On the regularity of the generalised golden ratio function

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Abstract

Given a finite set of real numbers A, the generalised golden ratio is the unique real number $\mathcal{G}(A) > 1$ for which we only have trivial unique expansions in smaller bases, and have non-trivial unique expansions in larger bases. We show that $\mathcal{G}(A)$ varies continuously with the alphabet A (of fixed size). What is more, we demonstrate that as we vary a single parameter m within A, the generalised golden ratio function may behave like $m^{1/h}$ for any positive integer h. These results follow from a detailed study of $\mathcal{G}(A)$ for ternary alphabets, building upon the work of Komornik, Lai, and Pedicini (2011). We give a new proof of their main result, that is we explicitly calculate the function $\mathcal{G}(\{0,1,m\})$. (For a ternary alphabet, it may be assumed without loss of generality that $A = \{0,1,m\}$ with $m \in (1,2)$].) We also study the set of $m \in (1,2]$ for which $\mathcal{G}(\{0,1,m\}) = 1 + \sqrt{m}$, we prove that this set is uncountable and has Hausdorff dimension 0. We show that the function mapping m to $\mathcal{G}(\{0,1,m\})$ is of bounded variation yet has unbounded derivative. Finally, we show that it is possible to have unique expansions as well as points with precisely two expansions at the generalised golden ratio.

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1 Introduction and statement of results

Let $A := \{a_0, a_1, \ldots, a_d\}$ be a set of real numbers satisfying $a_0 < a_1 < \cdots < a_d$. We call A an alphabet. Given $\beta > 1$ and $x \in \mathbb{R}$, we say that a sequence $(u_k)_{k=1}^{\infty} \in A^{\mathbb{N}}$ is a β-expansion for x over the alphabet A if

$$x = \sum_{k=1}^{\infty} \frac{u_k}{\beta^k}.$$

When the underlying alphabet is obvious we may simply refer to (u_k) as a β-expansion. Expansions in non-integer bases were introduced by Rényi [8]. Perhaps the most well studied case is when $\beta \in (1,2]$ and $A = \{0,1\}$. For $\beta \in (1,2]$ and this choice of alphabet, x has a β-expansion over A if and only if $x \in [0, \frac{1}{\beta-1}]$. Moreover, a result of Erdős, Joó, and Komornik [3] states that if $\beta \in (1, \frac{1+\sqrt{5}}{2})$ then every $x \in (0, \frac{1}{\beta-1})$ has a continuum of β-expansions. This result is complemented by a theorem of Daróczy and Katai [2] which states that if $\beta \in (\frac{1+\sqrt{5}}{2}, 2]$ then there exists $x \in (0, \frac{1}{\beta-1})$ with a unique β-expansion. Note that the end points of the interval $[0, \frac{1}{\beta-1}]$ trivially have a unique β-expansion for any $\beta \in (1,2]$. The above demonstrates that the golden ratio acts as a natural boundary between the possible cardinalities the set of expansions can take. It is natural to ask whether such a boundary exists for more general alphabets.

Before we state the definition of a generalised golden ratio it is necessary to define the univoque set. Given an alphabet A and $\beta > 1$ we set

$$\mathcal{U}_\beta(A) := \Big\{(u_k)_{k=1}^\infty \in A^{\mathbb{N}} : \sum_{k=1}^{\infty} \frac{u_k}{\beta^k} \text{ has a unique } \beta\text{-expansion}\Big\}.$$

We call $\mathcal{U}_\beta(A)$ the univoque set. Note that for any $\beta > 1$ and alphabet $A = \{a_0, \ldots, a_d\}$ satisfying $a_0 < a_1 < \cdots < a_d$, the points

$$\sum_{k=1}^{\infty} \frac{a_0}{\beta^k} \text{ and } \sum_{k=1}^{\infty} \frac{a_d}{\beta^k}$$

both have a unique expansion, so $\overline{a_0}$ and $\overline{a_d}$ are always contained in the univoque set. Here and throughout $\overline{w}$ denotes the infinite periodic word with period w. We are now in a position to define a generalised golden ratio for an arbitrary alphabet. Given an alphabet A, we call $\mathcal{G}(A) \in (1, \infty)$ the generalised golden ratio for A if whenever $\beta \in (1, \mathcal{G}(A))$ we have $\mathcal{U}_\beta(A) = \{\overline{a_0}, \overline{a_d}\}$, and if $\beta > \mathcal{G}(A)$ then $\mathcal{U}_\beta(A)$ contains a non-trivial element.

Komornik, Lai, and Pedicini [4] were the first authors to make a thorough study of generalised golden ratios over arbitrary alphabets. Importantly they proved that for any alphabet A a generalised golden ratio exists. For ternary alphabets, they showed that the generalised golden ratio varies continously with the alphabet. We extend this result to alphabets of arbitrary size.

Theorem 1. *Let $\Delta_d := \{(a_0, a_1, \ldots, a_d) \in \mathbb{R}^{d+1} : a_0 < a_1 < \cdots < a_d\}$, $d \geq 1$. The map $(a_0, a_1, \ldots, a_d) \mapsto \mathcal{G}(\{a_0, a_1, \ldots, a_d\})$ is continuous on Δ_d.*

We prove this theorem in Section 2. In the rest of the paper, we restrict our attention to ternary alphabets. Every ternary alphabet can be assumed to be of the form $A = \{0, 1, m\}$ for some $m > 1$ because shifting the alphabet and multiplying by a constant does not affect the generalised golden ratio. We thus set

$$\mathcal{G}(m) := \mathcal{G}(\{0, 1, m\}) \text{ and } \mathcal{U}_\beta(m) := \mathcal{U}_\beta(\{0, 1, m\}).$$

Moreover,

$$\mathcal{G}(m) = \mathcal{G}(\{0, 1, m\}) = \mathcal{G}(\{-m, 1-m, 0\}) = \mathcal{G}\Big(\Big\{\frac{m}{m-1}, 1, 0\Big\}\Big) = \mathcal{G}\Big(\frac{m}{m-1}\Big).$$

By the above $\mathcal{G}(m) = \mathcal{G}(\frac{m}{m-1})$ and we may therefore assume $m \in (1, 2]$. The authors of [4] considered $m \geq 2$, and their results read as follows in our setting.

Theorem KLP. *The function $\mathcal{G} : (1, 2] \to \mathbb{R}$ is continuous and satisfies*

$$2 \leq \mathcal{G}(m) \leq 1 + \sqrt{m}$$

for all $m \in (1, 2]$. Moreover, the following statements hold.

- *$\mathcal{G}(m) = 2$ for $m \in (1, 2]$ if and only if $m = \frac{2^k}{2^k - 1}$ for some positive integer k.*
- *The set $\mathfrak{M} := \{m \in (1, 2] : \mathcal{G}(m) = 1 + \sqrt{m}\}$ is a Cantor set, its largest element is $x^2 \approx 1.7548$ where $x \approx 1.3247$ is the smallest Pisot number.*
- *Each connected component (m_1, m_2) of $(1, x^2) \setminus \mathfrak{M}$ has a point μ such that $\mathcal{G}$ is strictly decreasing on $[m_1, \mu]$ and strictly increasing on $[\mu, m_2]$; $\mathcal{G}$ is strictly increasing on $[x^2, 2]$.*

In Section 3, we reprove all these results, making some of the statements more explicit and simplifying several proofs. An approximation of the graph of $\mathcal{G}$ can be found in Figure 3.1. The function $\mathcal{G}$ is given by implicit equations on subintervals of $(1, 2]$, and it has the following unusual regularity properties.

Theorem 2. *The function $\mathcal{G} : (1, 2] \to \mathbb{R}$ is differentiable except on the set $\mathfrak{M}$ and on the countable set of points μ defined in Theorem KLP. Its derivative is unbounded, but its total variation is less than 2.*

We have the following result on the size of $\mathfrak{M}$.

Theorem 3. *The set $\mathfrak{M}$ is an uncountable Cantor set with Hausdorff dimension* 0.

On certain intervals, the function $\mathcal{G}$ has the following simple form.

Theorem 4. *Let h be a positive integer and $2^h \le m \le \big(1+\sqrt{\frac{m}{m-1}}\big)^h$. Then we have*

$$\mathcal{G}(m) = \mathcal{G}\Big(\frac{m}{m-1}\Big) = m^{1/h}.$$

Note that if $m = 2^h$ then $m < \big(1+\sqrt{\frac{m}{m-1}}\big)^h$, thus the set of m defined within Theorem 4 is non-empty.

Moreover, we have the following result on the size of the set of expansions at the generalised golden ratio.

Theorem 5. *There exists $m \in (1,2]$ such that:*

- *$\mathcal{U}_{\mathcal{G}(m)}(m)$ contains non-trivial elements.*
- *When $\beta = \mathcal{G}(m)$ there exists an x with precisely two β-expansions.*

Finally, we remark that the problem of calculating $\mathcal{G}(A)$ remains wide open. Only $\mathcal{G}(\{0,1,\dots,m\})$ has been calculated for any positive integer m in [1].

2 Continuity of $\mathcal{G}(A)$

Before proving Theorem 1, we recall results of Pedicini [7, Proposition 2.1 and Theorem 3.1] on (unique) expansions in non-integer bases over arbitrary alphabets; see also [4, Theorem 2.2].

Theorem P. *Let $\beta \in (1, q(A)]$, with*

$$q(A) := 1 + \frac{a_d - a_0}{\max\{a_1-a_0, a_2-a_1, \dots, a_d-a_{d-1}\}}.$$

Every $x \in [\frac{a_0}{\beta-1}, \frac{a_d}{\beta-1}]$ has a β-expansion over A. We have $u_1u_2\cdots \in \mathcal{U}_\beta(A)$ if and only if, for all $i \ge 1$,

$$\sum_{k=0}^{\infty} \frac{u_{i+k}}{\beta^k} < a_{j+1} + \frac{a_0}{\beta-1} \qquad \text{when } u_i = a_j \ne a_d, \tag{2.1}$$

and

$$\sum_{k=0}^{\infty} \frac{u_{i+k}}{\beta^k} > a_{j-1} + \frac{a_d}{\beta-1} \qquad \text{when } u_i = a_j \ne a_0. \tag{2.2}$$

Remark 2.1. The conditions (2.1) and (2.2) can be restated in terms of uniqueness regions E_a: Let

$$E_{a_0} = \Big[\frac{a_0\beta}{\beta-1}, a_1+\frac{a_0}{\beta-1}\Big),\ E_{a_j} = \Big(a_{j-1}+\frac{a_d}{\beta-1}, a_{j+1}+\frac{a_0}{\beta-1}\Big),\ 1 \le j < d,\ E_{a_d} = \Big(a_{d-1}+\frac{a_d}{\beta-1}, \frac{a_d\beta}{\beta-1}\Big].$$

Then (2.1) and (2.2) hold if and only if $\sum_{k=0}^{\infty} \frac{u_{i+k}}{\beta^k} \in E_{u_i}$.

By the following lemma, it is sufficient to consider $\beta \le q(A)$.

Lemma 2.2. *We have $\mathcal{G}(A) \le q(A)$.*

Proof. If $\beta > 1 + \frac{a_d-a_0}{a_{j+1}-a_j}$ for some $0 \le j < d$, then $a_j + \frac{a_d}{\beta-1} < a_{j+1} + \frac{a_0}{\beta-1}$ and thus $a_j\,\overline{a_d} \in \mathcal{U}_\beta(A)$. ☐

Remark 2.3. This upper bound is attained for certain alphabets. For example, let $A = \{0,1,4,5\}$. For $\beta = q(A) = 8/3$, the uniqueness regions are $E_0 = [0,1)$, $E_1 = (3,4)$, $E_4 = (4,5)$ and $E_5 = (7,8]$. If $\sum_{k=0}^{\infty} \frac{u_{i+k}}{\beta^k} \in E_{u_i}$, then $\sum_{k=0}^{\infty} \frac{u_{i+1+k}}{\beta^k} \in (E_{u_i} - u_i)\beta$; the latter intervals are $[0,8/3)$, $(16/3,8)$, $(0,8/3)$ and $(16/3,8]$ respectively. Therefore, the only unique expansions are $\overline{0}$ and $\overline{5}$.

Proof of Theorem 1. As $\mathcal{G}(A) = \mathcal{G}\big(\frac{A-a_0}{a_d-a_0}\big)$, we have $\mathcal{G}(\{a_0, a_1, \ldots, a_d\}) = \mathcal{G}(\iota \circ r(a_0, a_1, \ldots, a_d))$, with

$$\begin{aligned} r : \Delta_d \to \Delta'_d, \quad & (a_0, a_1, \ldots, a_d) \mapsto \Big(\frac{a_1 - a_0}{a_d - a_0}, \frac{a_2 - a_0}{a_d - a_0}, \ldots, \frac{a_{d-1} - a_0}{a_d - a_0}\Big), \\ \iota : \Delta'_d \to \mathcal{P}(\mathbb{R}), \quad & (a_1, a_2, \ldots, a_{d-1}) \mapsto \{0, a_1, a_2, \ldots, a_{d-1}, 1\}, \end{aligned}$$

and $\Delta'_d = \{(a_1, a_2 \ldots, a_{d-1}) \in \mathbb{R}^{d-1} : 0 < a_1 < a_2 < \cdots < a_{d-1} < 1\}$. As r is continuous on Δ_d, it is sufficient to prove that $\mathcal{G} \circ \iota$ is continuous on Δ'_d.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_{d-1}) \in \Delta'_d$ and $\varepsilon > 0$ arbitrary but fixed. We will show that $|\mathcal{G}(\iota(\mathbf{b})) - \mathcal{G}(\iota(\mathbf{a}))| \le 3\varepsilon$ for all $\mathbf{b}$ in a neighbourhood of $\mathbf{a}$. Let first $X \subset \Delta'_d$ be a closed neighbourhood of $\mathbf{a}$ such that $|q(\iota(\mathbf{b})) - q(\iota(\mathbf{a}))| \le \varepsilon$ for all $\mathbf{b} \in X$. (Note that $q \circ \iota$ is continuous on Δ'_d.) Set

$$\alpha = \min_{\mathbf{b} \in X} q(\iota(\mathbf{b})) - \varepsilon, \qquad Y = \{\mathbf{b} \in X : \mathcal{G}(\iota(\mathbf{b})) < \alpha\}.$$

If $Y = \emptyset$, then X is a neighbourhood of $\mathbf{a}$ with $|\mathcal{G}(\iota(\mathbf{b})) - \mathcal{G}(\iota(\mathbf{a}))| \le 2\varepsilon$ for all $\mathbf{b} \in X$. Otherwise, let $\ell \ge 2$ be such that $\sum_{k=1}^{\ell} \alpha^{-k} \ge (\alpha + \varepsilon - 1)^{-1}$. Then

$$b_{j+1} - b_j \le \frac{1}{q(\iota(\mathbf{b})) - 1} \le \frac{1}{\alpha + \varepsilon - 1} \le \sum_{k=1}^{\ell} \frac{1}{\alpha^k} \tag{2.3}$$

for all $(b_1, \ldots, b_{d-1}) \in Y$, $0 \le j < d$, with $b_0 = 0$, $b_d = 1$. Set

$$\delta(\mathbf{a}, \mathbf{b}) = \min_{0 \le j < d} \big((a_{j+1} - a_j) - (b_{j+1} - b_j)\big)$$

(with $b_0 = a_0 = 0$, $b_d = a_d = 1$), and let $Z \subset X$ be a neighbourhood of $\mathbf{a}$ such that

$$\frac{a_j}{(\alpha+\varepsilon)^k} - \frac{b_j}{\alpha^k} \le \delta(\mathbf{a}, \mathbf{b}), \quad \frac{b_j}{(\alpha+\varepsilon)^k} - \frac{a_j}{\alpha^k} \le \delta(\mathbf{a}, \mathbf{b}) \quad \text{for all } 1 \le j \le d,\ 1 \le k \le \ell, \tag{2.4}$$

$$\frac{1-a_j}{(\alpha+\varepsilon)^k} - \frac{1-b_j}{\alpha^k} \le \delta(\mathbf{a}, \mathbf{b}), \quad \frac{1-b_j}{(\alpha+\varepsilon)^k} - \frac{1-a_j}{\alpha^k} \le \delta(\mathbf{a}, \mathbf{b}) \quad \text{for all } 0 \le j < d,\ 1 \le k \le \ell, \tag{2.5}$$

for all $\mathbf{b} = (b_1, \ldots, b_{d-1}) \in Z$. Note that $\delta(\mathbf{a}, \mathbf{b}) \le 0$, thus we also have

$$\frac{a_j}{\alpha+\varepsilon} \le \frac{b_j}{\alpha}, \quad \frac{1-a_j}{\alpha+\varepsilon} \le \frac{1-b_j}{\alpha}, \quad \frac{b_j}{\alpha+\varepsilon} \le \frac{a_j}{\alpha}, \quad \frac{1-b_j}{\alpha+\varepsilon} \le \frac{1-a_j}{\alpha} \quad \text{for all } 0 \le j \le d. \tag{2.6}$$

For $\mathbf{b} \in Y \cap Z$ and $\beta \in (\mathcal{G}(\iota(\mathbf{b})), \alpha]$, choose $\mathbf{u} = u_1 u_2 \cdots \in \mathcal{U}_\beta(\iota(\mathbf{b}))$. Assume, w.l.o.g., that $u_1 u_2 \notin \{00, 11\}$. We show first that $\mathbf{u}$ does not contain ℓ consecutive zeros or ones. Indeed, suppose that $u_{i+1} = u_{i+2} = \cdots = u_{i+\ell} = 1$ for some $i \ge 1$; then we have

$$\sum_{k=1}^{\infty} \frac{u_{i+k}}{\beta^k} \ge \sum_{k=1}^{\ell} \frac{1}{\beta^k} \ge \sum_{k=1}^{\ell} \frac{1}{\alpha^k} \ge b_{j+1} - b_j$$

for all $0 \le j < d$, hence $u_i = 1$ because of (2.1); recursively we would obtain that $u_{i-1} = \cdots = u_1 = 1$, contradicting that $u_1 u_2 \ne 11$. Similarly, $u_{i+1} = u_{i+2} = \cdots = u_{i+\ell} = 0$ implies that $\sum_{k=1}^{\infty} (1 - u_{i+k}) \beta^{-k} \ge b_j - b_{j-1}$ for all $1 \le j \le d$, hence $u_i = 0$ because of (2.2), eventually contradicting that $u_1 u_2 \ne 00$.

We define the sequence $\tilde{\mathbf{u}}$ via the relation $\tilde{u}_{i+k} = a_j$ if $u_{i+k} = b_j$. Let now $i \ge 1$. We have $\tilde{u}_{i+k} (\beta + \varepsilon)^{-k} \le u_{i+k} \beta^{-k}$ for all $k \ge 1$ because (2.6) implies that $a_j (\beta + \varepsilon)^{-k} \le b_j \beta^{-k}$ for all $0 \le j \le d$. Moreover, (2.4) and (2.6) give that

$$\frac{a_j}{(\beta+\varepsilon)^k} - \frac{b_j}{\beta^k} \le \frac{a_j}{(\alpha+\varepsilon)^k} - \frac{b_j}{\alpha^k} \le \delta(\mathbf{a}, \mathbf{b})$$

for all $1 \le k \le \ell$, $1 \le j \le d$. Since $u_{i+k} \ne 0$ for some $1 \le k \le \ell$, we have $\tilde{u}_{i+k} (\beta + \varepsilon)^{-k} \le u_{i+k} \beta^{-k} + \delta(\mathbf{a}, \mathbf{b})$ for some $k \ge 1$. Using (2.1), we get

$$\sum_{k=1}^{\infty} \frac{\tilde{u}_{i+k}}{(\beta+\varepsilon)^k} \le \sum_{k=1}^{\infty} \frac{u_{i+k}}{\beta^k} + \delta(\mathbf{a}, \mathbf{b}) < b_{j+1} - b_j + \delta(\mathbf{a}, \mathbf{b}) \le a_{j+1} - a_j \quad \text{when } u_i = b_j \ne 1.$$

Similarly, we obtain from (2.2), (2.5) and (2.6) that

$$\sum_{k=1}^{\infty} \frac{1-\tilde{u}_{i+k}}{(\beta+\varepsilon)^k} \le \sum_{k=1}^{\infty} \frac{1-u_{i+k}}{\beta^k} + \delta(\mathbf{a},\mathbf{b}) < b_j - b_{j-1} + \delta(\mathbf{a},\mathbf{b}) \le a_j - a_{j-1} \quad \text{when } u_i = b_j \neq 0.$$

Therefore, we have $\tilde{\mathbf{u}} \in \mathcal{U}_{\beta+\varepsilon}(\iota(\mathbf{a}))$, thus $\mathcal{G}(\iota(\mathbf{a})) \le \mathcal{G}(\iota(\mathbf{b})) + \varepsilon$ for all $\mathbf{b} \in Y \cap Z$.

For $\mathbf{b} \in X \setminus Y$, recall that $\mathcal{G}(\iota(\mathbf{a})) \le q(\iota(\mathbf{a})) \le \alpha + 2\varepsilon \le \mathcal{G}(\iota(\mathbf{b})) + 2\varepsilon$. Similarly, we obtain for all $\mathbf{b} \in Z$ that $\mathcal{G}(\iota(\mathbf{b})) \le \mathcal{G}(\iota(\mathbf{a})) + \varepsilon$ when $\mathbf{a} \in Y$, $\mathcal{G}(\iota(\mathbf{b})) \le \mathcal{G}(\iota(\mathbf{a})) + 3\varepsilon$ when $\mathbf{a} \notin Y$. This gives that $|\mathcal{G}(\iota(\mathbf{b})) - \mathcal{G}(\iota(\mathbf{a}))| \le 3\varepsilon$ for all $\mathbf{b} \in Z$, thus $\mathcal{G} \circ \iota$ is continuous at $\mathbf{a}$. □

3 Generalised golden ratios over ternary alphabets

3.1 Statements

Komornik, Lai and Pedicini [4] described the function $m \mapsto \mathcal{G}(m)$ on the interval $(1,2]$. We provide more details for this function, in particular for the set

$$\mathfrak{M} := \{m \in (1,2] : \mathcal{G}(m) = 1 + \sqrt{m}\}.$$

For $h \ge 0$, let τ_h be the substitution on the alphabet $\{0,1\}$ defined by

$$\tau_h(0) = 0^{h+1}1, \qquad \tau_h(1) = 0^h 1,$$

and set $S = \{\tau_h : h \ge 0\}$. A (right) infinite word $\mathbf{u}$ is a *limit word* of a sequence of substitutions $(\sigma_n)_{n\ge0}$ if there exist words $\mathbf{u}^{(n)}$ with $\mathbf{u}^{(0)} = \mathbf{u}$ and $\mathbf{u}^{(n)} = \sigma_n(\mathbf{u}^{(n+1)})$ for all $n \ge 0$. A sequence $(\sigma_n)_{n\ge0} \in S^{\mathbb{N}}$ is *primitive* if $\sigma_n \neq \tau_0$ for infinitely many $n \ge 0$. A limit word of a primitive sequence in $S^{\mathbb{N}}$ starts with $\sigma_0\sigma_1\cdots\sigma_n(0)$ for all $n \ge 0$ and is therefore unique. If $\sigma_n = \tau_0$ for all $n \ge 0$, then $1^k0\overline{1}$, $k \ge 0$, and $\overline{1}$ are limit words of $(\sigma_n)_{n\ge0}$; we are only interested in $0\overline{1}$ and $\overline{1}$. Therefore, we define the following sets of limit words (or S-*adic words*), where $S^* = \bigcup_{n\ge0} S^n$ denotes the set of finite products of substitutions in S:

$$\mathcal{S} = \mathcal{S}_\infty \cup \mathcal{S}_{0\overline{1}} \cup \mathcal{S}_{\overline{1}} \quad \text{with} \quad \mathcal{S}_{0\overline{1}} = \{\sigma(0\overline{1}) : \sigma \in S^*\}, \quad \mathcal{S}_{\overline{1}} = \{\sigma(\overline{1}) : \sigma \in S^*\},$$
$$\mathcal{S}_\infty = \{\mathbf{u} : \mathbf{u} \text{ is the limit word of a primitive sequence of substitutions in } S^{\mathbb{N}}\}.$$

Remark 3.1. Komornik, Lai and Pedicini [4] observed that the sequences $\mathbf{u} \in \mathcal{S}_\infty$ with the leading 0 removed are exactly the standard Sturmian sequences. However, they omitted the word "standard".

For $\mathbf{u} = u_0u_1\cdots \in \{0,1\}^{\mathbb{N}}$, we define $\mathfrak{m}_{\mathbf{u}} \ge 1$ as the unique solution to

$$\mathfrak{m}_{\mathbf{u}} = 1 + \sum_{k=0}^{\infty} \frac{u_k}{(1+\sqrt{\mathfrak{m}_{\mathbf{u}}})^k}. \tag{3.1}$$

Remark 3.2. We can rewrite (3.1) as

$$1 + \sqrt{\mathfrak{m}_{\mathbf{u}}} = 2 + \sum_{k=0}^{\infty} \frac{u_k}{(1+\sqrt{\mathfrak{m}_{\mathbf{u}}})^{k+1}},$$

i.e., Parry's [6] β-expansion of $\beta = 1 + \sqrt{\mathfrak{m}_{\mathbf{u}}}$ is $2\mathbf{u}$. We have $\mathfrak{m}_{\mathbf{u}} = 1$ if and only if $\mathbf{u} = \overline{0}$.

For $\sigma \in S^*$, we define the interval $I_\sigma = \big[\mathfrak{m}_{\sigma(0\overline{1})}, \mathfrak{m}_{\sigma(\overline{1})}\big] \subset \big(1, \frac{3+\sqrt{5}}{2}\big]$. We define $\beta_\sigma \ge 2$ implicitly via the equation

$$1 + \sum_{k=1}^{\infty} \frac{\tilde{u}_k^{(\sigma)}}{\beta_\sigma^k} = (\beta_\sigma - 1)\bigg(1 + \sum_{k=0}^{\infty} \frac{u_k^{(\sigma)}}{\beta_\sigma^{k+1}}\bigg),$$

where

$$\tilde{u}_0^{(\sigma)}\tilde{u}_1^{(\sigma)}\tilde{u}_2^{(\sigma)}\cdots = \sigma(0\overline{1}), \qquad u_0^{(\sigma)}u_1^{(\sigma)}u_2^{(\sigma)}\cdots = \sigma(\overline{1}).$$

Moreover, we let μ_σ denote the coinciding value, i.e.,

$$\mu_\sigma := 1 + \sum_{k=1}^{\infty} \frac{\tilde{u}_k^{(\sigma)}}{\beta_\sigma^k}.$$

Note that all the numbers and sequences do not change if we replace σ by $\sigma\tau_0$ since $\tau_0(0\overline{1}) = 0\overline{1}$ and $\tau_0(\overline{1}) = \overline{1}$. Therefore, we can assume that $\sigma \in S^* \setminus S^*\tau_0$.

The following propositions recover the main part of Theorem KLP, adding explicit equations giving the generalised golden ratios, which are used for drawing the graph of $\mathcal{G}$ in Figure 3.1.

Proposition 3.3. *The interval* $\big(1, \frac{3+\sqrt{5}}{2}\big]$ *admits the partition*

$$\{I_\sigma : \sigma \in S^* \setminus S^*\tau_0\} \cup \{\{\mathfrak{m}_{\mathbf{u}}\} : \mathbf{u} \in \mathcal{S}_\infty\}. \tag{3.2}$$

Proposition 3.4. *For* $\sigma \in S^*$ *and* $m \in \big[\mathfrak{m}_{\sigma(0\overline{1})}, \mu_\sigma\big]$, $\mathcal{G}(m)$ *is given by*

$$m = 1 + \sum_{k=1}^{\infty} \frac{\tilde{u}_k^{(\sigma)}}{\mathcal{G}(m)^k}.$$

For $\sigma \in S^*$ *with* $\sigma(1) \neq 1$ *and* $m \in \big[\mu_\sigma, \mathfrak{m}_{\sigma(\overline{1})}\big]$, $\mathcal{G}(m)$ *is given by*

$$\frac{m}{\mathcal{G}(m) - 1} = 1 + \sum_{k=1}^{\infty} \frac{u_k^{(\sigma)}}{\mathcal{G}(m)^{k+1}}.$$

Proposition 3.5. *We have*

$$\mathfrak{M} = \big\{\mathfrak{m}_{\mathbf{u}} : \mathbf{u} \in \mathcal{S} \setminus \{\overline{1}\}\big\}.$$

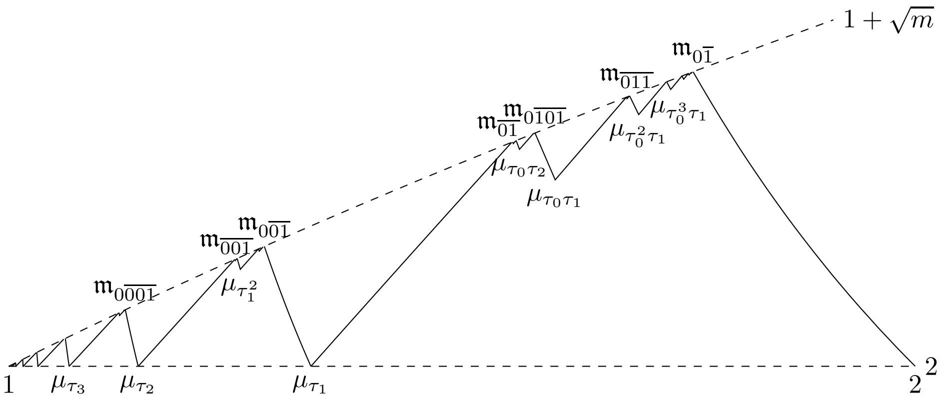

Figure 3.1: A graph of $\mathcal{G}(m)$.

3.2 Partition of $(1, \frac{3+\sqrt{5}}{2}]$

We first prove Proposition 3.3, using the following lemmas.

Lemma 3.6. *Let* $\sigma \in S^*$. *Then* σ *preserves the lexicographic order on infinite words.*

Proof. The lexicographic order on infinite words is preserved by the identity and by $\sigma \in S$. By induction on n, this holds for all $\sigma \in S^n$, $n \geq 0$. $\square$

Lemma 3.7. *Let* $\sigma \in S^*$ *with* $\sigma(1) \neq 1$, *and write* $\sigma(1) = 0w1$. *Then*

$$\sigma(0\overline{1}) = 0\,\overline{w01}.$$

In particular, $1w0$ *is a circular shift of* $\sigma(1)$.

Proof. Since $\sigma(1) = \sigma\tau_0(1)$ and $\sigma(0\overline{1}) = \sigma\tau_0(0\overline{1})$, we assume w.l.o.g. that $\sigma = \tau_{h_0}\tau_{h_1}\cdots\tau_{h_n}$ with $n \geq 0$, $h_n \neq 0$. Let $\sigma_k = \tau_{h_k}$. Then $\sigma_{[0,n]}(1) = 0w1$ with

$$w = \sigma_0(1)\,\sigma_{[0,1]}(1)\cdots\sigma_{[0,n-1]}(1)\sigma_{[0,n-1]}(0^{h_n-1})\,\sigma_{[0,n-2]}(0^{h_{n-1}})\cdots\sigma_0(0^{h_1})0^{h_0}.$$

Let $v = \sigma_0(1)\,\sigma_{[0,1]}(1)\cdots\sigma_{[0,n-1]}(1)$. Then we have $\sigma_{[0,n]}(0) = 0w01v$ and $v\sigma_{[0,n]}(1) = w01v$. Therefore, $1w0$ is a circular shift of $\sigma(1)$ and $\sigma(0\overline{1}) = 0\,\overline{w01}$. $\square$

Lemma 3.8. *Let $\mathbf{u} = u_0u_1\cdots \in \{0,1\}^{\mathbb{N}} \setminus \{\overline{0}\}$. We have $\mathbf{u} \in \mathcal{S}$ if and only if*

$$u_0u_1u_2\cdots \le u_iu_{i+1}u_{i+2}\cdots \le 1u_1u_2\cdots \qquad \text{for all } i \ge 0. \tag{3.3}$$

Proof. Assume that (3.3) holds. Then $u_0 = 0$ or $\mathbf{u} = \overline{1} = \tau_0(\overline{1})$. If $u_0 = 0$, let $h \ge 0$ be minimal such that $u_{h+1} = 1$. Then each 1 is followed by $0^{h+1}1$ or 0^h1, i.e., $\mathbf{u} = \tau_h(\mathbf{u}')$ for some word $\mathbf{u}' = u'_0u'_1\cdots$. Moreover, we have $\mathbf{u}' \le u'_iu'_{i+1}\cdots \le 1u'_1u'_2\cdots$ for all $i \ge 0$. In case $\mathbf{u}' = \overline{0}$, we have $\mathbf{u} = \tau_{h+1}(\overline{1})$. Therefore, we can repeat the arguments and obtain recursively that $\mathbf{u}$ is the limit word of a sequence $(\sigma_n)_{n\ge0} \in S^{\mathbb{N}}$. More precisely, we have $\mathbf{u} \in \mathcal{S}_{0\overline{1}}$ or $\mathbf{u}$ starts with $\sigma_{[0,n]}(0)$ for all $n \ge 0$, i.e., $\mathbf{u} \in \mathcal{S}_\infty \cup \mathcal{S}_{\overline{1}}$.

Consider now $\mathbf{u} \in \mathcal{S}_\infty \cup \mathcal{S}_{0\overline{1}}$, limit word of $(\sigma_n)_{n\ge0} \in S^{\mathbb{N}}$. Then $\mathbf{u}$ starts with $\sigma_{[0,n]}(0)$ for all $n \ge 0$. Denote the preimage of $\mathbf{u}$ by σ_0 by $\mathbf{u}' = u'_0u'_1\cdots$, i.e., $\sigma_0(\mathbf{u}') = \mathbf{u}$. Suppose that $u_iu_{i+1}\cdots \le \mathbf{u}$. Then $u_iu_{i+1}\cdots$ starts with $\sigma_0(0)$, and $u_iu_{i+1}\cdots = \sigma_0(u'_{i'}u'_{i'+1}\cdots)$ for some $i' \ge 0$. This implies that $u'_{i'}u'_{i'+1}\cdots \le \mathbf{u}'$, thus $u'_{i'}u'_{i'+1}\cdots$ starts with $\sigma_1(0)$. Inductively, we obtain that $u_iu_{i+1}\cdots$ starts with $\sigma_{[0,n]}(0)$ for all $n \ge 0$, i.e., $u_iu_{i+1}\cdots = \mathbf{u}$. Suppose now that $u_iu_{i+1}\cdots \ge 1u_1u_2\cdots$. Then $u_i = 1$ and $u_{i+1}u_{i+2}\cdots = \sigma_0(u'_{i'}u'_{i'+1}\cdots)$ for some $i' \ge 0$, with $u'_{i'}u'_{i'+1}\cdots \ge 1u'_1u'_2\cdots$. We get that

$$u_iu_{i+1}\cdots = 1\sigma_0(1)\sigma_{[0,1]}(1)\sigma_{[0,2]}(1)\cdots = 1u_1u_2\cdots.$$

Therefore, (3.3) holds.

Finally, let $\mathbf{u} \in \mathcal{S}_{\overline{1}}$. If $\mathbf{u} = \overline{1}$, then (3.3) holds trivially. Otherwise, we have $\mathbf{u} = \sigma\tau_h(\overline{1})$ with $\sigma \in S^*$, $h \ge 1$. Then $\sigma\tau_{h-1}\tau_j(0\overline{1}) \in \mathcal{S}_{0\overline{1}}$ converges to $\mathbf{u}$ for $j \to \infty$ (in the usual topology of infinite words). By the previous paragraph, (3.3) holds for these words. Hence, it also holds for the limit word $\mathbf{u}$. □

Proof of Proposition 3.3. For $m \in \big(1, \frac{3+\sqrt{5}}{2}\big]$, Parry's $(1+\sqrt{m})$-expansions of $1+\sqrt{m}$ are of the form $2\mathbf{u} \in \{0,1,2\}^{\mathbb{N}}$ with $\overline{0} < \mathbf{u} \le \overline{1}$ and are ordered with respect to the lexicographic order. We show that the interval $(\overline{0}, \overline{1}] \subset \{0,1,2\}^{\mathbb{N}}$ admits the partition

$$\{[\sigma(0\overline{1}), \sigma(\overline{1})] : \sigma \in S^* \setminus S^*\tau_0\} \cup \{\{\mathbf{u}\} : \mathbf{u} \in \mathcal{S}_\infty\}.$$

Assume that $\mathbf{u} = u_0u_1\cdots \notin \mathcal{S}$, i.e., (3.3) does not hold. Let $i \ge 1$ be minimal such that one of the equalities is not satisfied. Suppose first that $u_iu_{i+1}\cdots < \mathbf{u}$; then $u_{i-1} = 1$. Similarly to the proof of Lemma 3.8, let $\sigma_0 \in S$ be such that $u_0\cdots u_{i-1} = \sigma_0(u'_0\cdots u'_{i'-1})$ with $u'_0\cdots u'_{i'-1} \ne 0\cdots0$. By minimality of i, we have $u'_{i'-1} = 1$, and $u'_0\cdots u'_{i'-1} = 1\cdots1$ implies $i' = 1$. Therefore, we can define recursively substitutions $\sigma_j \in S$ until

$$\sigma_{[0,n]}(1) = u_0u_1\cdots u_{i-1}.$$

Then we have $\mathbf{u} < \sigma_{[0,n]}(1)\,\mathbf{u} < \cdots < \sigma_{[0,n]}(\overline{1})$. By Lemma 3.6, we have $\sigma_{[0,n]}(0\overline{1}) < \mathbf{u}$.

Suppose now that $u_iu_{i+1}\cdots > 1u_1u_2\cdots$. Then we have substitutions $\sigma_k = \tau_{h_k}$ such that

$$\sigma_{[0,n]}(0) = u_0u_1\cdots u_{i-1}\,1\,\sigma_0(1)\,\sigma_{[0,1]}(1)\cdots\sigma_{[0,n-1]}(1),$$

with $h_n \ne 0$. We have $\mathbf{u} > u_0\,\overline{u_1\cdots u_{i-1}1}$, and the latter word is equal to $\sigma_{[0,n]}(0\overline{1})$ by Lemma 3.7 and its proof. Since $u_0\cdots u_{i-1} < \sigma_{[0,n]}(1)$, we also have $\mathbf{u} < \sigma_{[0,n]}(\overline{1})$.

We have seen that each $\mathbf{u}$ is the limit word of a primitive sequence of substitutions $\boldsymbol{\sigma} \in S^{\mathbb{N}}$ or between the extremal limit words of a non-primitive sequence $\boldsymbol{\sigma} \in S^{\mathbb{N}}$. To see that $\boldsymbol{\sigma}$ is unique, let $\mathbf{u}$ and $\tilde{\mathbf{u}}$ be limit words of two different sequences $(\sigma_n)_{n\ge0}$ and $(\tilde\sigma_n)_{n\ge0}$. Let $n \ge 0$ be minimal such that $\sigma_n \ne \tilde\sigma_n$. Let $\sigma_n = \tau_h$, $\tilde\sigma_n = \tau_j$, and assume w.l.o.g. that $h < j$. Then we have $\tilde{\mathbf{u}} \le \tilde\sigma_{[0,n]}(\overline{1}) \le \sigma_{[0,n]}(\overline{0}) < \mathbf{u}$. Therefore, the intervals are disjoint. □

3.3 Calculating the generalised golden ratio

We now prove that $\mathcal{G}(m)$ is as in Theorem KLP and Proposition 3.4.

Lemma 3.9. *Let $m \in (1,2]$, $\beta \in [m, m+1]$, and $\mathbf{u} = u_0u_1\cdots \in \{0,1\}^{\mathbb{N}} \setminus \{\overline{0}\}$. Then $\mathbf{u} \in \mathcal{U}_\beta(m)$ if and only if*

$$\frac{m}{\beta-1} < 1 + \sum_{k=1}^{\infty} \frac{u_{i+k}}{\beta^k} < m \quad \text{for all } i \ge 0 \text{ such that } u_i = 1.$$

Proof. As $q(\{0,1,m\}) = 1+m$, Theorem P and Remark 2.1 give that $\mathbf{u} \in \mathcal{U}_\beta(m)$ if and only if $\sum_{k=0}^\infty \frac{u_{i+k}}{\beta^k} \in E_{u_i}$ for all $i \ge 0$, with $E_0 = [0,1)$ and $E_1 = (\frac{m}{\beta-1}, m)$. If $u_i = 0$, then we either have $u_j = 0$ for all $j \ge i$ and thus $\sum_{k=0}^\infty \frac{u_{i+k}}{\beta^k} = 0 \in E_0$ or there exists $j > i$ such that $u_i = \cdots = u_{j-1} = 0$, $u_j = 1$. In the latter case, $\sum_{k=0}^\infty \frac{u_{j+k}}{\beta^k} \in E_1$ implies that $\sum_{k=0}^\infty \frac{u_{i+k}}{\beta^k} \in \beta^{i-j} E_1 \subset E_0$ since $\beta \ge m$. This proves the lemma. ☐

Lemma 3.10. *Let $\mathbf{u} \in \mathcal{S}_\infty$, $m \in (1,2]$. Then we have $\mathbf{u} \in \mathcal{U}_{1+\sqrt{m}}(m)$ if and only if $m = \mathfrak{m}_\mathbf{u}$. In particular, we have $\mathcal{G}(\mathfrak{m}_\mathbf{u}) \le 1 + \sqrt{\mathfrak{m}_\mathbf{u}}$.*

Proof. By Lemma 3.9, we have $\mathbf{u} = u_0 u_1 \cdots \in \mathcal{U}_{1+\sqrt{m}}(m)$ if and only if

$$\sqrt{m} < 1 + \sum_{k=1}^\infty \frac{u_{i+k}}{(1+\sqrt{m})^k} < m$$

for all $i \ge 0$ such that $u_i = 1$. By Lemma 3.8 and since $\mathbf{u}$ is aperiodic, $u_i = 1$ implies that $\mathbf{u} < u_{i+1}u_{i+2}\cdots < u_1 u_2 \cdots$. Here, the bounds $\mathbf{u}$ and $u_1u_2\cdots$ cannot be improved because, for all $n \ge 0$, $1\sigma_{[0,n]}(0)$ and $1\sigma_0(1)\cdots\sigma_{[0,n-1]}(1)$ (which is a suffix of $\sigma_{[0,n]}(0)$) are factors of $\mathbf{u}$. Therefore, we have $\mathbf{u} \in \mathcal{U}_{1+\sqrt{m}}(m)$ if and only if

$$\sqrt{m} \le 1 + \sum_{k=1}^\infty \frac{u_k}{(1+\sqrt{m})^{k+1}} \quad \text{and} \quad 1 + \sum_{k=1}^\infty \frac{u_k}{(1+\sqrt{m})^k} \le m.$$

This means that $1 + \sum_{k=1}^\infty u_k\,(1+\sqrt{m})^{-k} = m$, i.e., $m = \mathfrak{m}_\mathbf{u}$. ☐

Lemma 3.11. *Let $\sigma \in S^*$ and $m > 1$. There is a unique number $f_\sigma(m) > 1$ such that*

$$m = 1 + \sum_{k=1}^\infty \frac{\tilde{u}_k^{(\sigma)}}{f_\sigma(m)^k}. \tag{3.4}$$

We have $f'_\sigma(m) < 0$, $f_\sigma(\mathfrak{m}_{\sigma(0\overline{1})}) = 1 + \sqrt{\mathfrak{m}_{\sigma(0\overline{1})}}$, $f_\sigma(m) < 1+\sqrt{m}$ if and only if $m > \mathfrak{m}_{\sigma(0\overline{1})}$, and $\sigma(\overline{1}) \notin \mathcal{U}_{f_\sigma(m)}(m)$ if $m \le 2$.

Proof. Let $h_m(x) = 1 + \sum_{k=1}^\infty \tilde{u}_k^{(\sigma)} x^{-k} - m$. Then $\lim_{x\to 1} h_m(x) = \infty$, $\lim_{x\to\infty} h_m(x) = 1-m < 0$, $h_m(x)$ is continuous and strictly monotonically decreasing, thus $f_\sigma(m)$ is the unique solution of $h_m(x) = 0$. We have

$$\frac{1}{f'_\sigma(m)} = -\sum_{k=1}^\infty \frac{k\tilde{u}_k^{(\sigma)}}{f_\sigma(m)^{k+1}} < 0,$$

in particular $f_\sigma(m) < f_\sigma(\mathfrak{m}_{\sigma(0\overline{1})}) = 1 + \sqrt{\mathfrak{m}_{\sigma(0\overline{1})}} < 1 + \sqrt{m}$ for $m > \mathfrak{m}_{\sigma(0\overline{1})}$.

By Lemma 3.7, $1\tilde{u}_1^{(\sigma)}\tilde{u}_2^{(\sigma)}\cdots$ is a periodic word with the same period as $\sigma(\overline{1})$. Therefore, (3.4) and Lemma 3.9 imply that $\sigma(\overline{1}) \notin \mathcal{U}_{f_\sigma(m)}(m)$. ☐

Lemma 3.12. *Let $\sigma \in S^*$ and $m > 1$. There is a unique number $g_\sigma(m) > 1$ such that*

$$\frac{m}{g_\sigma(m)-1} = 1 + \sum_{k=0}^\infty \frac{u_k^{(\sigma)}}{g_\sigma(m)^{k+1}}. \tag{3.5}$$

We have $g'_\sigma(m) > 0$, $g_\sigma(\mathfrak{m}_{\sigma(\overline{1})}) = 1 + \sqrt{\mathfrak{m}_{\sigma(\overline{1})}}$, $g_\sigma(m) < 1 + \sqrt{m}$ if and only if $m < \mathfrak{m}_{\sigma(\overline{1})}$, and $\sigma(\overline{1}) \notin \mathcal{U}_{g_\sigma(m)}(m)$ if $m \le 2$.

Proof. Setting $h_m(x) = \frac{m}{x-1} - 1 - \sum_{k=0}^\infty \frac{u_k^{(\sigma)}}{x^{k+1}}$, we have

$$h'_m(x) = \sum_{k=0}^\infty \frac{(k+1)u_k^{(\sigma)}}{x^{k+2}} - \frac{m}{(x-1)^2} \le \sum_{k=0}^\infty \frac{k+1}{x^{k+2}} - \frac{m}{(x-1)^2} = \frac{1-m}{(x-1)^2} < 0 \tag{3.6}$$

for $x > 1$. Similarly to the proof of Lemma 3.11, $g_\sigma(m)$ is the unique solution of $h_m(x) = 0$. (Note that $g_\sigma(m) = m$ if $\sigma(\overline{1}) = \overline{1}$.) We have

$$\frac{1}{g_\sigma'(m)} = \big(1 - g_\sigma(m)\big)\, h_m'(g_\sigma(m)) > 0.$$

Now, $g_\sigma(m) < 1 + \sqrt{m}$ is equivalent to $h_m(1+\sqrt{m}) < 0$, i.e., $1 + \sum_{k=0}^\infty u_k^{(\sigma)} (1+\sqrt{m})^{-k-1} > \sqrt{m}$. By Remark 3.2, we obtain that $m < \mathfrak{m}_{\sigma(\overline{1})}$. Since $1\sigma(\overline{1})$ is a suffix of $\sigma(\overline{1})$, (3.5) and Lemma 3.9 give that $\sigma(\overline{1}) \notin \mathcal{U}_{g_\sigma(m)}(m)$. ☐

Lemma 3.13. *Let $\sigma \in S^*$. There is a unique $\mu_\sigma \in \big(\mathfrak{m}_{\sigma(0\overline{1})}, \mathfrak{m}_{\sigma(\overline{1})}\big)$ with $f_\sigma(\mu_\sigma) = g_\sigma(\mu_\sigma)$. We have $f_\sigma(\mu_\sigma) = g_\sigma(\mu_\sigma) \geq 2$, with equality if and only if $\sigma(\overline{1}) = \overline{0^n 1}$ for some $n \geq 0$.*

If $m \in \big[\mathfrak{m}_{\sigma(0\overline{1})}, \mu_\sigma\big]$, then $\sigma(\overline{1}) \in \mathcal{U}_\beta(m)$ for all $\beta > f_\sigma(m)$, in particular $\mathcal{G}(m) \leq f_\sigma(m)$.

If $m \in \big[\mu_\sigma, \mathfrak{m}_{\sigma(\overline{1})}\big]$, then $\sigma(\overline{1}) \in \mathcal{U}_\beta(m)$ for all $\beta > g_\sigma(m)$, in particular $\mathcal{G}(m) \leq g_\sigma(m)$.

Proof. The number μ_σ is well defined since $f'(m) < 0$, $g'(m) > 0$ on I_σ,

$$f_\sigma(\mathfrak{m}_{\sigma(0\overline{1})}) = 1 + \sqrt{\mathfrak{m}_{\sigma(0\overline{1})}} > g_\sigma(\mathfrak{m}_{\sigma(0\overline{1})}) \text{ and } f_\sigma(\mathfrak{m}_{\sigma(\overline{1})}) < 1 + \sqrt{\mathfrak{m}_{\sigma(\overline{1})}} = g_\sigma(\mathfrak{m}_{\sigma(\overline{1})}).$$

If $\sigma(\overline{1}) = \overline{1}$, then $\mu_\sigma = \beta_\sigma = 2$. Assume in the following that $\sigma(\overline{1}) \neq \overline{1}$ and let $m = 1 + \sum_{k=0}^\infty u_k^{(\sigma)} 2^{-k-1}$, i.e., $g_\sigma(m) = 2$. By Lemma 3.7, we have $\sigma(\overline{1}) = \overline{0w1} \leq \overline{w01} = \tilde{u}_1^{(\sigma)} \tilde{u}_2^{(\sigma)} \cdots$ for some finite word w, thus

$$1 + \sum_{k=1}^\infty \frac{\tilde{u}_k^{(\sigma)}}{2^k} \geq 1 + \sum_{k=0}^\infty \frac{u_k^{(\sigma)}}{2^{k+1}} = m,$$

hence $f_\sigma(m) \geq 2 = g_\sigma(m)$. This implies that $\beta_\sigma \geq 2$. If $\beta_\sigma = 2$, then we must have $0w = w0$, i.e., $w = 0 \cdots 0$. Therefore, $\beta_\sigma = 2$ is equivalent to $\sigma(\overline{1}) = \overline{0^n 1}$ for some $n \geq 0$.

Let now $m \in \big[\mathfrak{m}_{\sigma(0\overline{1})}, \mu_\sigma\big]$ and $\beta > f_\sigma(m)$, or $m \in \big[\mu_\sigma, \mathfrak{m}_{\sigma(\overline{1})}\big]$ and $\beta > g_\sigma(m)$. Then we also have $\beta > g_\sigma(m)$ and $\beta > f_\sigma(m)$ respectively. For $i \geq 0$ with $u_i^{(\sigma)} = 1$, we get

$$\frac{m}{\beta - 1} < 1 + \sum_{k=0}^\infty \frac{u_k^{(\sigma)}}{\beta^{k+1}} \leq 1 + \sum_{k=1}^\infty \frac{u_{i+k}^{(\sigma)}}{\beta^k} \leq 1 + \sum_{k=1}^\infty \frac{\tilde{u}_k^{(\sigma)}}{\beta^k} < m,$$

where the first inequality follows from $\beta > g_\sigma(m)$ and (3.6), the last inequality from $\beta > f_\sigma(m)$, and the middle inequalities are direct consequences of $\beta \geq 2$ and $\sigma(\overline{1}) \leq u_{i+1}^{(\sigma)} u_{i+2}^{(\sigma)} \cdots \leq \tilde{u}_1^{(\sigma)} \tilde{u}_2^{(\sigma)} \cdots$, which holds by Lemmas 3.8 and 3.7. Thus $\sigma(\overline{1}) \in \mathcal{U}_\beta(m)$. ☐

The preceding lemmas show that $\mathcal{G}(m) \leq 1 + \sqrt{m}$ for all $m \in (1, 2]$. The next lemma justifies why we have restricted our attention to sequences in $\{0, 1\}^{\mathbb{N}}$.

Lemma 3.14. *Let $m \in (1, 2]$, $\beta \leq 1 + \sqrt{m}$, $u_0 u_1 \cdots \in \mathcal{U}_\beta(m)$. Then $u_i = m$ implies $u_0 \cdots u_i = m \cdots m$.*

Proof. If $u_i = m$, $i \geq 1$, then (2.2) implies that $u_{i-1} + \sum_{k=1}^\infty \frac{u_{i-1+k}}{\beta^k} > u_{i-1} + \frac{1}{\beta}\big(1 + \frac{m}{\beta - 1}\big) \geq u_{i-1} + 1$, thus (2.1) excludes that $u_{i-1} = 0$ or $u_{i-1} = 1$. Recursively, we obtain that $u_k = m$ for all $0 \leq k \leq m$. ☐

Lemma 3.15. *Let $m \in (1, 2]$, $\beta < 2$. Then $\mathcal{U}_\beta(m)$ is trivial.*

Proof. Let $u_0 u_1 \cdots \in \mathcal{U}_\beta(m)$. By Theorem P, we have $u_i \neq 1$ for all $i \geq 0$. Since $m \leq 1 + \frac{m}{\beta - 1}$, we have $m\overline{0} \notin \mathcal{U}_\beta(m)$, thus Lemma 3.14 implies that $\mathcal{U}_\beta(m) = \{\overline{0}, \overline{m}\}$. ☐

Lemma 3.16. *Let $m \in (1, 2]$, $\beta \leq 1 + \sqrt{m}$, and $u_0 u_1 \cdots \in \mathcal{U}_\beta(m) \cap (\{0, 1\}^{\mathbb{N}} \setminus \{\overline{0}\})$. Then we have $\inf\{u_{i+1} u_{i+2} \cdots : i \geq 0,\, u_i = 1\} \in \mathcal{S}_\infty \cup \mathcal{S}_{\overline{1}}$.*

Proof. Let $\tilde{\mathbf{u}} = \tilde{u}_0 \tilde{u}_1 \cdots = \inf\{u_{i+1} u_{i+2} \cdots : i \geq 0,\, u_i = 1\}$. Since $\tilde{\mathbf{u}} = \overline{1} \in \mathcal{S}_{\overline{1}}$ when $\tilde{u}_0 = 1$, we assume in the following that $\tilde{u}_0 = 0$. For all $i \geq 0$ with $\tilde{u}_i = 1$, we have

$$\sum_{k=1}^\infty \frac{\tilde{u}_{i+k}}{\beta^k} < m - 1 \leq \beta \left(\frac{m}{\beta - 1} - 1\right) \leq \beta \inf_{i \geq 0:\, u_i = 1} \sum_{k=1}^\infty \frac{u_{i+k}}{\beta^k} \leq \beta \sum_{k=0}^\infty \frac{\tilde{u}_k}{\beta^{k+1}} = \sum_{k=1}^\infty \frac{\tilde{u}_k}{\beta^k},$$

since $E_1 = \big(\frac{m}{\beta - 1}, m\big)$ and $\beta \leq 1 + \sqrt{m}$. As $\beta \geq 2$ by Lemma 3.15, we obtain that $\tilde{u}_i \tilde{u}_{i+1} \cdots < 1 \tilde{u}_1 \tilde{u}_2 \cdots$ for all $i \geq 0$. By the definition of $\tilde{\mathbf{u}}$, we also have $\tilde{u}_i \tilde{u}_{i+1} \cdots \geq \tilde{\mathbf{u}}$, thus $\tilde{\mathbf{u}} \in \mathcal{S}$ by Lemma 3.8. Moreover, we have $\tilde{\mathbf{u}} \notin \mathcal{S}_{0\overline{1}}$ by Lemma 3.7 and the fact that $\tilde{u}_i \tilde{u}_{i+1} \cdots < 1 \tilde{u}_1 \tilde{u}_2 \cdots$ for all $i \geq 0$. ☐

Remark 3.17. One obtains similarly that $\sup\{0u_{i+1}u_{i+2}\cdots : i \ge 0,\, u_i = 1\} \in \mathcal{S}_\infty \cup \mathcal{S}_{0\overline{1}}$.

Proposition 3.18. *We have*

$$\mathcal{G}(m) = \begin{cases} 1+\sqrt{m} & \text{if } m \in \{\mathfrak{m}_{\mathbf{u}} : \mathbf{u} \in \mathcal{S}_\infty\}, \\ f_\sigma(m) & \text{if } m \in \big[\mathfrak{m}_{\sigma(0\overline{1})}, \mu_\sigma\big],\ \sigma \in S^*, \\ g_\sigma(m) & \text{if } m \in \big[\mu_\sigma, \mathfrak{m}_{\sigma(\overline{1})}\big],\ \sigma \in S^*. \end{cases}$$

Proof. Let $\beta \ge 2$, $\mathbf{u} \in \mathcal{U}_\beta(m) \cap \{0,1\}^{\mathbb{N}}$ and $\tilde{\mathbf{u}}$ as in Lemma 3.16. Then $\tilde{\mathbf{u}} \in \mathcal{U}_{\tilde\beta}(m)$ for all $\tilde\beta > \beta$. If $\tilde{\mathbf{u}} \in \mathcal{S}_\infty$, then Lemma 3.10 gives that $\beta \ge 1+\sqrt{m}$. If $\tilde{\mathbf{u}} = \sigma(\overline{1})$, $\sigma \in S^*$, then $\beta \ge \max\{f_\sigma(m), g_\sigma(m)\}$ by Lemmas 3.11 and 3.12. This implies that $\beta \ge f_\sigma(m)$ if $m \in \big[\mathfrak{m}_{\sigma(0\overline{1})}, \mu_\sigma\big]$, $\beta \ge g_\sigma(m)$ if $m \in \big[\mu_\sigma, \mathfrak{m}_{\sigma(\overline{1})}\big]$, and $\beta \ge 1+\sqrt{m}$ otherwise. The opposite inequalities are also proved in Lemmas 3.10, 3.11 and 3.12. □

The previous lemmas prove Propositions 3.4 and 3.5 and the main part of Theorem KLP.

Proof of Theorem 2. By Lemmas 3.11 and 3.12, $\mathcal{G}(m)$ is differentiable on $(1,2] \setminus \big(\mathfrak{M} \cup \{\mu_\sigma : \sigma \in S^*\}\big)$. By Propositions 3.3 and 3.18, Lemmas 3.11 and 3.12, and the continuity of $\mathcal{G}$ on $(1,2]$, the total variation is

$$\sum_{\sigma \in S^* \setminus S^*\tau_0} \big(\mathcal{G}(\mathfrak{m}_{\sigma(0\overline{1})}) - \mathcal{G}(\mu_\sigma)\big) + \sum_{\sigma \in S^* \setminus S^*\tau_0 :\, \sigma(1) \ne 1} \big(\mathcal{G}(\mathfrak{m}_{\sigma(\overline{1})}) - \mathcal{G}(\mu_\sigma)\big).$$

As $\lim_{m\to 1^+} \mathcal{G}(m) = 2 = \mathcal{G}(2)$, the two sums are equal. For $m \in \big(\mu_\sigma, \mathfrak{m}_{\sigma(\overline{1})}\big)$, $\sigma \in S^*$, $\sigma(1) \ne 1$, we have

$$\begin{aligned} \frac{1}{\mathcal{G}'(m)} &= \frac{m}{\mathcal{G}(m)-1} - \big(\mathcal{G}(m)-1\big) \sum_{k=1}^\infty \frac{(k+1)u_k^{(\sigma)}}{\mathcal{G}(m)^{k+2}} = 1 - \sum_{k=1}^\infty \left(k - \frac{k+1}{\mathcal{G}(m)}\right) \frac{u_k^{(\sigma)}}{\mathcal{G}(m)^{k+1}} \\ &> 1 - \sum_{k=1}^\infty \left(k - \frac{k+1}{\mathcal{G}(m)}\right) \frac{1}{\mathcal{G}(m)^{k+1}} = 1 - \frac{1}{\mathcal{G}(m)^2} \ge \frac{3}{4}, \end{aligned}$$

using that $k - \frac{k+1}{\mathcal{G}(m)} \ge 0$ for all $k \ge 1$. Therefore, we have

$$\sum_{\sigma \in S^* \setminus S^*\tau_0 :\, \sigma(1) \ne 1} \big(\mathcal{G}(\mathfrak{m}_{\sigma(\overline{1})}) - \mathcal{G}(\mu_\sigma)\big) < \frac{4}{3} \sum_{\sigma \in S^* \setminus S^*\tau_0 :\, \sigma(1) \ne 1} (\mathfrak{m}_{\sigma(\overline{1})} - \mu_\sigma) = \frac{4}{3}\left(1 - \sum_{\sigma \in S^* \setminus S^*\tau_0} (\mu_\sigma - \mathfrak{m}_{\sigma(0\overline{1})})\right).$$

We have $\beta_{\tau_h} = 2$ for all $h \ge 0$ since $\tau_h(\overline{1}) = \overline{0^h 1}$ and $\tau_h(0\overline{1}) = 0\,\overline{0^h 1}$, thus $\mu_{\tau_h} = 2^{h+1}/(2^{h+1}-1)$, and $\beta = 1+\sqrt{\mathfrak{m}_{\tau_h(0\overline{1})}}$ satisfies $\beta^{h+3} - 2\beta^{h+2} - 1 = \beta^2 - 2\beta$, hence $\mu_{\tau_0} - \mathfrak{m}_{\tau_0(0\overline{1})} \approx 0.24512$ (and $\mu_\sigma = \mu_{\tau_0}$ when σ is the identity, $\mathfrak{m}_{0\overline{1}} = \mathfrak{m}_{\tau_0(0\overline{1})}$), $\mu_{\tau_1} - \mathfrak{m}_{\tau_1(0\overline{1})} \approx 0.05136$. This gives that $\sum_{\sigma \in S^* \setminus S^*\tau_0} (\mu_\sigma - \mathfrak{m}_{\sigma(0\overline{1})}) > 1/4$, thus the total variation is less than 2.

The derivative is unbounded because we have, for all $m \in \big(\mathfrak{m}_{\sigma(0\overline{1})}, \mu_\sigma\big)$ with $\sigma \in \tau_h S^*$,

$$\left|\frac{1}{\mathcal{G}'(m)}\right| = \sum_{k=1}^\infty \frac{k\tilde{u}_k^{(\sigma)}}{\mathcal{G}(m)^{k+1}} \le \sum_{k=h+1}^\infty \frac{k}{2^{k+1}} = \frac{h+2}{2^{h+1}}.$$

□

Proof of Theorem 4. Note that the map $\iota : m \mapsto \frac{m}{m-1}$ is an order-reversing involution on $(1,\infty)$. By Proposition 3.18, we have $m = \iota(\mathcal{G}(m)^h)$ and thus $\mathcal{G}(m) = \iota(m)^{1/h}$ for all $m \in [\mathfrak{m}_{\tau_{h-1}(0\overline{1})}, \mu_{\tau_{h-1}}]$, $h \ge 1$. Moreover, $\iota(m) \ge \mathfrak{m}_{\tau_{h-1}(0\overline{1})}$ is equivalent to $1 + \sqrt{\iota(m)} \ge m^{1/h}$ by Lemma 3.11, and $\mu_{\tau_{h-1}} = 2$ by the proof of Theorem 2. For $2^h \le m \le \big(1+\sqrt{\iota(m)}\big)^h$, we have thus $\mathcal{G}(\iota(m)) = m^{1/h}$. □

3.4 Hausdorff dimension of $\mathfrak{M}$

In this section we show that the Hausdorff dimension of $\mathfrak{M}$ is 0.

Proof of Theorem 3. It suffices to show that $\dim_H(\mathcal{G}(\mathfrak{M})) = 0$ because $\mathcal{G} : \mathfrak{M} \to \mathbb{R}$ is given by $\mathcal{G}(m) = 1+\sqrt{m}$, and $1+\sqrt{m}$ is bi-Lipschitz on the interval $(1,2]$.

Given $m \in \mathfrak{M}$, we know by Proposition 3.5 that $m = \mathfrak{m}_{\mathbf{u}}$ for some $\mathbf{u} \in \mathcal{S} \setminus \{\overline{1}\}$. Remark 3.2 states that $2\mathbf{u}$ is also the β-expansion of $\beta = 1 + \sqrt{\mathfrak{m}_{\mathbf{u}}}$. Therefore, for each $n \in \mathbb{N}$ we have

$$1 + \sqrt{\mathfrak{m}_{\mathbf{u}}} \in C_{2u_1 \cdots u_n} := \{\beta > 1 : \text{the } \beta\text{-expansion of } \beta \text{ starts with } 2u_1 \cdots u_n\}.$$

We have $C_{2u_1 \cdots u_n} \subset [2, \infty)$ and, hence, the diameter of $C_{2u_1 \cdots u_n}$ is at most 2^{-n}, e.g., by a lemma of Schmeling [10, Lemma 4.1].

We now prove $\dim_H(\mathcal{G}(\mathfrak{M})) = 0$ by explicitly constructing a cover. We introduce the set

$$L_n := \{u_1 \cdots u_n \in \{0,1\}^n : u_1 \cdots u_n \text{ is a prefix of an element of } \mathcal{S}\}.$$

For each $n \in \mathbb{N}$ we have

$$\mathcal{G}(\mathfrak{M}) \subset \bigcup_{u_1 \cdots u_n \in L_n} C_{2u_1 \cdots u_n}.$$

So the set $\{C_{2u_1 \cdots u_n} : u_1 \cdots u_n \in L_n\}$ is a cover of $\mathcal{G}(\mathfrak{M})$. Let $s > 0$ be arbitrary and $\mathcal{H}^s(\cdot)$ denote the s-dimensional Hausdorff measure. We observe

$$\mathcal{H}^s(\mathcal{G}(\mathfrak{M})) \leq \lim_{n \to \infty} \sum_{u_1 \cdots u_n \in L_n} \mathrm{Diam}(C_{2u_1 \cdots u_n})^s \leq \lim_{n \to \infty} \frac{\# L_n}{2^{ns}}.$$

As was pointed out in Remark 3.1, every element of $\mathcal{S}$ is a Sturmian sequence. Thus it is a consequence of [5, Theorem 2.2.36] that $\# L_n$ grows at most polynomially in n. Therefore $\lim_{n \to \infty} \# L_n 2^{-ns} = 0$ and $dim_H(\mathfrak{M}) \leq s$. Since s is arbitrary we are done. $\square$

4 Behaviour at the generalised golden ratio

In this section we discuss the behaviour of the univoque set at the generalised golden ratio. It was observed in [1] that when $\beta = \mathcal{G}(\{0, 1, \ldots, m\})$ for some positive integer m, then every $x \in (0, \frac{m}{\beta - 1})$ either has a countable infinite of expansions, or a continuum of expansions. In other words $\mathcal{U}_{\mathcal{G}(\{0,1,\ldots,m\})}(\{0, 1, \ldots, m\})$ is still trivial. However, Lemma 3.10 demonstrates that this is not always the case. Indeed the following result is an immediate consequence of this lemma.

Proposition 4.1. *If $\mathbf{u} \in \mathcal{S}_\infty$ then $\mathcal{U}_{\mathcal{G}(\mathfrak{m}_{\mathbf{u}})}(\mathfrak{m}_{\mathbf{u}})$ is non-trivial.*

In [9] it was shown that the smallest $\beta \in (1, 2)$ for which an x has precisely two expansions over the alphabet $\{0, 1\}$ was $\beta_2 \approx 1.71064$. As such, there is a small gap between the golden ratio for the alphabet $\{0, 1\}$, and the smallest β for which an x has precisely two expansion. As we show below, for certain alphabets it is possible that an x has precisely two expansions at the golden ratio.

Proposition 4.2. *For every $\mathbf{u} \in \mathcal{S}_\infty$, the number $\mathfrak{m}_{\mathbf{u}}/\mathcal{G}(\mathfrak{m}_{\mathbf{u}})$ has precisely two expansions in base $\mathcal{G}(\mathfrak{m}_{\mathbf{u}})$ over the alphabet $\{0, 1, \mathfrak{m}_{\mathbf{u}}\}$.*

Proof. Let $\beta = \mathcal{G}(\mathfrak{m}_{\mathbf{u}}) = 1 + \sqrt{\mathfrak{m}_{\mathbf{u}}}$ and let $\mathfrak{m}_{\mathbf{u}}/\beta = \sum_{k=1}^{\infty} v_k \beta^{-k}$ be an expansion of $\mathfrak{m}_{\mathbf{u}}/\beta$ over the alphabet $\{0, 1, \mathfrak{m}_{\mathbf{u}}\}$. Since $\mathfrak{m}_{\mathbf{u}} > \frac{\mathfrak{m}_{\mathbf{u}}}{\beta - 1}$, we have $v_1 \in \{1, \mathfrak{m}_{\mathbf{u}}\}$, thus $\sum_{k=1}^{\infty} v_{k+1} \beta^{-k}$ equals $\mathfrak{m}_{\mathbf{u}} - 1$ and 0 respectively. Clearly, 0 has a unique expansion, and $\mathfrak{m}_{\mathbf{u}} - 1$ has the expansion $u_1 u_2 \cdots$ by (3.1), which is also unique by Lemma 3.10. $\square$

Proposition 4.1 and Proposition 4.2 imply Theorem 5.

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