unique, for almost all* α.

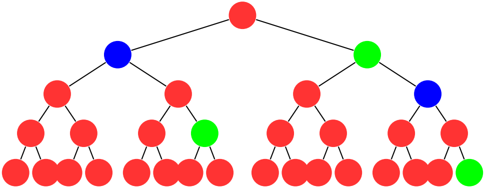
Equivalences?

What are the equivalences between definitions?

Equivalences between Definitions

Theorem (Mechanical $\sim$ strongly balanced)

- *A mechanical tree is strongly balanced*
- *A strongly balanced tree with irrational density is mechanical*
- *A strongly balanced tree with rational density is ultimately mechanical.*



Example: Ultimately mechanical tree

Sketch of Proof

Mechanical implicates strongly balanced.

The number of 1 in a subtree of size n and width k is bounded by $\lfloor (2^n - 2^k)\alpha \rfloor$ and $\lfloor (2^n - 2^k)\alpha \rfloor + 1$ □

Strongly Balanced implicates mechanical.

$\forall \tau \in [0;1)$, if h_n is the number of 1 in the subtree of size n, at least one of these properties is true:

1. for all n: $h_n \leq \lfloor (2^n - 1)\alpha + \tau \rfloor$,
2. for all n: $h_n \geq \lfloor (2^n - 1)\alpha + \tau \rfloor$.

We choose ϕ the maximal τ such that 1 is true. □

Theorem

- *An irrational mechanical tree is a sturmian tree: it has $n+1$ subtrees of height n.*

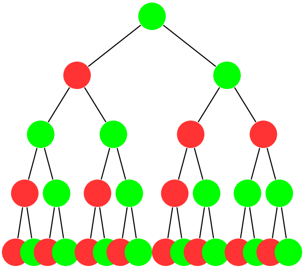
Proof.

- A subtree of size n depends only on its phase
- In fact, it depends on $((2^1-1)\alpha+\phi,\dots,(2^n-1)\alpha+\phi)$ which takes $n+1$ values when $\phi \in [0;1)$.

□

Limit of the Equivalences

- Balanced $\not\Rightarrow$ strongly balanced (whether the density is rational or not).
- Sturmian $\not\Rightarrow$ balanced.
- Irrational Balanced tree $\not\Rightarrow$ sturmian.



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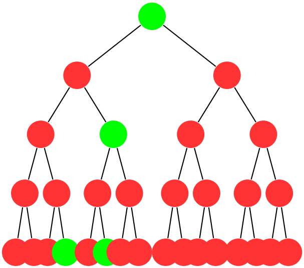

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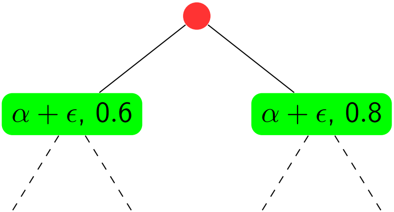

Example: Balanced tree non sturmian

Optimization Issues

Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a convex function. For each node n and each height $k > 0$, we define a cost $C_{[n,k]}$:

$$C_{[n,k]} = g(d(\mathcal{A}_{[n,k]})).$$

cost of order k of the tree is:

$$C_k = \limsup_{\ell} \frac{\sum_{n \ \mathcal{A}_{[0,l]}} C_{[n,k]}}{2^{\ell} - 1}.$$

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This has potential applications in optimization problem in distributed systems with a binary causal structure and generalizes some results presented in Part 1, based on the same principle.

Conclusion

- Non-planar definition better?
- Constructive definition
- Strict inclusions
- Good characterization

but:

- what are exactly balanced trees?
- how many balanced trees of size n?