Université de Paris
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Numeration systems: automata, combinatorics, dynamical systems, number theory

Mémoire d'habilitation à diriger des recherches de

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soutenu publiquement le 6 avril 2021 devant le jury constitué de

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CHAPTER 1

INTRODUCTION

The main object of this thesis are *numeration systems*, sometimes also called *number systems* or *numeral systems*, which are different ways of representing numbers using (finite or infinite) sequences of symbols (digits). These are important ingredients of computers, in particular for computer arithmetic. For instance, Knuth starts the chapter on *Arithmetic* of *The art of computer programming* [**105**, Chapter 4] by a section on *Positional number systems* because "The way we do arithmetic is intimately related to the way we represent the numbers we deal with, so it is appropriate to begin our study of the subject with a discussion of the principal means for representing numbers". Fraenkel [**93**] writes that "The proper choice of a counting system may solve mathematical problems or lead to improved algorithms". For a history of numeration systems starting from the Babylonian hexagesimal system (ca. 1750 BC), we refer to [**105**, Section 4.1]. Of course, integer base systems like the binary, decimal and hexadecimal systems are the most important ones but real base systems have also found applications, e.g. in the golden ratio encoder [**84**] or in β-encoders [**106**].

Another important application of numeration systems is *Diophantine approximation*, i.e., approximation of real numbers by rational numbers. The most efficient way to obtain such approximations is usually the *continued fraction algorithm*, which is related to Euclid's algorithm (*Elements*, ca. 300 BC). In their modern form, continued fractions go back to Euler [**89**]. For a survey on Diophantine approximation, we refer to Bugeaud [**75**], for multidimensional continued fraction algorithms to Schweiger [**129**]. Other applications of numeration systems that were considered by the author are in *cryptography* (see e.g. Morain and Olivos [**118**]) and in the construction of *low discrepancy sequences* (see Drmota and Tichy [**86**]).

They are connected to many areas of mathematics and theoretical computer science: Besides the obvious connections to *number theory*, we are concerned with (symbolic) *dynamical systems* (since the representations of numbers are usually infinite sequences and shifting a representation sequence gives the representation of another number), *combinatorics on words* plays a role in the study of these sequences, the sets of representations are often recognised by *automata* and arithmetic operations can be performed by *transducers*, *ergodic theory* often tells us expected properties of the dynamical system, and *fractal sets* appear in a natural way, e.g. the triadic Cantor set or Rauzy fractals.

All these connections are covered by books [**68, 69, 79, 91, 115, 116**] but there seems to be no monograph devoted exclusively to numeration systems. The present thesis highlights some of these relations, and the bibliography positions the author's research within its context. A brief overview over β-expansions can also be found in the Ph.D.

thesis of Tomaš Hejda [**99**], which was supervised by the author together with Edita Pelantová from the Czech Technical University in Prague.

The majority of the author's publications deals with β-expansions, i.e., representations of numbers with respect to a real base β, and associated expansions of integers; see [**1**]–[**28**]. This subject, which was initiated by Rényi [**125**] and Parry [**122**], has been very popular at least since the 1990 paper of Erdős, Jóo and Komornik [**88**]. We cover a wide (although not exhaustive) range of problems on β-expansions. Another research focus lies on (one- and multidimensional) continued fractions and S-adic sequences; see [**29**]–[**39**]. Other numeration systems like canonical and rational base number systems as well as abstract numeration systems are treated in [**40**]–[**45**]. A few of the author's publications do not concern (and are not immediately motivated by) numeration systems. They deal with m-ary search trees [**46**], return words [**47**], the similar dissection of sets [**48**], and permutations [**50**]; we do not discuss them further.

Many results are not stated exactly as in the literature, but in a way that seems more appropriate now, several years after their publication. Therefore, the thesis gives not only an overview of the author's work but also some slight ameliorations of existing results. We choose to state only one theorem (and one open question) per subsection. Other results and questions are mentioned in the text. The thesis concludes with an outlook on possible research directions for the future.

CHAPTER 2

EXPANSIONS IN POSITIVE REAL BASES

Expansions in positive real bases are natural generalisations of expansions in positive integer bases, which have a lot of interesting properties and some applications. This chapter deals with the publications [**1**]–[**21**], which cover various aspects of β-expansions such as lexicographic characterisations, sofic shifts and shifts of finite type, natural extensions and associated tilings, periodic and finite expansions, unique expansions, minimal weight expansions, representations of integers, low discrepancy sequences, etc.

2.1. Definitions and basic results

In order to write a real number $x \in [0,1)$ in base $\beta > 1$ with a sequence of integer *digits* $d_\beta(x) = a_1 a_2 \cdots$, one can use the *greedy β-expansion*, i.e.,

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots \quad \text{with} \quad 0 \le \frac{a_k}{\beta} + \frac{a_{k+1}}{\beta^2} + \cdots < 1 \text{ for all } k \ge 1.$$

By multiplying by powers of β, this allows representing each real number $x \ge 0$ in base β; adding a sign provides representations of all real numbers. One way to obtain this expansion is to use the *β-transformation*

$$T_\beta : [0,1) \to [0,1), \quad x \mapsto \beta x - \lfloor \beta x \rfloor,$$

where the floor function $\lfloor x \rfloor$ gives the largest integer not exceeding x. Indeed, one has

$$x = \frac{a_1}{\beta} + \frac{T_\beta(x)}{\beta} = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{T_\beta^2(x)}{\beta^2} = \cdots = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots,$$

with $a_k = \lfloor \beta T_\beta^{k-1}(x) \rfloor$, thus $a_k \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ for all $k \ge 1$. When $\beta \ge 2$ is an integer, this gives the classical β-ary expansions, e.g., the binary expansions for $\beta = 2$ and the decimal expansions for $\beta = 10$. The prototype of a non-integer base is the golden ratio $\beta = \frac{1+\sqrt{5}}{2}$, where $a_k = 1$ implies that $a_{k+1} = 0$; see Figure 2.1.

The β-transformation was first studied by Rényi [**125**] (for its ergodic properties) and Parry [**122**]. Parry showed that $a_1 a_2 \cdots$ is a greedy β-expansion if and only if

$$00 \cdots \le a_k a_{k+1} \cdots < d_\beta(1^-) \quad \text{for all } k \ge 1,$$

where $d_\beta(1^-)$ is the limit of $d_\beta(x)$ (w.r.t. the usual product topology on infinite words) for x tending to 1 from below, and $\le$ denotes the lexicographic order. For an integer $\beta \ge 2$, we have $d_\beta(1^-) = (\beta - 1)^\omega$, i.e., all sequences $a_1 a_2 \cdots \in \{0, 1, \ldots, \beta - 1\}^\omega$ that do not end with $(\beta - 1)^\omega$ are greedy β-expansions. For the golden ratio $\beta = \frac{1+\sqrt{5}}{2}$, we have

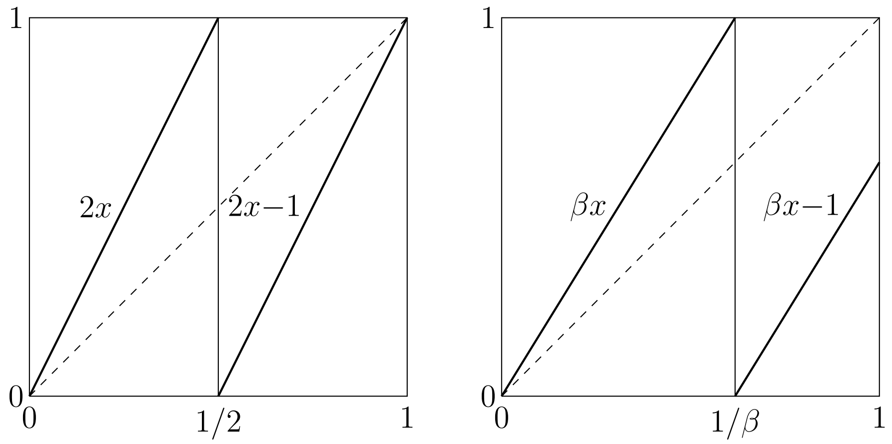

FIGURE 2.1. The greedy β-transformations for $\beta = 2$ and $\beta = (1+\sqrt{5})/2$.

$d_\beta(1^-) = (10)^\omega$, i.e., all sequences $a_1a_2\cdots \in \{0,1\}^\omega$ without the factor 11 and not ending with $(10)^\omega$ are greedy β-expansions.

There are many other transformations that generate β-expansions, e.g. the symmetric β-transformations defined by Akiyama and Scheicher [**58**] as $T(x) = \beta x - \lfloor \beta x + 1/2 \rfloor$ for $x \in [-1/2, 1/2)$ (see Figure 2.2), or the intermediate β-transformations $\beta x + \alpha \bmod 1$. These two examples (and many others) are of the following form.

Definition 2.1. — A *right-continuous β-transformation* is a surjective map

$$T : X \to X, \quad x \mapsto \beta x - a(x),$$

where $X \subset \mathbb{R}$ admits a finite partitition $X = \bigcup_{i\in I} [\ell_i, r_i)$ and the digit function $a : X \to \mathbb{R}$ is constant on each interval $[\ell_i, r_i)$, $i \in I$. The *T-expansion* of $x \in X$ is the sequence

$$d(x) = a(x)\, a(T(x))\, a(T^2(x)) \,\cdots,$$

and a sequence $a_1a_2\cdots$ is *T-admissible* if and only if $a_1a_2\cdots = d(x)$ for some $x \in X$.

With Kalle [**13**], we extended Parry's characterisation to these transformations. Usually, we can choose $a(x) = i$ for $x \in [\ell_i, r_i)$, i.e., I is the digit set of the expansions, and a is usually a non-decreasing map. Then [**13**, Theorem 2.5] states that a sequence $a_1a_2\cdots$ is T-admissible if and only if

$$d(\ell_{a_k}) \le a_k a_{k+1}\cdots < d(r_{a_k}^-) \quad \text{for all } k \ge 1.$$

In the general case of right-continuous β-transformations with non-decreasing digit function a, a sequence $a_1a_2\cdots$ is T-admisssible if and only if

$$\forall\, k \ge 1\ \exists\, i \in I : \ d(\ell_i) \le a_k a_{k+1}\cdots < d(r_i^-).$$

A sequence is *weakly T-admissible* if the right inequality is replaced by a weak inequality.

Throughout this section, we state many results for right-continuous β-transformation even if they can be found in the literature only for the greedy β-transformation; the generalisation of the proofs is straightforward. Symmetric results hold for *left-continuous β-transformations* (where $[\ell_i, r_i)$ is replaced by $(\ell_i, r_i]$), such as the *lazy β-transformation*

$$\tilde{T}_\beta : \big(\tfrac{\lceil\beta\rceil-\beta}{\beta-1}, \tfrac{\lceil\beta\rceil-1}{\beta-1}\big] \to \big(\tfrac{\lceil\beta\rceil-\beta}{\beta-1}, \tfrac{\lceil\beta\rceil-1}{\beta-1}\big], \quad x \mapsto \beta x - \big\lceil \beta x - \tfrac{\lceil\beta\rceil-1}{\beta-1} \big\rceil.$$

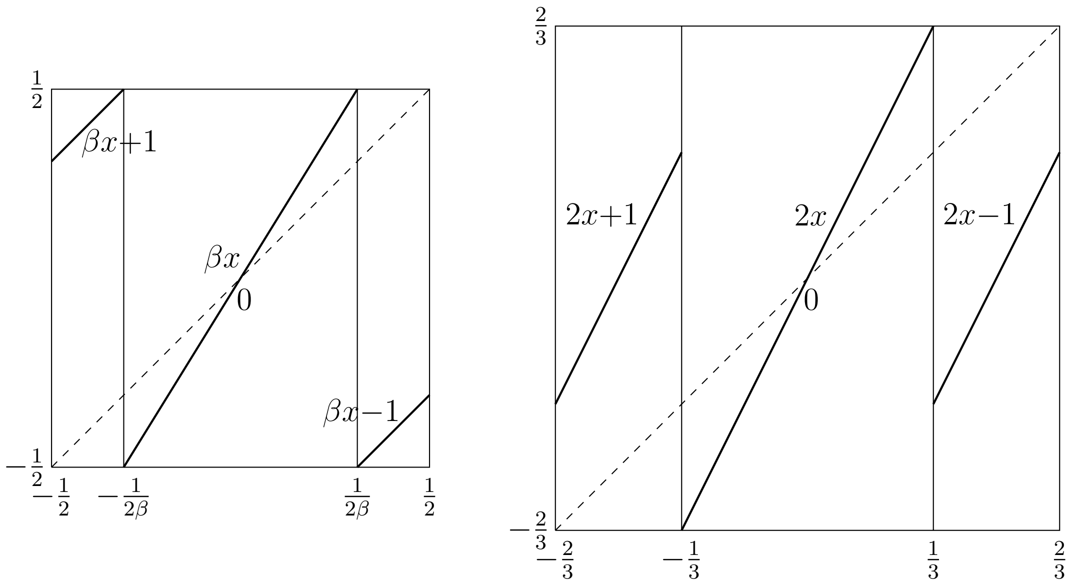

FIGURE 2.2. The symmetric β-transformation for $\beta = (1+\sqrt{5})/2$ and a transformation producing 2-expansions with digits $-1, 0, 1$ where each non-zero digit is followed by 0 (non-adjacent form).

While the greedy β-expansion of $x \in [0, 1)$ is the lexicographically largest sequence $a_1a_2 \cdots \in A_\beta^\omega$ satisfying $x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots$, with $A_\beta = \{0, 1, \ldots, \lceil\beta\rceil - 1\}$, the lazy β-expansion of $x \in \big(\frac{\lceil\beta\rceil-1}{\beta-1} - 1, \frac{\lceil\beta\rceil-1}{\beta-1}\big]$ is the lexicographically smallest such sequence.

The β-*shift* is the set of sequences

$$\Omega_\beta = \overline{\{d_\beta(x) : x \in [0, 1)\}} \subset A_\beta^\omega,$$

i.e., the closure of the set of greedy β-expansions (again w.r.t. the usual product topology on infinite words), which is invariant under the left shift $a_1a_2 \cdots \mapsto a_2a_3 \cdots$. We have thus $a_1a_2 \cdots \in \Omega_\beta$ if and only if $00 \cdots \le a_ka_{k+1} \cdots \le d_\beta(1^-)$ for all $k \ge 1$, i.e., if $a_1a_2 \cdots$ is weakly T_β-admissible. Therefore, Ω_β is a *sofic shift* (i.e., it consists of all infinite paths in a finite labelled directed graph) if and only if $d_\beta(1^-)$ is eventually periodic. Similarly, the weakly T-admissible sequences of a right-continuous β-transformation T form a sofic shift if and only if $d(\ell_i)$ and $d(r_i)$ are eventually periodic for all $i \in I$; see [**13**, Proposition 2.14]. The shift space Ω_β is a *shift of finite type* (i.e., it is obtained from the full shift by forbidding a finite set of patterns) if and only if $d_\beta(1^-)$ is purely periodic. For right-continuous β-transformations, we have the following theorem.

Theorem 2.1. — *Let T be a right-continuous β-transformation with $a(r_i^-) \ne a(r_i)$ for all $r_i \in X$. Then the closure of the set of T-admissible sequences $\overline{\{d(x) : x \in X\}}$ is a shift of finite type if and only if for each $i \in I$ there exist $j, k \in I$ and $m, n \ge 1$ such that $T^m(\ell_i) = \ell_j$ and $T^n(r_i^-) = r_k$.*

Since this theorem is not stated in the literature, we sketch its proof: If there is an $i \in I$ such that $T^m(\ell_i) \notin \{\ell_j : j \in I\}$ for all $m \ge 1$, then for small $\varepsilon > 0$ the sequence $a(\ell_i)\, d(T(\ell_i) - \varepsilon)$ is a β-expansion of $\ell_i - \varepsilon/\beta$ that is not equal to its T-expansion, with length of the shortest forbidden factor tending to infinity as $\varepsilon \to 0$. In the same way, we see that $\overline{\{d(x) : x \in X\}}$ is not a shift of finite type if there is some $i \in I$ such that

$T^n(r_i^-) \notin \{r_k : k \in I\}$ for all $n \geq 1$. On the other hand, if for each $i \in I$ there exist $j, k \in I$ and $1 \leq m, n \leq N$ such that $T^m(\ell_i) = \ell_j$ and $T^n(r_i^-) = r_k$, and N is chosen sufficiently large such that $a(x) \cdots a(T^N(x)) \neq a(y) \cdots a(T^N(y))$ for all $x \in [\ell_i, r_i)$, $y \in [\ell_j, r_j)$ with $i \neq j$, then $\overline{\{d(x) : x \in X\}}$ is a shift of finite type (in $a(X)^\omega$) with forbidden set $F = \{w \in a(X)^{N+1} : w \neq a(x) \cdots a(T^N(x)) \text{ for all } x \in X\}$.

If the β-shift is sofic, then β is called a *Parry number*; if the β-shift is of finite type, then β is a *simple Parry number*. Each Parry number is the Perron–Frobenius eigenvalue of a primitive non-negative integer matrix and thus a Perron number [**122**], i.e., an algebraic integer with all conjugates (besides itself) of absolute value less than β. If β is a Pisot number, i.e., a Perron number with all conjugates of absolute value less than 1, then β is a Parry number [**70, 128**]. There seems to be no simple algebraic characterisation of Parry numbers, and the following question is a long standing open problem.

Open question 2.1. — Are all Salem numbers, i.e., algebraic integers with conjugates on the unit circle and no conjugates (except themselves) outside the closed unit disk, Parry numbers?

Boyd proved in [**73**] that Salem numbers of degree 4 are Parry numbers but he also gives heuristics in [**74**] that this might not be true for degree larger than 6.

The following sections deal with properties of β-expansions, but it is usually simpler to state (and prove) the results in terms of β-transformations. Note that x has a finite (greedy) β-expansion if and only if $T_\beta^k(x) = 0$ for some $k \geq 0$, it has a purely periodic β-expansion if and only if $T_\beta^k(x) = x$ for some $k \geq 1$, and it has an eventually periodic β-expansion if and only if $\{T_\beta^k(x) : k \geq 0\}$ is a finite set.

2.2. Natural extensions

The β-transformation is clearly not bijective, and it is often convenient to extend it to a bijective map, i.e., we consider a natural extension of it. More precisely, let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system, which means that $(X, \mathcal{B}, \mu)$ is a probability space and the map $T : X \to X$ preserves the measure μ, i.e., $\mu \circ T^{-1} = \mu$. A measure-preserving dynamical system $(\hat{X}, \hat{\mathcal{B}}, \hat{\mu}, \hat{T})$ is called a *natural extension* of $(X, \mathcal{B}, \mu, T)$ if $\hat{T} : \hat{X} \to \hat{X}$ is bijective (up to $\hat{\mu}$-measure zero), there is a surjective map $\pi : \hat{X} \to X$ such that $\pi \circ \hat{T} = T \circ \pi$ and $\mu = \hat{\mu} \circ \pi^{-1}$, and $\bigcup_{k=0}^{\infty} \hat{T}^k(\pi^{-1}(\mathcal{B}))$ generates the σ-algebra $\hat{\mathcal{B}}$. In the following, $\mathcal{B}$ and $\hat{\mathcal{B}}$ will always be the Borel σ-algebras and we will omit them.

For a one-sided shift, a natural extension is given by the corresponding two-sided shift. In this section, we are interested in geometric realisations of natural extensions, which can be used e.g. for characterising the purely periodic β-expansions. Dajani, Kraaikamp and Solomyak [**81**] constructed a planar natural extension of the β-transformation for all $\beta > 1$, which is rather intricated and does not respect algebraic properties of numbers. The following construction, which works only for Pisot numbers, turns out to be more useful for our purposes.

If β is a Pisot number and (T, X) is a right-continuous β-transformation with digit set $a(X) \subset \mathbb{Z}[\beta]$, then we can define a natural extension in the following way. If the algebraic conjugates of β are $\beta_1, \ldots, \beta_r \in \mathbb{R}$, $\beta_{r+1}, \ldots, \beta_{r+s} \in \mathbb{C}$ (with $r + 2s$ being the degree of $\beta_1 = \beta$), then the representation space is

$$\mathbb{K}_\beta = \mathbb{R}^r \times \mathbb{C}^s \times \mathbb{Z}_\beta,$$

where $\mathbb{Z}_\beta$ is the set of β-*adic integers* (not to be confused with the set of β-integers considered in Section 3.4), which is the closure of $\mathbb{Z}[\beta]$ w.r.t. to the topology where $x, y \in \mathbb{Z}[\beta]$ are close if $x - y \in \beta^k\mathbb{Z}[\beta]$ for large k. Similarly to the p-adic integers for integer p, the β-adic integers can be considered as inverse limit $\varprojlim \mathbb{Z}[\beta]/\beta^k\mathbb{Z}[\beta]$, and $\mathbb{Z}_\beta$ is topologically conjugate to $\{0, 1, \dots, |N(\beta)|-1\}^\omega$, where $N(\beta)$ denotes the norm of the algebraic integer β, i.e., the constant coefficient p_0 of the minimal polynomial $x^d + p_{d-1}x^{d-1} + \cdots + p_0 \in \mathbb{Z}[x]$ of β. Indeed, the map $\psi : \{0, 1, \dots, |N(\beta)| - 1\}^\omega \to \mathbb{Z}_\beta$, $(a_k)_{k\ge0} \mapsto \sum_{k=0}^\infty a_k\beta^k$, is a bijection, with $\beta\,\psi(a_0, a_1, \dots) = \psi(0, a_0, a_1, \dots)$. The Haar measure ν of the compact group $(\mathbb{Z}_\beta, +)$ assigns measure $|N(\beta)|^{-k}$ to every cylinder of length k in $\{0, 1, \dots, |N(\beta)|-1\}^\omega$, and we have $\nu(\beta E) = \nu(E)/|N(\beta)|$ for every measurable set $E \subseteq \mathbb{Z}_\beta$.

We embed $\mathbb{Z}[\beta]$ in $\mathbb{K}_\beta$ and in $\mathbb{K}'_\beta = \mathbb{R}^{r-1} \times \mathbb{C}^s \times \mathbb{Z}_\beta$ by

$$\varphi_\beta : \mathbb{Z}[\beta] \to \mathbb{K}_\beta,\ z \mapsto (z^{(1)}, z^{(2)}, \dots, z^{(r+s)}, z).$$
$$\varphi'_\beta : \mathbb{Z}[\beta] \to \mathbb{K}'_\beta,\ z \mapsto (z^{(2)}, \dots, z^{(r+s)}, z),$$

where $z^{(i)} \in \mathbb{Z}[\beta_i]$ denotes the conjugate of $z \in \mathbb{Z}[\beta]$ obtained by replacing β by β_i. When β is an algebraic unit, i.e., $|N(\beta)| = 1$, then $\beta^k\mathbb{Z}[\beta] = \mathbb{Z}[\beta]$ for all $k \in \mathbb{N}$, hence $\mathbb{Z}_\beta$ contains only one point and we can omit it in $\mathbb{K}_\beta$. In particular, when β is a quadratic unit such as the golden ratio $\beta = \frac{1+\sqrt{5}}{2}$, the representation space is just $\mathbb{K}_\beta = \mathbb{R}^2$.

The past of a point $x \in X$ under the transformation T can be represented by its *Rauzy fractal*, which is the Hausdorff limit

$$\begin{aligned}\mathcal{R}(x) &= \lim_{k\to\infty} \varphi'_\beta\big(x - \beta^k T^{-k}(x)\big) \\ &= \bigg\{-\sum_{k=0}^{\infty} \varphi'_\beta(a_k\beta^k) :\ a_{-k}a_{-k+1}\cdots a_0 d(x) \text{ is } T\text{-admissible for all } k \ge 0\bigg\}.\end{aligned}$$

Since $\varphi'_\beta(\beta^k)$ tends to 0 exponentially, $\mathcal{R}(x)$ is compact. The *natural extension domain* is

$$\hat{X} = \bigcup_{x\in X} \{x\} \times \mathcal{R}(x) \subset \mathbb{K}_\beta,$$

i.e., we associate each point $x \in X$ with its past, and the *natural extension map* is

$$\hat{T} :\ \hat{X} \to \hat{X},\ (x_1, \dots, x_{r+s}, y) \mapsto (\beta_1 x_1, \dots, \beta_{r+s}x_{r+s}, \beta y) - \varphi_\beta(a(x_1)).$$

We have $\pi_1 \circ \hat{T} = T \circ \pi_1$, where $\pi_1 : \mathbb{K}_\beta \to \mathbb{R}$ is the projection on the first coordinate. Note that $\hat{T}$ is piecewise expanding in the first coordinate (by the factor $\beta_1 = \beta$) and piecewise contracting in the other coordinates (by the factor $\beta_2 \cdots \beta_{r+s}/|N(\beta)| = 1/\beta$). Denote by $\hat{X}_\beta$ and $\mathcal{R}_\beta(x)$ the natural extension domain and the Rauzy fractals of the greedy β-transformation T_β. For an integer $\beta \ge 2$, we have $\hat{X}_\beta = [0, 1) \times \mathbb{Z}_\beta$; examples for quadratic Pisot numbers are given in Figure 2.3 and for cubic Pisot units in Figure 2.4.

Let $\hat{\mu}$ be the product measure of the Lebesgue measure on $\mathbb{R}^r \times \mathbb{C}^s$ and the Haar measure on $\mathbb{Z}_\beta$, normalised so that $\hat{\mu}(\hat{X}) = 1$ (if $\hat{X}$ has positive measure). Since $\hat{\mu}(\hat{T}(\hat{B})) = \hat{\mu}(\hat{B})$ for all measurable sets $\hat{B} \subset \hat{X}$ with $\pi_1(\hat{B}) \subset [\ell_i, r_i)$ for some $i \in I$, and $\hat{T}(\hat{X}) = \hat{X}$, we obtain that $\hat{T}$ is bijective on $\hat{X}$ up to a set of measure zero. For the following theorem, it mainly remains to show that $\hat{X}$ has positive measure. This was done using the (multiple) tilings of the following section in [**13**, Theorem 3.7] for Pisot units (where $\mathbb{Z}_\beta$ is a singleton) and by Berthé and Siegel in [**67**, Theorem 2 (2)] for (non-unit) Pisot greedy β-transformations T_β; see also [**16**, Theorem 2] in a work with Minervino. The general

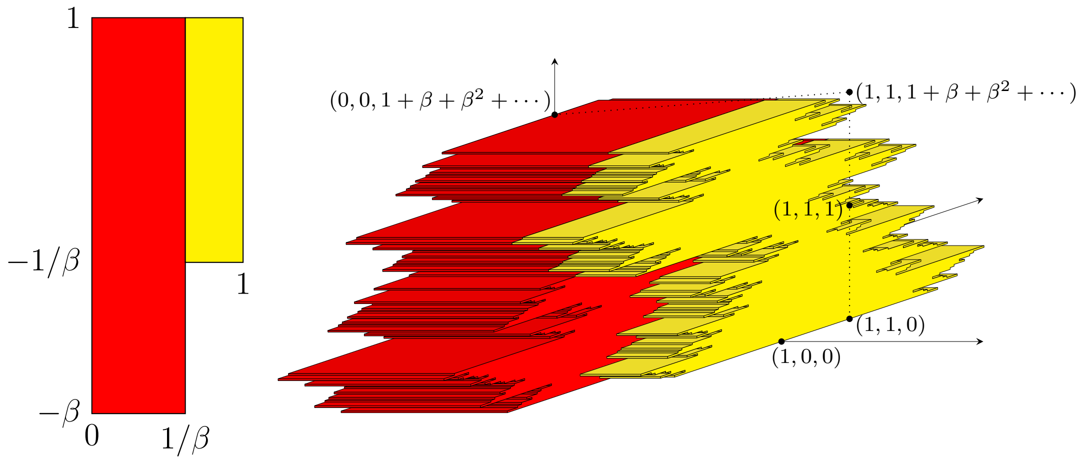

FIGURE 2.3. Natural extension domain $\hat{X}_\beta$ of the greedy β-transformation T_β for the golden ratio $\beta^2 = \beta + 1$ and for $\beta^2 = 2\beta + 2$. For $\beta^2 = \beta + 1$, we have $\beta_2 = -1/\beta$ and $\mathbb{Z}_\beta$ is a singleton, thus $\hat{T}(x_1, x_2) = (\beta x_1 - \lfloor \beta x_1 \rfloor, -x_2/\beta - \lfloor \beta x_1 \rfloor)$. For $\beta^2 = 2\beta + 2$, we have $\beta_2 = -2/\beta$ and elements $\sum_{k=0}^\infty a_k \beta^k \in \mathbb{Z}_\beta$, with $a_k \in \{0, 1\}$, are represented as $\sum_{k=0}^\infty a_k 2^{-k-1}$.

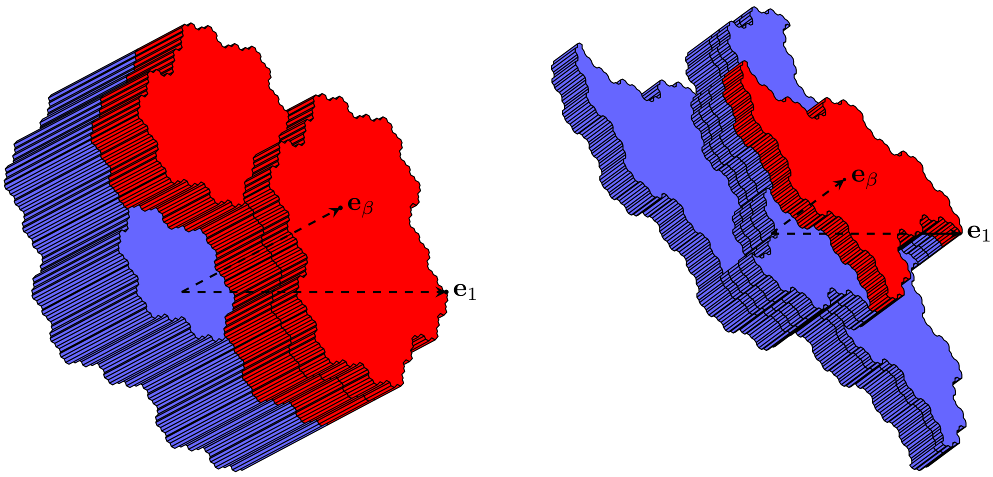

FIGURE 2.4. Natural extension $\hat{X}_\beta$ of the greedy β-transformation T_β for the Tribonacci number ($\beta^3 = \beta^2+\beta+1$) and the smallest Pisot number ($\beta^3 = \beta^2+1$). Here, $\mathbf{e}_1 = \varphi_\beta(1)$, $\mathbf{e}_\beta = (1, 0, \ldots, 0)$, and $\mathbb{K}_\beta$ is represented as in [**13**].

result follows by a combination of these proofs. Note that in [**67, 16**] the representation space is not defined using $\mathbb{Z}_\beta$ but by finite places of the number field $\mathbb{Q}(\beta)$; this adds some additional algebraic structure which is not needed here, and the equivalence of the two representations for our purposes is proved with my student Tomaš Hejda in [**19**].

Theorem 2.2. — *Let T be a right-continuous β-transformation with a Pisot number β and $a(X) \subset \mathbb{Z}[\beta]$. Then the dynamical system $(\hat{X}, \hat{\mu}, \hat{T})$ is a natural extension of the dynamical system $(X, \hat{\mu} \circ \pi_1^{-1}, T)$.*

Note that we do not determine first an invariant measure μ of (X, T) and then show that $(\hat{X}, \hat{\mu}, \hat{T})$ is a natural extension (X, μ, T), but our construction of the natural extension also provides the invariant measure $\mu = \hat{\mu} \circ \pi_1^{-1}$, which is absolutely continuous w.r.t. the Lebesgue measure. In many cases, the absolutely continuous invariant measure is unique.

Theorem 2.2 can certainly be extended to Salem numbers provided that $\hat{X}$ is bounded.

Open question 2.2. — Is there a Salem number β such that $\hat{X}_\beta$ is bounded? Is there a Salem number β such that $\hat{X}_\beta$ is not bounded?

2.3. Periodic β-expansions

We know from Bertrand [**70**] and Schmidt [**128**] that, when the base β is a Pisot number, $x \in [0,1)$ has an eventually periodic β-expansion if and only if $x \in \mathbb{Q}(\beta)$. This result easily extends to right-continuous β-transformations with digit set $a(X) \subset \mathbb{Q}(\beta)$. One of the main features of the natural extension presented in the previous section is that it characterises the purely periodic β-expansions. Similarly to Theorem 2.2, the following result is a generalisation of [**13**, Theorem 3.2] and [**67**, Theorem 3]; for the greedy β-transformation with a quadratic Pisot unit, this was proved by Hama and Imahashi [**98**].

Theorem 2.3. — *Let T be a right-continuous β-transformation with a Pisot number β and $a(X) \subset \mathbb{Z}[\beta]$. Then $T^n(x) = x$ for some $n \geq 1$ if and only if $qx \in \mathbb{Z}[\beta]$ for some $q \in \mathbb{Z}$ with* $\gcd(q, N(\beta)) = 1$ *and $\varphi_\beta(x) \in \hat{X}$.*

Here, φ_β is extended canonically to elements $x \in \mathbb{Z}[\beta]/q$ with $\gcd(q, N(\beta)) = 1$; indeed, we can regard such an x as an element of $\mathbb{Z}_\beta$ because there is a unique sequence $a_0a_1\cdots \in \{0,1,\ldots,|N(\beta)|-1\}^\omega$ satisfying $\sum_{k=0}^\infty qa_k\beta^k = qx$ (in $\mathbb{Z}_\beta$).

For integers $\beta \geq 2$ and the greedy β-transformation, Theorem 2.3 states the well-known fact that the β-ary expansion of $x \in [0,1)$ is purely periodic if and only if x is a rational number p/q with $\gcd(q, \beta) = 1$, i.e., $p/q \in \mathbb{Z}_\beta$.

Theorem 2.3 allows us to study the quantity

$$\gamma(\beta) = \inf\{p/q \in \mathbb{Q} \cap [0,1) : \gcd(q, N(\beta)) = 1,\ T_\beta^n(p/q) \neq p/q \text{ for all } n \geq 1\} \cup \{1\},$$

in other words, the maximal interval starting at 0 where all rational numbers with denominator coprime to the norm of β have purely periodic greedy β-expansion. If $\beta \geq 2$ is an integer, then $\gamma(\beta) = 1$. Schmidt [**128**] proved that $\gamma(\beta) = 1$ also holds for $\beta^2 = a\beta + 1$, $a \geq 1$, while we know from Hama and Imahashi [**98**] that $\gamma(\beta) = 0$ if $\beta^2 = a\beta - 1$, $a \geq 3$. By Akiyama [**52**], we have $\gamma(\beta) > 0$ if β is a Pisot unit satisfying the finiteness property

$$(\mathrm{F}_\beta) \qquad \text{for each } x \in \mathbb{Z}[\beta] \cap [0,1) \text{ we have } T_\beta^n(x) = 0 \text{ for some } n \geq 0;$$

this condition implies that the origin is an inner point of $\mathcal{R}_\beta(0)$. For the smallest Pisot number ($\beta^3 = \beta + 1$), Akiyama and Scheicher [**57**] found the surprising value $\gamma(\beta) = 0.666666666086\cdots$, which is not equal to $2/3$ or another rational number. Indeed, with Adamczewski, Frougny and Siegel [**11**, Theorem 1.2] we proved that $\gamma(\beta)$ is irrational for all cubic Pisot units satisfying (F_β) such that the number field $\mathbb{Q}(\beta)$ is not totally real. We also proved in [**11**, Theorem 1.1] that $\gamma(\beta) = 0$ if β is a cubic Pisot unit not satisfying (F_β). This is due to the fact that, at each point $\varphi'_\beta(x)$ on the boundary of the Rauzy fractal (with $x \in \mathbb{Q}(\beta)$), this Rauzy fractal looks like a spiral.

For Pisot non-units that are not in $\mathbb{Z}$ (but that satisfy (F_β)), the first values of $\gamma(\beta)$ were calculated by Akiyama, Barat, Berthé and Siegel [**53**]: $\gamma(2+\sqrt{7}) = 0$, $\gamma(5+2\sqrt{7}) = (7-\sqrt{7})/12$. With Hejda [**19**], we give a simple algorithm to determine or approximate $\gamma(\beta)$ for quadratic Pisot numbers. For $\beta^2 = a\beta + b$ with $a \geq b \geq 1$ and b dividing a, we obtain in [**19**, Theorem 3] the surprising result that $\gamma(\beta) = 1$ if and only if $a \geq b^2$ or $(a, b) \in \{(24, 6), (30, 6)\}$; moreover we have $\gamma(\beta) = 0$ if $a = b \geq 3$.

Open question 2.3. — Is it possible that $\gamma(\beta) > 0$ when β does not satisfy (F_β)?

By completely different methods (exponential sums over finite fields), we consider with Shparlinski [**15**] expansions in integer bases that are purely periodic but contain all possible factors up to a certain length. In [**15**, Theorem 1], we show that, for any integer base $\beta \geq 2$, fixed $\varepsilon > 0$, for almost all primes p and all $1 \leq m < p$, the number of different factors of length k in the expansion of m/p is $(1 + o(1))\beta^k$ as $p \to \infty$, provided that $k \leq (17/72 - \varepsilon) \log_\beta p$. This is motivated by applications to pseudorandom number generators; see Blum, Blum and Shub [**72**].

2.4. Tilings

The natural extension defined in Section 2.2 gives rise to several (multiple) tilings. A collection $\mathcal{C}$ of tiles in K is a *multiple tiling* of K with multiplicity m if every point of K lies in at least m tiles and almost all points lie in exactly m tiles. (Often additional conditions are imposed, e.g. that the tiles are compact, that they are the closures of their interiors, that there are finitely many tiles up to translation, etc. While all our tiles are compact, they sometimes contain only one point and hence are not the closures of their interiors.) If $m = 1$, then we say that $\mathcal{C}$ is a *tiling*.

The first collection of tiles is

$$\mathcal{C}_{\text{ext}} = \{\varphi_\beta(z) + \hat{X} : z \in \mathbb{Z}[\beta]\};$$

note that $\varphi_\beta(\mathbb{Z}[\beta])$ is a lattice in $\mathbb{K}_\beta$. By intersecting these tiles with the "hyperplane" $\pi_1(x) = 0$ (and omitting the first coordinate), we obtain the aperiodic collection

$$\mathcal{C}_{\text{aper}} = \{\mathcal{R}(z) - \varphi'_\beta(z) : z \in \mathbb{Z}[\beta] \cap X\};$$

see Figure 2.5.

A sufficient condition for having finitely many different Rauzy fractals is that all endpoints of the intervals $[\ell_i, r_i)$, $i \in I$, are in $\mathbb{Q}(\beta)$ and have thus eventually periodic T-expansion, i.e., $\{T^k(\ell_i) : k \geq 0\}$ and $\{T^k(r_i^-) : k \geq 0\}$ are finite sets. This holds in particular for the greedy β-transformation. However, we can weaken this condition. For a point x in the interior X° of X, we say that *matching* holds if $T^{n_x}(x) = T^{n_x}(x^-)$ for some $n_x \geq 1$; we define n_x as the smallest positive integer with this property and $n_x = \infty$ if we do have no matching. (Since $n_x = 1$ when x is continuous, we are only interested in the discontinuity points of T.) The *pre-matching set* is

$$\mathcal{P} = \bigcup_{x \in X^\circ} \{T^k(x) : 1 \leq k < n_x\} \cup \{T^k(x^-) : 1 \leq k < n_x\}.$$

By [**13**, Proposition 3.9], we have $\mathcal{R}(x) = \mathcal{R}(y)$ if $(x, y] \cap \mathcal{P} = \emptyset$. This implies that the density of the invariant measure $\hat{\mu} \circ \pi_1^{-1}$ is constant between consecutive points of $\mathcal{P}$ (and equal to the measure of $\mathcal{R}(x)$ up to a normalising constant). The following theorem is again proved only for Pisot units in [**13**, Theorem 4.10 and Proposition 4.20], but can be easily extended to non-units. Note that the finiteness of the pre-matching set $\mathcal{P}$ is needed for having finitely many different Rauzy fractals, which are then given by a graph-directed iterated function system, but the finiteness is probably not needed for the tiling theorem.

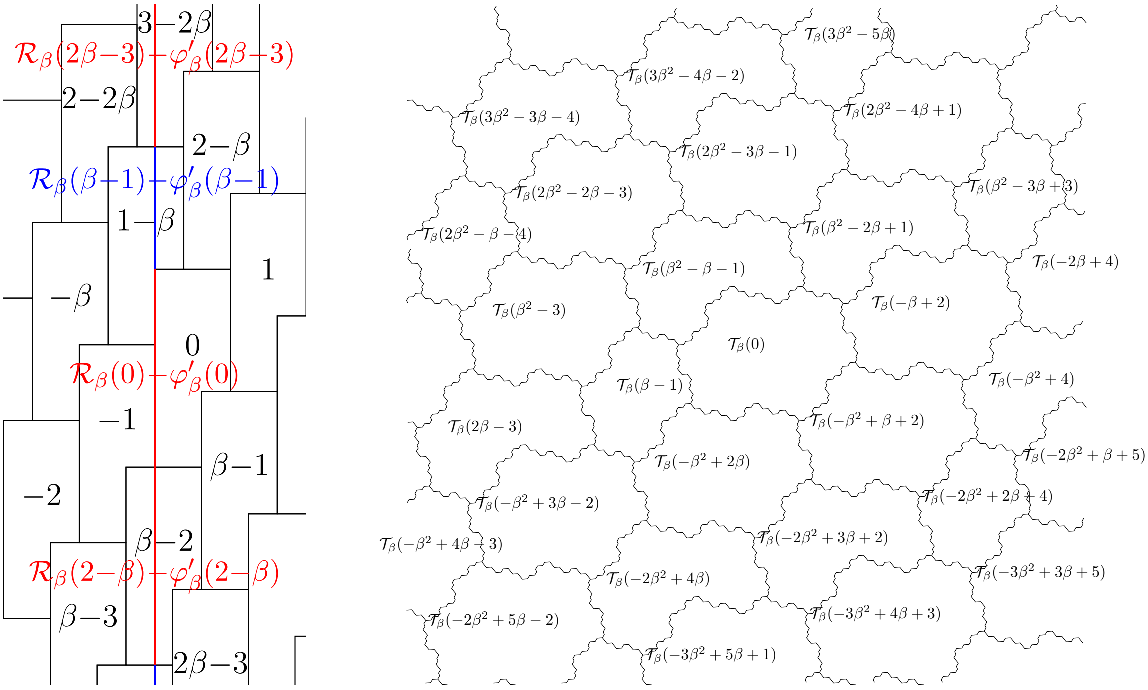

FIGURE 2.5. The tilings $\mathcal{C}_{\rm ext}$ and $\mathcal{C}_{\rm aper}$ for $\beta^2 = \beta+1$ (left, $\varphi_\beta(z)+\hat{X}_\beta$ is labeled by z) and the tiling $\mathcal{C}_{\rm aper}$ for $\beta^3 = \beta^2+\beta+1$ (right, $\mathcal{T}_\beta(z) = \varphi'_\beta(z)-\mathcal{R}_\beta(z)$).

Theorem 2.4. — *Let T be a right-continuous β-transformation with a Pisot number β and digit set $a(X) \subset \mathbb{Z}[\beta]$ such that the pre-matching set $\mathcal{P}$ is finite. Then $\mathcal{C}_{\rm ext}$ and $\mathcal{C}_{\rm aper}$ form multiple tilings of $\mathbb{K}_\beta$ and $\mathbb{K}'_\beta$ with the same multiplicity. If* (W) *holds, then the multiplicity is 1.*

The condition (W), which is a weakening of the finiteness condition (F), is given in [**13**, p. 750]. We only state it here for T_β:

$$(\mathrm{W}_\beta) \qquad \forall\, x \in P_\beta\ \exists\, y \in [0, 1-x),\ n \ge 0:\ T_\beta^n(x+y) = T_\beta^n(y) = 0,$$

where P_β denotes the set of purely periodic points in $\mathbb{Z}[\beta] \cap [0,1)$. Recall from Section 2.3 that (F_β) means that $P_\beta = \{0\}$, which obviously implies (W_β). Barge [**63**] proved that condition (W_β) holds for all Pisot numbers β; his proof is rather intricate. Previously, this was known for large classes of Pisot numbers, in particular due to results with Akiyama and Rao [**5**], which are proved combinatorially.

Surprisingly, the condition (W) does not hold for certain symmetric β-transformations, and we showed in [**13**] that the multiplicity of the tilings for the Tribonacci number ($\beta^3 = \beta^2 + \beta + 1$) and for the smallest Pisot number ($\beta^3 = \beta + 1$) is 2. Hejda [**100**] extended this result to multinacci numbers and proved that the multiplicity is $d-1$ for $\beta^d = \beta^{d-1} + \beta^{d-2} + \cdots + 1$.

Open question 2.4. — Is it possible that (W_β) holds for a Salem number?

The original motivation of the author for studying the collection of tiles $\mathcal{C}_{\rm ext}$ was to determine digits in the β-expansion of a number without determining the whole expansion: If $a(X) \in \mathbb{Z}[\beta]$, then we have $T^k(x) - \beta^k x \in \mathbb{Z}[\beta]$, thus

$$\varphi_\beta(\beta^k x) \in \{T^k(x)\} \times \mathcal{R}(T^k(x)) + \varphi_\beta(\mathbb{Z}[\beta])$$

provided that $0 \in \mathcal{R}(x)$. If $\varphi'_\beta(\beta^k x)$ is in the interior of $\mathcal{R}(T^k(x))$, in particular if 0 is in the interior of $\mathcal{R}(x)$, and $\mathcal{C}_{\mathrm{ext}}$ is a tiling, then we obtain that $\varphi_\beta(\beta^k x) \mod \varphi_\beta(\mathbb{Z}[\beta])$ determines $T^k(x)$ and thus the $(k+1)$-st digit of the T-expansion of x; see [**13**, Theorem 4.23]. For quadratic Pisot units, this was already applied with Drmota in [**1**] and in [**2**] to prove distribution results for digits of polynomial sequences.

In certain cases, the *central tile* $\mathcal{R}_\beta(0)$ given by the greedy β-transformation T_β also induces a periodic tiling of $\mathbb{K}'_\beta$. More precisely, this holds when condition (QM) in [**16**] is satisfied, i.e., when the $\mathbb{Z}$-module generated by $\{T_\beta^{k+1}(1^-)-T_\beta^k(1^-) : k \geq 0\}$ has rank $\deg(\beta)-1$, in particular when the cardinality of $\{T_\beta^k(1^-) : k \geq 0\}$ is $\deg(\beta)$. This periodic tiling is important for Rauzy fractals defined by substitutions (see Section 4.3 below), but its relevance to β-expansions is less clear.

Finally, choosing only points $(0, x_2, \ldots, x_{r+s}, 0) \in \varphi_\beta(z) + \hat{X}_\beta$, $z \in \mathbb{Z}[\beta]$, where $\hat{X}_\beta$ is the natural extension domain of $(T_\beta, [0,1))$ gives an SRS tiling; see Section 5.2 below.

2.5. Fibonacci expansions

We now turn to (positional) representations of integers. Let $U = (U_k)_{k\geq 0}$ be a strictly increasing sequence of integers with $U_0 = 1$. From Fraenkel [**92**], we know that each integer $N \geq 0$ has a (greedy) U-expansion,

$$N = \sum_{k=1}^{n} a_k U_{n-k} \quad \text{with} \quad 0 \leq \sum_{k=1}^{j} a_k U_{j-k} < U_j \text{ for all } 1 \leq j \leq n.$$

If $\lim_{k\to\infty} U_{k+1}/U_k = \beta$, then the language of the U-expansions is essentially the language of the (greedy) β-expansions, and for certain sequences U these two languages are equal. A classical example of these numeration systems is given by the Zeckendorf expansions, where U is the Fibonacci sequence $U_0 = 1$, $U_1 = 2$, $U_k = U_{k-1} + U_{k-2}$ for $k \geq 2$, and β is the golden ratio. In this case, the sequences of digits are characterised by the fact that each 1 is always followed by a 0.

Similarly, to lazy β-expansions, one can define lazy U-expansions. In [**7**], we compared the sum of digits of the greedy Fibonacci expansion $s_g(N)$ to that of the lazy Fibonacci expansion $s_\ell(N)$. Even if s_ℓ can be much larger than s_g, e.g. $s_g(U_k) = 1$ and $s_\ell(U_k) = \lfloor k/2 \rfloor + 1$, they are usually correlated. We proved a central limit theorem for their joint distribution and showed that their correlation is about 0.90983.

Theorem 2.5. — *We have, as $N \to \infty$,*

$$\frac{1}{N}\#\Big\{0 \leq M < N : \frac{s_g(M) - \mu_g \log_\beta N}{\sigma\sqrt{\log_\beta N}} < x_g, \frac{s_\ell(M) - \mu_\ell \log_\beta N}{\sigma\sqrt{\log_\beta N}} < x_\ell\Big\}$$
$$\to \frac{1}{2\pi\sqrt{1-C^2}} \int_{-\infty}^{x_\ell}\int_{-\infty}^{x_g} e^{-\frac{1}{2(1-C^2)}(t_g^2+t_\ell^2-2Ct_g t_\ell)} \mathrm{d}t_g \mathrm{d}t_\ell$$

with $\beta = \frac{1+\sqrt{5}}{2}$, $\mu_g = \frac{1}{\beta^2+1}$, $\mu_\ell = \frac{\beta^2}{\beta^2+1}$, $\sigma = 5^{-3/4}$ and $C = 9 - 5\beta \approx 0.90983$.

The proof uses a Markov chain defined by the two expansions.

Open question 2.5. — Can we use natural extensions to study U-expansions defined by other Pisot numbers β?

2.6. Minimal weight expansions

In public key cryptosystems based on the Diffie–Hellman scheme, one has to calculate scalar multiples Ng of a group element g, where N is a large integer and the group is often an elliptic curve or the multiplicative group of a finite field. In order to calculate Ng, one uses a U-expansion $N = \sum_{k=1}^{n} a_k U_{n-k}$ (with integer digits), usually with $U_k = 2^k$. The number of operations depends then heavily on the Hamming weight (number of non-zero digits) of the expansion or on the sum of the absolute values of the digits, $\sum_{k=1}^{n} |a_k|$. In the most interesting case, $a_k \in \{0, \pm 1\}$, the two notions are equivalent.

Each integer N has a unique expansion $N = \sum_{k\geq 0} a_k 2^k$ with $a_k \in \{-1, 0, 1\}$ where each non-zero digit is followed and preceded by zero digits; this is the non-adjacent form (NAF). The weight of these expasions is minimal among all expansions in base 2, it is on average $(\log_2 N)/3$. Heuberger [**101**] determined all 2-expansions with the same (minimal) weight. For U-expansions with the Fibonacci sequence $U = F$, he showed that each integer has an expansion with $a_k \in \{0, \pm 1\}$ with forbidden factors 11, $1(-1)$, 101, $10(-1)$, 1001 and their inverses, and the weight of this expansion is minimal. With Frougny [**9**], we determined all expansions of minimal weight, for U-expansions in Fibonacci base (Figure 2.6) as well as for β-expansions with $\beta = \frac{1+\sqrt{5}}{2}$. The weight of these U-expansions is on average $(\log_\beta N)/5 \approx 0.288 \log_2 N$.

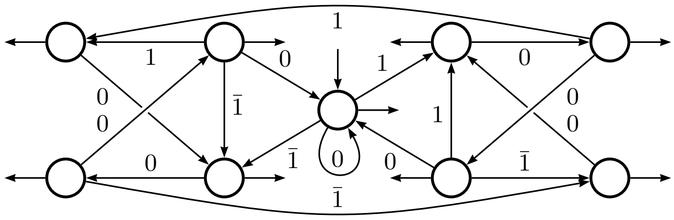

FIGURE 2.6. Automaton recognising the F-expansions of minimal weight in $\{-1, 0, 1\}^*$.

We obtained similar results for $U_k = U_{k-1} + U_{k-2} + U_{k-3}$ (Tribonacci) and the corresponding β-expansions ($\beta^3 = \beta^2 + \beta + 1$). An interesting case is $U_k = U_{k-2} + U_{k-3}$. The corresponding β is the smallest Pisot number ($\beta^3 = \beta + 1$), and the weight of these U-expansions is about $0.235 \log_2 N$. The main result of [**9**] is that the language of β-expansions with minimal sum of absolute values of digits is regular when β is an algebraic number without conjugates on the unit circle. More precisely, this is stated in [**9**] only for Pisot numbers; its extension to the more general case is immediate from Frougny, Pelantová and Svobodová [**94**, Theorem 5.7].

Open question 2.6. — Does the sequence U defined by $U_k = U_{k-2} + U_{k-3}$ minimise the average weight of minimal weight expansions?

In order to avoid side-channel attacks on cryptographic algorithms like analysis of power consumption, one can use the redundancy of the U-expansions of minimal weight. With Grabner [**12**], we proved that the language of such expansions is regular if U is (eventually) recurrent with a characteristic polynomial that is the minimal polynomial of a Pisot number β.

Theorem 2.6. — *Let $U = (U_k)_{k\geq 0}$ be a strictly increasing sequence of integers with $U_0 = 1$, satisfying eventually a linear recurrence with characteristic polynomial equal to* *the minimal polynomial of a Pisot number. Then the set of U-expansions of minimal weight is recognised by a finite automaton.*

Here, it is probably not possible to replace the Pisot condition by a hyperbolic one.

The average number of representations in these systems grows with $N^{\log_\beta \alpha}$, where α is the dominating eigenvalue of the corresponding automaton. Moreover, there exists a transducer that outputs all U-expansions of minimal weight of a number given a certain U-expansion of this number as input. The largest number of different U-expansions of minimal weight of the same number is therefore given by the joint spectral radius of a family of matices.

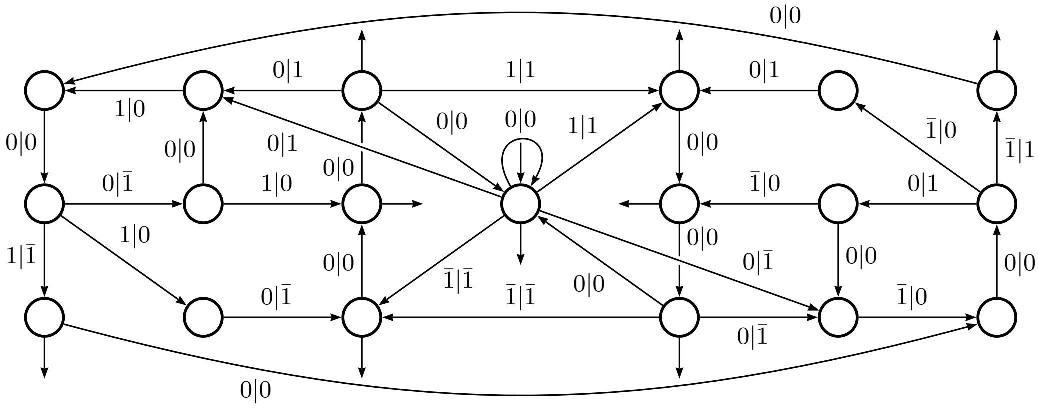

FIGURE 2.7. Transducer normalising F-expansions of minimal weight in $\{-1, 0, 1\}^*$.

2.7. Unique β-expansions

We have already seen that β-expansions are not unique. In fact, if β is not an integer, then almost all numbers have uncountably many β-expansions with digits $\{0, 1, \ldots, \lfloor\beta\rfloor\}$. However, it is also interesting to study numbers with *unique β-expansions* for a given digit set A. Let

$$\mathcal{U}_\beta(A) = \{u \in A^\infty : \pi_\beta(u) \neq \pi_\beta(v) \text{ for all } v \in A^\infty \setminus \{u\}\},$$

where $\pi_\beta(u_1u_2\cdots) = \sum_{k=1}^\infty u_k\beta^{-k}$. We know from Daróczy and Kátai [**83**] that $\mathcal{U}_\beta(\{0,1\})$ is trivial if and only if $\beta \leq (1+\sqrt{5})/2$, where trivial means that $\mathcal{U}_\beta(\{0,1\}) = \{\overline{0}, \overline{1}\}$, $\overline{a}$ being the infinite repetition of a. Therefore, the number $\mathcal{G}(A) = \inf\{\beta > 1 : |\mathcal{U}_\beta(A)| > 2\}$ is called *generalised golden ratio* of A. Glendinning and Sidorov [**95**] showed that the set $\mathcal{U}_\beta(\{0,1\})$ is uncountable if and only if β is larger than the Komornik–Loreti constant $\beta_{\mathrm{KL}} \approx 1.787$; we call $\mathcal{K}(A) = \inf\{\beta > 1 : \mathcal{U}_\beta(A) \text{ is uncountable}\}$ *generalised Komornik–Loreti constant* of A. Since multiplying and translating the digit set by constants does not change the structure of unique β-expansions, the only two-letter alphabet to consider is $\{0,1\}$, and we can restrict to $\{0,1,m\}$, $m \in (1,2]$, for three-letter alphabets. By Komornik, Lai and Pedicini [**107**] (see also my work with Baker [**18**]), the generalised golden ratio $\mathcal{G}(\{0,1,m\})$ is given by mechanical words, i.e., Sturmian words and their periodic counterparts; in particular, we can restrict to sequences $u \in \{0,1\}^\infty$. Calculating $\mathcal{K}(\{0,1,m\})$ seems to be much harder since this restriction is not possible. Therefore, we studied in [**21**]

$$\mathcal{L}(\{0,1,m\}) = \inf\{\beta > 1 : \mathcal{U}_\beta(\{0,1,m\}) \cap \{0,1\}^\infty \text{ is uncountable}\},$$

following Komornik and Pedicini [**108**], where this quantity was determined for certain intervals.

We use words that are generated by the substitutions (or morphisms)

$$L: 0 \mapsto 0, \qquad M: 0 \mapsto 01, \qquad R: 0 \mapsto 01,$$
$$ 1 \mapsto 01, \qquad 1 \mapsto 10, \qquad 1 \mapsto 1.$$

More precisely, u is a limit word of a sequence of substitutions $(\sigma_k)_{k\ge 0}$ if u is an image of $\sigma_0\sigma_1\cdots\sigma_k$ for all $k \ge 0$. The sequence $(\sigma_n)_{n\ge 0}$ is primitive if for each $k \ge 0$ there exists $n \ge k$ such that the image of each letter by $\sigma_k\sigma_{k+1}\cdots\sigma_n$ contains all letters. (This generalises the notion of primitivity for a single substitution.) If we denote by $\mathcal{S}_S$ the set of limit words of primitive sequences of substitutions in a set S, then $\mathcal{S}_{\{L,R\}}$ consists of *Sturmian words*, and $\mathcal{S}_{\{M\}}$ consists of the *Thue-Morse word* $0u = 0110100110010110\cdots$, which defines the Komornik–Loreti constant by $\pi_{\beta_{\mathrm{KL}}}(u) = 1$, and its reflection by $0 \leftrightarrow 1$. We call the elements of $\mathcal{S}_{\{L,M,R\}}$, which to our knowledge have not been studied yet, *Thue–Morse–Sturmian words*. For $u \in \{0,1\}^\infty$ and $m \in (1,2]$, define $f_u(m)$ (if u contains at least two ones) and $g_u(m)$ as the unique positive solutions of

$$f_u(m)\, \pi_{f_u(m)}(\sup \mathcal{O}(u)) = m \quad \text{and} \quad (g_u(m)-1)(1+\pi_{g_u(m)}(\inf \mathcal{O}(u))) = m$$

respectively, where $\mathcal{O}(u_1u_2\cdots) = \{u_ku_{k+1}\cdots :\ k \ge 1\}$ denotes the shift orbit, and infinite words are ordered by the lexicographic order. Define μ_u by $f_u(\mu_u) = g_u(\mu_u)$. Then the following characterisation of $\mathcal{G}(\{0,1,m\})$ and $\mathcal{L}(\{0,1,m\})$ is due to [**107, 18, 21**].

Theorem 2.7. — *For $m \in (1,2]$, the generalised golden ratio is*

$$\mathcal{G}(\{0,1,m\}) = \begin{cases} f_{\sigma(\overline{0})}(m) & \text{if } m \in [\mu_{\sigma(1\overline{0})}, \mu_{\sigma(\overline{0})}],\ \sigma \in \{L,R\}^*M, \\ g_{\sigma(\overline{0})}(m) & \text{if } m \in [\mu_{\sigma(\overline{0})}, \mu_{\sigma(0\overline{1})}],\ \sigma \in \{L,R\}^*M, \\ f_{\overline{1}}(m) & \text{if } m \in [\mu_{0\overline{1}}, 2], \\ 1+\sqrt{m} & \text{if } m = \mu_u,\ u \in \mathcal{S}_{\{L,R\}}, \end{cases}$$

and the modified Komornik-Loreti constant is

$$\mathcal{L}(\{0,1,m\}) = \begin{cases} g_{\sigma(1\overline{0})}(m) & \text{if } m \in [\mu_{\sigma(1\overline{0})}, \mu_{\sigma(01\overline{0})}],\ \sigma \in \{L,M,R\}^*M, \\ f_{\sigma(0\overline{1})}(m) & \text{if } m \in [\mu_{\sigma(10\overline{1})}, \mu_{\sigma(0\overline{1})}],\ \sigma \in \{L,M,R\}^*M, \\ g_{0\overline{1}}(m) & \text{if } m \in [\mu_{0\overline{1}}, 2], \\ f_u(m) & \text{if } m = \mu_u,\ u \in \mathcal{S}_{\{L,M,R\}}. \end{cases}$$

Here, S^* denotes the set of finite products of substitutions in S. We know that

$$2 \le \mathcal{G}(\{0,1,m\}) \le 1+\sqrt{m} \le \mathcal{K}(\{0,1,m\}) \le \mathcal{L}(\{0,1,m\}) \le 1+m$$

for all $m \in (1,2]$, with $\mathcal{G}(\{0,1,m\}) = \mathcal{L}(\{0,1,m\})$ if and only if $m \in \{\mu_{\sigma(1\overline{0})}, \mu_{\sigma(0\overline{1})}\}$, $\sigma \in \{L,R\}^*M$, or $m = \mu_u$, $u \in \mathcal{S}_{\{L,R\}}$. Otherwise, $\mathcal{K}(\{0,1,m\})$ is known only for $m = 2$:

$$\mathcal{K}(\{0,1,2\}) \approx 2.536 < \tfrac{3+\sqrt{5}}{2} = \mathcal{L}(\{0,1,2\}).$$

Note that $\mathcal{G}(\{0,1,m\})$, $\mathcal{K}(\{0,1,m\})$ and $\mathcal{L}(\{0,1,m\})$ are continuous in m; see [**108**].

Open question 2.7. — When is $\mathcal{K}(\{0,1,m\}) < \mathcal{L}(\{0,1,m\})$? Determine $\mathcal{K}(\{0,1,m\})$.

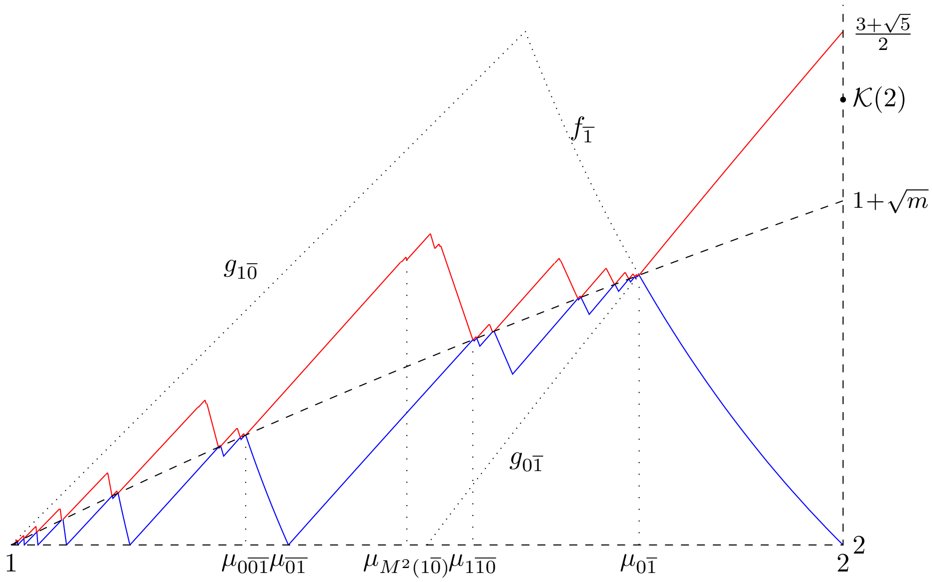

FIGURE 2.8. The critical bases $\mathcal{G}(\{0,1,m\})$ (blue) and $\mathcal{L}(\{0,1,m\})$ (red).

2.8. Discrepancy and bounded remainder sets

For a sequence of real numbers $x_n \in [0,1)$, $n \ge 0$, the discrepancy function

$$D(N,[a,b)) = \frac{1}{N}\big|\#\{n < N : x_n \in [a,b)\} - N(b-a)\big|$$

measures the distance to uniform distribution in the interval $[0,1)$. For a *low discrepancy sequence* (or pseudo-random sequence), the *discrepancy* $\sup_{[a,b)\in[0,1)} D(N,[a,b))$ is $\mathcal{O}(\log N/N)$. These sequences can be used for numerical integration because the error is bounded by the discrepancy (multiplied by the total variation); see the books of Kuipers and Niederreiter [**109**] as well as Drmota and Tichy [**86**]. Van der Corput [**78**] constructed such sequences by reversing β-expansions with integer $\beta \ge 2$. Ninomiya [**121**] generalised this construction to real $\beta > 1$, by ordering the β-expansions by the lexicographic order least significant digit first. He proved that such a sequence has low discrepancy if β is a Pisot number and the cardinality of the T_β-orbit of 1^- is equal to the degree of β. In [**8**], we defined an inverse β-substitution that allows determining x_n without knowledge of its predecessors, thus giving precise results on the discrepancy function. In particular, we determined the *bounded remainder sets* of the form $[0,b)$.

Theorem 2.8. — *Let β be a Pisot number with $|\{T_\beta^k(1^-) : k \ge 0\}| = \deg(\beta)$. For the β-adic van der Corput sequence, $N\,D(N,[0,b))$ of is bounded (in N) for $b \in [0,1)$ if and only if $T_\beta^n(b^-) = T_\beta^k(1^-)$ for some $k,n \ge 0$.*

Open question 2.8. — Which intervals (not starting at 0) are bounded remainder sets?

In [**10**], we extended these results to abstract numeration sytems (that are defined in Section 5.3).

Halton sequences are d-dimensional sequences where each coordinate is given by a van der Corput sequence in an integer base $\beta_i \ge 2$, such that the β_i are pairwise coprime; these sequences have discrepancy $\mathcal{O}((\log N)^d/N)$. Drmota [**85**] obtained discrepancy estimates for generalised Halton sequences with $\beta_i = (1+\sqrt{5})/2$ for some i. Thuswaldner [**133**] considered Halton sequences with m-bonacci bases, using Rauzy fractals. For arbitrary Pisot bases as above (satisfying some independence condition), the Rauzy fractals defined by the inverse β-substitutions can probably be used to obtain similar results. It would also be interesting to find bounded remainder sets for these sequences.

2.9. Intermediate β-shifts of finite type

Similarly to the greedy β-transformation, the transformation $x \mapsto \beta x + \alpha \mod 1$, more precisely $x \mapsto \beta x + \alpha - \lfloor \beta x + \alpha \rfloor$, with a parameter $\alpha \in [0,1)$ is a right continuous β-transformation and provides β-expansions of real numbers with digits in $\mathbb{Z}-\alpha$. This transformation is conjugate to the map $x \mapsto \beta x - \lfloor \beta x - \frac{\alpha}{\beta-1} \rfloor$ on $\big[\frac{\alpha}{\beta-1}, \frac{\alpha+\beta-1}{\beta-1}\big)$, which gives β-expansions with integer digits; see Figure 2.9.

It is well known that the set of bases $\beta > 1$ such that the β-shift is of finite type is dense in $[1,\infty)$. With Li, Sahlsten and Samuel [**20**], we extend this to parameters (β,α). For simplicity, we considered only parameters (β,α) where the map $x \mapsto \beta x + \alpha - \lfloor \beta x + \alpha \rfloor$ has only one discontinuity in $[0,1)$.

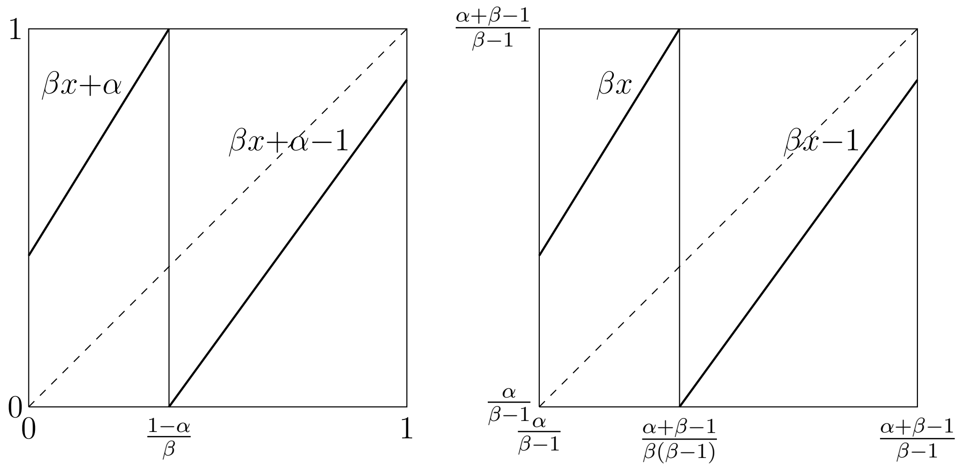

Figure 2.9. The transformations $\beta x + \alpha \mod 1$ and $x \mapsto \beta x - \lfloor \beta x - \frac{\alpha}{\beta-1} \rfloor$ for $\beta = (1+\sqrt{5})/2$ and $\alpha = 1/5$.

Theorem 2.9. — *The set of parameters (β,α) for which $\Omega_{\beta,\alpha}$ is a subshift of finite type is dense in $\{(x,y) : 1 < x \le 2,\ 0 \le y \le 2-x\}$.*

Open question 2.9. — For a Pisot number β, is the set of α for which $\Omega_{\beta,\alpha}$ is a subshift of finite type dense in $[0, 2-\beta]$? Is it not dense when β is not a Pisot number?

The proof of Theorem 2.9 relies on a result with Barnsley and Vince [**26**], where we characterised the possible expansions of the point of discontinuity both for positive and negative β. This leads us to the next chapter.

CHAPTER 3

EXPANSIONS IN NEGATIVE REAL BASES

Once we are familiar with expansions in positive real bases, it is natural to consider negative bases. Negative integer bases can be traced back at least to Grünwald [**97**]. As expected, many of their properties are similar to positive ones. However, some properties are also fundamentally different, which justifies considering them in detail. For instance, the generalisation of Parry's result on quasi-greedy expansions of 1 is a little bit involved, the absolutely continuous invariant measure of the $(-\beta)$-transformation need not have full support, the set of numbers with $(-\beta)$-expansions containing no negative powers of $(-\beta)$ need not be relatively dense, etc. This chapter deals with the publications [**22**]–[**28**].

3.1. $(-\beta)$-expansions

In order to represent real numbers in negative base, Ito and Sadahiro [**102**] defined the *$(-\beta)$-transformation*

$$U_{-\beta} : \big[\tfrac{-\beta}{\beta+1}, \tfrac{1}{\beta+1}\big) \to \big[\tfrac{-\beta}{\beta+1}, \tfrac{1}{\beta+1}\big), \quad x \mapsto -\beta x - \big\lfloor \tfrac{\beta}{\beta+1} - \beta x \big\rfloor.$$

Setting $a_k = \big\lfloor \frac{\beta}{\beta+1} - \beta U_{-\beta}^{k-1}(x) \big\rfloor$, we have

$$x = \frac{a_1}{-\beta} + \frac{a_2}{(-\beta)^2} + \cdots \quad \text{with} \quad \frac{-\beta}{\beta+1} \le \frac{a_k}{-\beta} + \frac{a_{k+1}}{(-\beta)^2} + \cdots < \frac{1}{\beta+1} \text{ for all } k \ge 1,$$

similarly to greedy β-expansions. By multiplying by powers of $(-\beta)$, this allows representing each real number in base $(-\beta)$ with digits in $A_{-\beta} = \{0, 1, \ldots, \lfloor\beta\rfloor\}$. This is the first advantage of negative bases over positive ones: We can represent all real numbers using nonnegative digits, not only the nonnegative numbers.

Instead of $U_{-\beta}$, it is often more convenient to study the transformation

$$T_{-\beta} : (0,1] \to (0,1], \quad x \mapsto -\beta x + \lfloor \beta x \rfloor + 1.$$

Since $U_{-\beta}(\frac{1}{\beta+1} - x) = \frac{1}{\beta+1} - T_{-\beta}(x)$ for all $x \in (0,1]$, the maps $T_{-\beta}$ and $U_{-\beta}$ are topologically conjugate; see [**23**] and Figure 3.1. For $x \in (0,1]$, set $d_{-\beta}(x) = a_1 a_2 \cdots$ with $a_k = \lfloor \beta T_{-\beta}^{k-1}(x) \rfloor$. We know from [**102**] that $a_1 a_2 \cdots = d_{-\beta}(x)$ for some $x \in (0,1]$ if and only if $d_{-\beta}(0^+) \prec a_k a_{k+1} \cdots \preceq d_{-\beta}(1)$ for all $k \ge 1$, where $\preceq$ denotes the *alternating lexicographical order*. A result similar to the characterisation of T-admissible sequences in Section 2.1 can also be proved, even if to our knowledge it has not been stated in the literature.

The $(-\beta)$-shift $\Omega_{-\beta} = \overline{\{d_{-\beta}(x) : x \in (0,1]\}}$ is sofic if and only if $d_{-\beta}(1)$ is eventually periodic [**102**]. In this case, we say that β is an *Yrrap number*. (We coined this notion

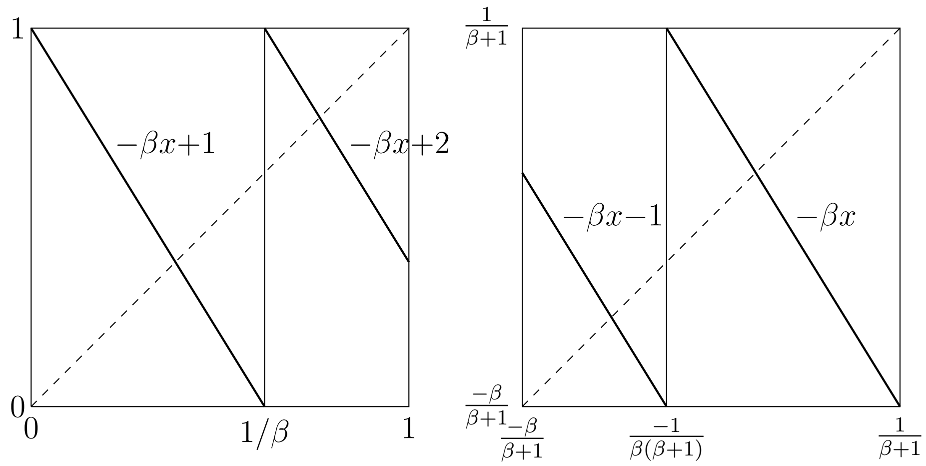

FIGURE 3.1. The transformations $T_{-\beta}$ and $U_{-\beta}$ for $\beta = \frac{1+\sqrt{5}}{2}$.

with Liao in [**23**]: Because $T_{-\beta}$ is order-reversing, we reversed the order of the letters in Parry's name.) As for Parry numbers, each Pisot number is an Yrrap number and each Yrrap number is a Perron number. However, the set of Parry numbers and the set of Yrrap numbers do not include each other, e.g. $\beta > 1$ with $\beta^4 = \beta + 1$ is Yrrap but not Parry, while $\beta > 1$ with $\beta^7 = \beta^6 + 1$ is Parry but not Yrrap [**23**].

Open question 3.1. — Which Salem numbers are Yrrap numbers?

For β-expansions, Parry [**122**] characterised the possible expansions of 1: One has $a_1a_2\cdots = d_\beta(1^-)$ for some $\beta > 1$ if and only if $00\cdots < a_ka_{k+1}\cdots \le a_1a_2\cdots$ for all $k \ge 1$. An analogue of Parry's characterisation to $(-\beta)$-expansions of 1 is given in [**25**, Theorem 2], but it is more complicated than in the positive case. The sequence

$$u = 1001110010010011100 11\cdots = \lim_{\beta\to 1^+} d_{-\beta}(1),$$

which is the fixed point of the morphism $0 \mapsto 1 \mapsto 100$, plays a role here. Note that in the positive case we simply have $\lim_{\beta\to1^+} d_\beta(1^-) = 1000\cdots$.

Theorem 3.1. — *For a sequence of positive integers $a_1a_2\cdots$, we have $d_{-\beta}(1) = a_1a_2\cdots$ for some $\beta > 1$ if and only if $a_ka_{k+1}\cdots \preceq a_1a_2\cdots$ for all $k \ge 2$,*

$$a_1a_2\cdots \notin \{a_1\cdots a_k,\, a_1\cdots a_{k-1}(a_k{-}1)1\}^\omega \setminus \{a_1\cdots a_k\}^\omega$$

for all $k \ge 1$ with $(a_1\cdots a_k)^\omega \succ u$, and

$$a_1a_2\cdots \notin \{a_1\cdots a_k1,\, a_1\cdots a_{k-1}(a_k{+}1)\}^\omega$$

for all $k \ge 1$ with $(a_1\cdots a_{k-1}(a_k{+}1))^\omega \succ u$.

3.2. Gaps in the $(-\beta)$-transformation

The β-transformation is always transitive, meaning that each two block of digits u and w can be joined to a block of digits uvw. (In certain cases, we even have specification, i.e., the block v can chosen to be of fixed length.) For the $(-\beta)$-transformation, this is no longer the case when $\beta < (1+\sqrt{5})/2$. Ito and Sadahiro [**102**] showed that the unique absolutely continuous invariant measure does not have full support in this case. For instance, setting $G_1 = (T_{-\beta}(1), T_{-\beta}^2(1))$, we have for each $x \in G_1 \setminus \{\frac{1}{\beta+1}\}$ some $k \geq 1$ such that $T_{-\beta}^k(x) \notin G_1$, and we have $T_{-\beta}((0,1] \setminus G_1) \subseteq T_{-\beta}((0,1] \setminus G_1)$. Therefore, the invariant measure of G_1 is zero, and we call G_1 a gap. With Liao [**23**, Theorem 2.1], we described the exact structure of the gaps. Their number increases as $\beta \to 1$.

Theorem 3.2. — *Let $\beta > 1$ be such that $\beta^{\lceil 2^n/3 \rceil} < \beta + 1 \leq \beta^{\lceil 2^{n+1}/3 \rceil}$, $n \geq 1$. Then there are $\lfloor 2^n/3 \rfloor$ gaps in the support of unique absolutely continuous invariant measure of $T_{-\beta}$.*

For intermediate β-transformations $\beta x + \alpha \mod 1$, we know from Flatto and Lagarias [**90**, Proposition 4.1] when they have gaps. For intermediate $(-\beta)$-transformations, it is probably harder to get such a characterisation.

Open question 3.2. — For which (β, α), the transformation $-\beta x + \alpha \bmod 1$ has a gap?

In [**23**], we showed that $T_{-\beta}$ is locally eventually onto on $(0,1]$ with the gaps removed, hence $T_{-\beta}$ is exact w.r.t. its unique absolutely continuous invariant measure. A formula for this measure was given by Ito and Sadahiro [**102**]; it consists of a series of weighted indicator functions. For β-expansions, the formula for the invariant measure can be deduced from the planar natural extension constructed by Dajani, Kraaikamp and Solomyak [**81**]. For $(-\beta)$-expansions, no natural extension of this form is known. For Pisot bases, the construction of a natural extension from Section 2.2 can certainly be extended to negative bases, although we cannot use half-open intervals, hence the tilings and the characterisation of purely periodic $(-\beta)$-expansions might not be as nice as for β-expansions.

3.3. Finite $(-\beta)$-expansions

In Sections 2.3 and 2.4, we have seen the importance of the finiteness condition (F_β). The corresponding condition for $(-\beta)$-expansions is

$$(\mathrm{F}_{-\beta}) \qquad \text{for each } x \in \mathbb{Z}[\beta] \cap \left[\tfrac{-\beta}{\beta+1}, \tfrac{1}{\beta+1}\right) \text{ we have } U_{-\beta}^k(x) = 0 \text{ for some } k \geq 0.$$

(Here, it makes no sense to replace $U_{-\beta}$ by $T_{-\beta}$.) With Krčmáriková and Vávra [**27**], we gave several necessary or sufficient conditions for β having property $(\mathrm{F}_{-\beta})$.

Theorem 3.3. — *Let $\beta > 1$.*
- *If $U_{-\beta}^n(\frac{-\beta}{\beta+1}) = 0)$ for some $n \geq 1$, then β does* not *satisfy $(\mathrm{F}_{-\beta})$.*
- *If $p(\beta) = 0$ and $|p(-1)| = 1$ for some $p(x) \in \mathbb{Z}[x]$, then β does* not *satisfy $(\mathrm{F}_{-\beta})$.*
- *If $\beta^d = m\beta^{d-1} + \cdots m\beta + m$ with positive integers d, m, then β satisfies $(\mathrm{F}_{-\beta})$ if and only if $d \in \{1, 3, 5\}$.*
- *If $\beta^d = (-1)^{d+1} \sum_{k=1}^d a_k(-\beta)^{d-k}$ with integers $a_k \geq 0$ and $a_1 \geq 2 + \sum_{k=2}^d a_k$, then β satisfies $(\mathrm{F}_{-\beta})$.*

We also characterised the cubic Pisot units satisfying $(\mathrm{F}_{-\beta})$. For $\beta^3 = m\beta^2 + m\beta + m$, we did not only show $(\mathrm{F}_{-\beta})$ but we also calculated the smallest n such that the length of the $(-\beta)$-fractional part of $x \pm y$ with two $(-\beta)$-integers x, y is at most n. Here, the $(-\beta)$-fractional part of $x \in (-\beta)^k\left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$, $k \geq 0$, is $U_{-\beta}^k(x(-\beta)^{-k})$, its length is the smallest $n \geq 0$ such that $U_{-\beta}^{k+n}(x(-\beta)^{-k}) = 0$, i.e., x has $(-\beta)$-expansion

$$x = a_1(-\beta)^{k-1} + \cdots - a_{k-1}\beta + a_k - a_{k+1}\beta^{-1} + \cdots + a_{k+n}(-\beta)^{-n},$$

and x is a $(-\beta)$-integer if $n = 0$; see also the following section. The corresponding length of the β-fractional parts for (positive) β-expansions was only determined for $m = 1$ by Bernat [**66**].

Open question 3.3. — What is the maximal length of β-fractional parts of $x \pm y$ for β-integers x, y, when $\beta^3 = m\beta^2 + m\beta + m$, $m \geq 2$?

3.4. $(-\beta)$-integers

Even when $(\mathrm{F}_{-\beta})$ does not hold, it is interesting to study the set of numbers with finite $(-\beta)$-expansions. The set of β-integers $Z_\beta = \bigcup_{k\geq 0} \beta^k T_{-\beta}^{-k}(0)$ (not to be confused with the set of β-adic integers defined in Section 2.2) was introduced by Gazeau in the 90's in connections with quasicrystals and studied by Burdík, Frougny, Gazeau and Krejcar [**76**] and many others. For negative bases, the set of $(-\beta)$-integers

$$Z_{-\beta} = \bigcup_{k\geq 0} (-\beta)^k U_{-\beta}^{-k}(0)$$

was first considered by Ambrož, Dombek, Masáková and Pelantová [**59**]. In [**22**], we show that the sequence of distances between consecutive points in $Z_{-\beta}$ is the fixed-point of an antimorphism (which is defined on a finite alphabet if and only if β is an Yrrap number), see also [**24**], and we discuss the Delone property of $Z_{-\beta}$. It is well known that Z_β is always relatively dense, and it is uniformly discrete if and only if 0 is not an accumulation point of $\{T_\beta^k(1^-) : k \geq 0\}$. From [**22**], we see that the situation is more complicated for $(-\beta)$-integers, and it is possible to find $\beta > 1$ where $Z_{-\beta}$ is not relatively dense.

Theorem 3.4. — *If $d_{-\beta}(1)$ is the fixed point of $3 \mapsto 30032$, $2 \mapsto 2$, $0 \mapsto 00$, starting with 3, then $Z_{-\beta}$ is not relatively dense. If 0 is not an accumulation point of $\{U_{-\beta}^{2k-1}(\frac{-\beta}{\beta+1}) < 0 : k \geq 1\}$, then the set $Z_{-\beta}$ is uniformly discrete.*

Open question 3.4. — For which numbers $\beta > 1$, $Z_{-\beta}$ is not uniformly discrete or relatively dense?

3.5. Permutations in $(-\beta)$-expansions

The complexity of a dynamical system is usually measured by its entropy. For symbolic dynamical systems, the (topological) entropy is the logarithm of the exponential growth rate of the number of distinct patterns of length n. Bandt, Keller and Pompe [**62**] proved for piecewise monotonic maps that the entropy is also given by the number of permutations defined by consecutive elements in the trajectory of a point. Note that the entropy of T_β and $T_{-\beta}$ (as well as all piecewise linear transformations with slope of absolute value β) is $\log \beta$. Elizalde [**87**] gave for each permutation π a characterisation of the infimum $B_+(\pi)$ of those β where π occurs as the ordering of consecutive elements in the trajectories of the β-shift. With Charlier [**28**], we do the same for the $(-\beta)$-shift and show that all the $B_-(\pi) = \inf\{\beta' > 1 : \pi \text{ occurs in the } (-\beta')\text{-shift}\}$ are Yrrap numbers.

Theorem 3.5. — *Let $\beta > 1$. There is some permutation π such that $B_-(\pi) = \beta$ if and only if β is an Yrrap number.*

Open question 3.5. — What is the number of permutations $\pi \in S_n$ with $B_-(\pi) = 1$?

CHAPTER 4

CONTINUED FRACTIONS AND S-ADIC SEQUENCES

In the two previous chapters, we have considered positional numeration systems. Continued fractions can be seen as a non-positional numeration system. Their main purpose is to approximate real numbers by rational numbers. We discuss variants of the regular continued fractions (in dimension one) as well as multidimensional continued fraction algorithms. As for β-expansions, we construct natural extensions and Rauzy fractals. However, here the Rauzy fractals are not building blocks of the natural extensions but they provide natural codings of torus translations; each sequence of partial quotients defines a symbolic (S-adic) dynamical system which is (almost always) conjugate to a torus translation. The publications for this chapter are [**29**]–[**39**].

4.1. α-continued fractions

The (regular) continued fraction expansion of $x \in [0,1)$ is

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}},$$

with positive integer digits a_i (usually called *partial quotients*); arbitrary real numbers can be represented by adding integers a_0. Nakada [**120**] defined a variant of these continued fractions by setting

$$x = \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \ddots}},$$

where, for given $\alpha \in [0,1]$, $\varepsilon_k = \mathrm{sgn}(T_\alpha^{k-1}(x))$ and $a_k = \lfloor 1/T_\alpha^{k-1}(x) \rfloor$, with the α-continued fraction transformation

$$T_\alpha : [\alpha-1,\alpha) \to [\alpha-1,\alpha), \quad x \mapsto \left|\frac{1}{x}\right| - \left\lfloor \left|\frac{1}{x}\right| + 1 - \alpha \right\rfloor \text{ if } x \neq 0, \quad 0 \mapsto 0;$$

see Figure 4.1. We can write $T_\alpha(x) = \begin{pmatrix} -a_1 & \varepsilon_1 \\ 1 & 0 \end{pmatrix} \cdot x$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}$ denotes a Möbius transformation. A slightly different kind of α-expansions was defined by Tanaka and Ito [**132**] by the map $T_\alpha : [\alpha-1,\alpha) \to [\alpha-1,\alpha)$, $x \mapsto \frac{1}{x} - \lfloor \frac{1}{x} + 1 - \alpha \rfloor$. Generalisations of these expansions were considered by Shallit [**130**].

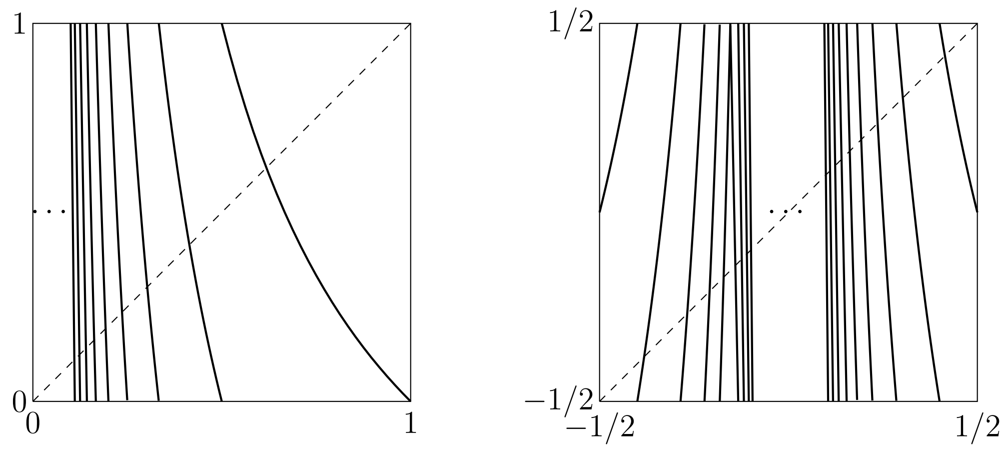

FIGURE 4.1. The Gauss map T_1 and the α-continued fraction transformation $T_{1/2}$.

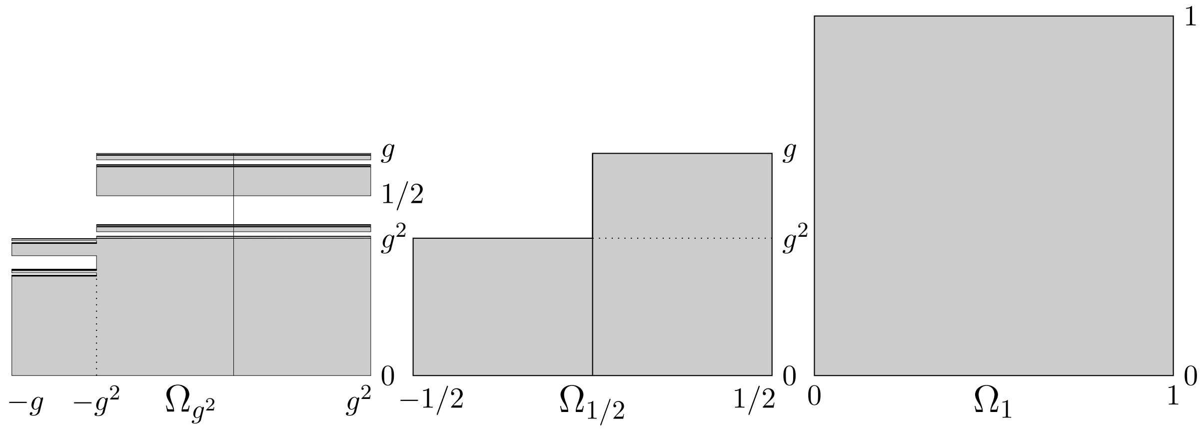

FIGURE 4.2. The natural extension domain Ω_α for Nakada's α-continued fractions with $\alpha \in \{g^2, 1/2, 1\}$, $g = \frac{1+\sqrt{5}}{2}$.

For both kinds of α-expansions, we can define a natural extension domain by

$$\hat{X}_\alpha = \bigcup_{x \in [\alpha-1,\alpha)} \{x\} \times \mathcal{D}_\alpha(x),$$

similarly to β-expansions, with

$$\mathcal{D}_\alpha(x) = \lim_{k\to\infty} \{{}^tM^{-1} \cdot 0 :\ M^{-1} \cdot x \in T_\alpha^{-k}(x)\},$$

and the natural extension map

$$\hat{T}_\alpha :\ \hat{X}_\alpha \to \hat{X}_\alpha, \quad (x,y) \mapsto (M \cdot x, {}^tM^{-1} \cdot y) \quad \text{if } T_\alpha(x) = M \cdot x.$$

For Nakada's α-continued fractions, we have $\mathcal{D}_\alpha(x) \subseteq \mathcal{D}_\alpha(y)$ if $\alpha-1 \le x \le y < \alpha$, $\hat{X}_\alpha$ is a union of at most three rectangles when $\sqrt{2}-1 \le \alpha \le 1$, and $\hat{X}_\alpha$ is a union of countably many rectangles when $0 \le \alpha < \sqrt{2}-1$; see Figure 4.2. With Kraaikamp and Schmidt [**30**], we gave explicit formulas for $\hat{X}_\alpha$ for all parameters $\alpha \in [0,1]$, extending works of Nakada [**120**] and Luzzi and Marmi [**117**]. Here, matching of the form $T_\alpha^m(\alpha-1) = T_\alpha^n(\alpha^-)$ (and pre-matching sets) play a big role. If matching occurs, then the natural extension can be described by a regular language, and the entropy $h(T_\alpha)$ (which gives the convergence rate of the continued fractions) is increasing or decreasing (in α) depending on the sign of $m-n$. Matching occurs for α in intervals covering $[0,1]$ save a set of Lebesgue measure 0 but Hausdorff dimension 1. Instead of the entropy, we rather study $\mu(\hat{X}_\alpha)$ with the (not normalised) invariant measure $\mathrm{d}\mu = \mathrm{d}x\mathrm{d}y/(1+xy)^2$ because of the following result of [**30**].

Theorem 4.1. — *We have $h(T_\alpha)\mu(\hat{X}_\alpha) = \pi^2/6$, $\mu(\hat{X}_\alpha)$ is continuous on $[0,1]$, and $\hat{X}_\alpha$ contains $[\alpha-1,\alpha) \times [0, 1/(d+1)]$ if $\alpha-1 \le 1/\alpha-d < \alpha$.*

For Tanaka–Ito α-continued fractions, we proved with Carminati and Langeveld [**38**] that matching also occurs for α in intervals covering $[0,1]$ save a set of Lebesgue measure 0 but Hausdorff dimension 1. It seems to be more difficult to give a description of the natural extension domain $\hat{X}_\alpha$ in terms of unions of rectangles than in Nakada's case, but we exhibited with Nakada [**37**] for each $\alpha \in [0,1]$ some rectangle contained in $\hat{X}_\alpha$. Interestingly, in this setting, we do not have $h(T_\alpha)\mu(\hat{X}_\alpha) = \pi^2/6$, but $h(T_\alpha)\mu(\hat{X}_\alpha)$ is a montonically increasing function for $\alpha \in [1/2, 1]$, with $h(T_{1/2})\mu(\hat{X}_{1/2}) = \pi^2/3$ and $h(T_1)\mu(\hat{X}_1) = \pi^2/6$.

Open question 4.1. — Is there a dynamical interpretation of $h(T_\alpha)\mu(\hat{X}_\alpha)$?

Another generalisation of continued fractions consists in choosing integer multiples of $\lambda = 2\cos(\pi/q)$ as partial quotients, with a fixed integer $q \ge 3$. This is related to Hecke groups. Rosen [**126**] used the transformation $T:\ \left[\frac{\lambda}{2}, \frac{\lambda}{2}\right) \to \left[\frac{\lambda}{2}, \frac{\lambda}{2}\right)$, $x \mapsto \left|\frac{1}{x}\right| - \lambda\left\lfloor\left|\frac{1}{x\lambda}\right| + \frac{1}{2}\right\rfloor$ to define continued fractions. Since $\lambda = 1$ when $q = 3$, these are generalisations of the nearest integer continued fractions. With Dajani and Kraaikamp [**29**], we introduced α-Rosen fractions with $\alpha \in \left[\frac{\lambda-1}{\lambda}, \frac{1}{\lambda}\right]$, generalising both Nakada's α-continued fractions and Rosen continued fractions. We determined natural extensions (which are finite unions of rectangles) for $\alpha \in \left[\frac{1}{2}, \frac{1}{\lambda}\right]$ and showed that the transformations are weakly Bernoulli.

4.2. Multidimensional continued fraction algorithms

A *multidimensional continued fraction algorithm* provides approximations to real vectors by rational vectors. However, there is no algorithm that shares all the good properties of the regular continued fraction algorithm (and the α-continued fractions). Instead, we have a wide range of different algorithms with different properties.

In this section, we consider (semi-ordered) d-dimensional continued fraction algorithms that are defined on a set $\Delta \subseteq [0,1]^d$ by

$$T:\ \Delta \to \Delta, \quad \mathbf{x} \mapsto \kappa\big(\iota(\mathbf{x})A(\mathbf{x})^{-1}\big)$$

where $\iota(x_1,\ldots,x_d) = (1, x_1, \ldots, x_d)$, $\kappa(x_0, x_1, \ldots, x_d) = (\frac{x_1}{x_0}, \ldots, \frac{x_d}{x_0})$, and

$$A:\ \Delta \to \mathrm{GL}(d{+}1, \mathbb{Z})$$

is a piecewise constant map assigning to each $\mathbf{x} \in \Delta$ an invertible integer matrix. Note that $T \circ \kappa = \kappa \circ L$ with the piecewise linear map $L: \kappa^{-1}(\Delta) \to \kappa^{-1}(\Delta)$, $\mathbf{y} \mapsto \mathbf{y}A(\kappa(\mathbf{y}))$. We usually assume that the algorithm is *positive*, i.e., that all $A(\mathbf{x})$ are nonnegative matrices; semi-ordered means that the first coordinate of $\iota(\mathbf{x})$ is the largest coordinate.

For example, the regular (1-dimensional) continued fractions are given for $x \neq 0$ by

$$A(x) = \begin{pmatrix} \lfloor \frac{1}{x} \rfloor & 1 \\ 1 & 0 \end{pmatrix}, \quad T(x) = \kappa\Big((1,x) \begin{pmatrix} 0 & 1 \\ 1 & -\lfloor \frac{1}{x} \rfloor \end{pmatrix}\Big) = \kappa\big(x, 1 - x\lfloor \tfrac{1}{x} \rfloor\big) = \tfrac{1}{x} - \lfloor \tfrac{1}{x} \rfloor.$$

The *cocycle associated with* A is

$$A^{(n)}(\mathbf{x}) = A(T^{n-1}\mathbf{x}) \cdots A(T\mathbf{x})\, A(\mathbf{x}),$$

i.e., we have $A^{(m+n)}(\mathbf{x}) = A^{(n)}(T^m\mathbf{x})A^{(m)}(\mathbf{x})$ for all $m, n \geq 0$. The sequences of rational convergents $\mathbf{p}_i^{(n)}/q_i^{(n)}$ to $\mathbf{x}$, $0 \leq i \leq d$, are then defined by

$$A^{(n)}(\mathbf{x}) = \begin{pmatrix} q_0^{(n)} & p_{0,1}^{(n)} & \cdots & p_{0,d}^{(n)} \\ q_1^{(n)} & p_{1,1}^{(n)} & \cdots & p_{1,d}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ q_d^{(n)} & p_{d,1}^{(n)} & \cdots & p_{d,d}^{(n)} \end{pmatrix}, \quad \mathbf{p}_i^{(n)} = (p_{i,1}^{(n)}, \ldots, p_{i,d}^{(n)}).$$

Their convergence to $\mathbf{x}$ is said to be *weak* if $\lim_{n\to\infty} \mathbf{p}_i^{(n)}/q_i^{(n)} = \mathbf{x}$ for all i with $0 \leq i \leq d$, and *strong* if $\lim_{n\to\infty} |\mathbf{p}_i^{(n)} - q_i^{(n)}\mathbf{x}| = 0$ for all $0 \leq i \leq d$. The matrices

$$D^{(n)}(x_1, x_2, \ldots, x_d) = \begin{pmatrix} p_{1,1}^{(n)} - q_1^{(n)}x_1 & \cdots & p_{1,d}^{(n)} - q_1^{(n)}x_d \\ \vdots & \ddots & \vdots \\ p_{d,1}^{(n)} - q_d^{(n)}x_1 & \cdots & p_{d,d}^{(n)} - q_d^{(n)}x_d \end{pmatrix}$$

also form a cocycle w.r.t. T. Under certain conditions (given by Lagarias [**110**]) that are satisfied by all continued fraction algorithms which we consider, the second *Lyapunov exponent* of A (with respect to a suitable invariant measure), which describes the rate of convergence of $\mathbf{p}_i^{(n)}$ to $q_i^{(n)}\mathbf{x}$, is given by the first Lyapunov exponent of D. Since it is usually much easier to determine the first Lyapunov exponent, which describes the growth rate, than the second one, we study D rather than A. For several continued fraction algorithms in dimension $d = 2$ (Brun, Jacobi–Perron, Selmer, ...), it is known that the second Lyapunov exponent of A is negative, i.e., that $|\mathbf{p}_i^{(n)} - q_i^{(n)}\mathbf{x}|$ tends to 0 exponentially. In [**36**], we give a simple proof of this fact for the Selmer algorithm and a lower bound that is of the same order as the numerically estimated value -0.07072.

Theorem 4.2. — *For $d = 2$, the second Lyapunov exponent of the Selmer algorithm is less than -0.05039. In particular, the Selmer algorithm is strongly convergent for almost all $\mathbf{x}$.*

For other dimensions and algorithms, we calculate $D^{(2^{30})}(\mathbf{x})$ for "randomly" chosen $\mathbf{x}$ in order to give heuristics for the first and second Lyapunov exponents. Table 4.1 contains the obtained value for the uniform approximation exponent, which is given by the ratio between the first and the second Lyapunov exponents. These heuristics indicate that the second Lyapunov exponent is positive for all considered continued fraction algorithms in dimension $d \geq 11$, except for the Arnoux–Rauzy algorithm, which however is only defined on a set Δ of measure zero. This contradicts several conjectures in the literature. We are currently not aware of any method that permits proving that the second Lyapunov exponent of a continued fraction algorithm (in high dimension) is positive.

From the assumption that A is positive, it is usually not difficult to infer that the cones $\mathbb{R}_+^d A^{(n)}(\mathbf{x})$ converge to $\mathbb{R}_+\mathbf{x}$, i.e., that we have weak convergence. If we drop the

d	Selmer	Brun	Jacobi–Perron	Intermediate
2	1.3871	1.3683	1.3735	1.3606
3	1.1444	1.2203	1.1922	1.2430
4	0.9866	1.1504	1.1114	1.1817
5	0.8577	1.1065	1.0676	1.1388
6	0.7442	1.0746	1.0413	1.1034
7	0.6437	1.0493	1.0243	1.0729
8	0.5561	1.0283	1.0127	1.0468
9	0.4810	1.0102	1.0044	1.0246
10	0.4173	0.9943	0.9981	1.0054
11	0.3636	0.9799	0.9933	0.9886

TABLE 4.1. Uniform approximation exponents $1 - \frac{\theta_2(A)}{\theta_1(A)}$ for the Selmer, Brun and Jacobi–Perron algorithms, and an intermediate algorithm defined in [**36**].

assumption of positivity, then weak convergence is more difficult to show (except when the algorithm is designed in a way that guarantees this property, e.g., Lagarias' geodesic multidemensional continued fractions [**111**]). Nevertheless, many α-continued fractions (as defined in Section 4.1) converge faster than the regular continued fractions; this is true in particular for the nearest integer continued fractions ($\alpha = 1/2$). It seems that the nearest integer variant of the Jacobi–Perron algorithm not only converges faster but also has strong convergence in higher dimensions than the usual algorithm, namely up to $d = 13$ instead of $d = 10$.

Open question 4.2. — Is there a "simple" continued fraction algorithm (defined on a subset of $[0,1]^d$ of positive Lebesgue measure) such that the second Lyapunov is negative for all $d \geq 2$, in particular for $d = 14$?

4.3. S-adic sequences and Rauzy fractals

The regular continued fractions are intimately related to Sturmian words. There is a wealth of equivalent definitions of Sturmian words, for example they are discretisations of lines with irrational slope. Standard Sturmian words can be written as $w = L(v)$ or $w = \tilde{R}(v)$, with the substitutions $L : 0 \mapsto 0,\ 1 \mapsto 01$, $\tilde{R} : 0 \mapsto 10,\ 1 \mapsto 1$, v another standard Sturmian word; L is as in Section 2.7 and $\tilde{R}$ is the reversal of R. (Each Sturmian word has the same language as a standard Sturmian word.) Therefore, w is the limit word of a sequence of substitutions $(\sigma_n)_{n\geq 0} \in \{L, \tilde{R}\}^\omega$. More precisely, we have $\sigma_0\sigma_1 \cdots = L^{a_1}\tilde{R}^{a_2}L^{a_3}\tilde{R}^{a_4}\cdots$, where $a_1a_2\cdots$ is the sequence of partial potients of the slope of w.

(Standard) Arnoux–Rauzy words generalise this definition to larger alphabets. They are limit words of primitive sequences of substitutions in $\{\alpha_0, \alpha_1, \ldots, \alpha_d\}^\omega$, with $\alpha_i : i \mapsto i,\ j \mapsto ij$ for $j \neq i$, $i, j \in \{0, 1, \ldots, d\}$. If the limit word of $(\sigma_n)_{n\geq 0}$ has the letter frequency vector $\mathbf{x}$, then $\mathbf{x} = M_{\sigma_0}\mathbf{x}'$, where M_{σ_0} is the incidence matrix of σ_0 and $\mathbf{x}'/\|\mathbf{x}'\|_1$ is the frequency vector of $(\sigma_n)_{n\geq 1}$. The cones $M_{\sigma_0}\cdots M_{\sigma_n}\mathbb{R}_+^d$ converge to the half line $\mathbb{R}_+\mathbf{x}$, and we say that $\mathbf{x}$ is a generalised right eigenvector of $(\sigma_n)_{n\geq 0}$. Therefore, $(M_{\sigma_n})_{n\geq 0}$ is a d-dimensional continued fraction expansion of $\mathbf{x}$, and the columns of $M_{\sigma_0}\cdots M_{\sigma_n}$ are integer approximations of $\mathbb{R}\mathbf{x}$. However, the approximation properties are not as good as in the onedimensional case. In particuar, one would like that the abelianised prefixes of Arnoux–Rauzy words always stay within bounded distance of its direction, but this *balancedness* property is not always true; see Section 4.4.

With a primitive sequence of substitutions $\boldsymbol{\sigma}$, we associate the symbolic dynamical system $(X_{\boldsymbol{\sigma}}, \Sigma)$, where Σ is the left shift and $X_{\boldsymbol{\sigma}}$ is the shift closure of a limit word of $\boldsymbol{\sigma}$. This dynamical system is called an *S-adic shift*. Under mild conditions, $\boldsymbol{\sigma}$ has a generalised right eigenvector $\mathbf{x}$. Therefore, shift-invariant sets of sequences of substitutions define a multidimensional continued fraction algorithm (provided that different sequences of substitutions have different generalised right eigenvectors); conversely, we associate with a positive continued fraction algorithm a set of substitutions and thus a set of S-adic shifts.

Instead of semi-ordered continued fraction algorithms as in Section 4.2, we often prefer here *symmetric* algorithms, which means that they are invariant under permutations of coordinates. We consider positive d-dimensional continued fraction algorithms defined on $\Delta \subseteq \{\mathbf{x} \in [0,1]^{d+1} : \|\mathbf{x}\|_1 = 1\}$ by

$$A : \Delta \to \mathrm{GL}(d{+}1, \mathbb{Z}), \qquad T : \Delta \to \Delta, \quad \mathbf{x} \mapsto \frac{{}^tA(\mathbf{x})^{-1}\mathbf{x}}{\|{}^tA(\mathbf{x})^{-1}\mathbf{x}\|_1};$$

since $\mathbf{x}$ is now a column vector, we need the transpose of $A(\mathbf{x})$. By choosing a *substitution selection* φ that assigns to each $\mathbf{x} \in \Delta$ a substition $\varphi(\mathbf{x})$ with incidence matrix $M_{\varphi(\mathbf{x})} = {}^tA(\mathbf{x})$, the continued fraction algorithm provides thus a sequence of substitutions $\boldsymbol{\varphi}(\mathbf{x}) = (\varphi(T^n(\mathbf{x})))_{n\ge0}$ with generalised right eigenvector $\mathbf{x}$.

For a primitive sequence of substitutions $\boldsymbol{\sigma}$ on the alphabet $\{0,1,\ldots,d\}$ with generalised right eigenvector $\mathbf{u}$, we can define the *Rauzy fractal* $\mathcal{R}_{\boldsymbol{\sigma}}$ by projecting the abelianisations of prefixes of a limit word of $\boldsymbol{\sigma}$ along $\mathbf{u}$ onto some hyperplane, and taking the closure; this is illustrated in Figure 4.3. It has subtiles $\mathcal{R}_{\boldsymbol{\sigma}}(i)$ defined by the prefixes that are followed by the letter i in the limit word. The original Rauzy fractal was defined by Rauzy [**124**] for the Tribonacci substitution $\tau : 0 \mapsto 01,\ 1 \mapsto 02,\ 2 \mapsto 0$; note that $\tau^3 = \alpha_0\alpha_1\alpha_2$. One motivation for Rauzy's construction was to show that $(X_{\tau^\omega}, \Sigma)$ is measure-theoretically isomorphic to a translation on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ (with $d = 2$). The *Pisot substitution conjecture* states that this is true for all Pisot unit substitutions, i.e., substitutions with unimodular incidence matrix having one eigenvalue > 1 and all other eigenvalues strictly inside the unit circle. This conjecture is still open, even if it has been confirmed for a lot of cases; see e.g. [**54**].

Arnoux, Mizutani and Sellami [**60**] defined Rauzy fractals associated to S-adic systems (by projecting the vertices of the broken line on a suitable hyperplance), but only when all substitutions have the same (Pisot) incidence matrix. With Berthé and Thuswaldner [**34**], we considered the general case. Under a set of conditions which we called *PRICE* (which stands for *P*rimitivity, *R*ecurrence and algebraic *I*rreducibility of the sequence of substitutions, *C*-balancedness of the limit word and the existence of a generalised left *E*igenvector), the Rauzy fractals form a multiple tiling. The conditions are not necessary, and we would like to remove in particular the recurrence condition from PRICE. As in the case of a single substitution, there are *geometric coincidence conditions* that guarantee a (simple) tiling, and we conjecture that they are always verified, i.e., that the S-adic shift is measure-theoretically conjugate to a translation on the torus if $\boldsymbol{\sigma}$ satisfies PRICE; this is our *S-adic Pisot conjecture*. With Berthé, Minervino and Thuswaldner [**33**], we proved the S-adic Pisot conjecture for words on 2 letters.

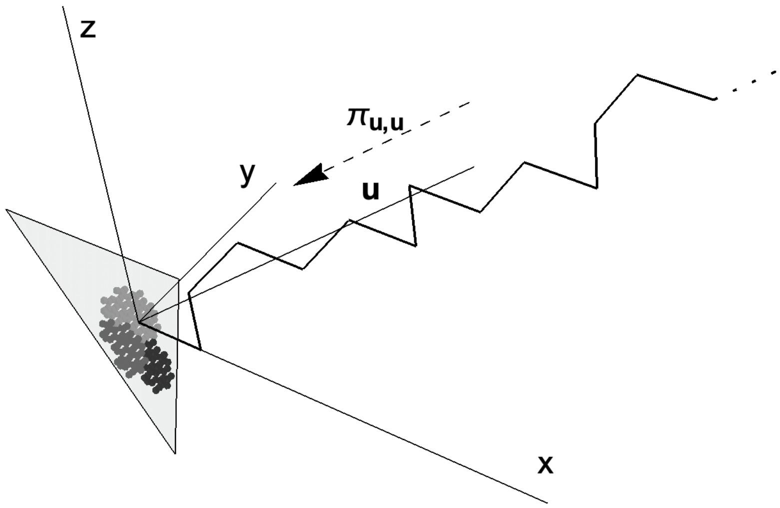

FIGURE 4.3. The construction of a Rauzy fractal $\mathcal{R}_{\boldsymbol{\sigma}}$.

For continued fraction algorithms, we prove the following result in [**39**]. Here, a continued fraction algorithm satisfies the *Pisot condition* if the second Lyapunov of its cocycle is negative. If the Pisot condition holds, then we have a multiple tiling for almost all sequences [**34**]. Moreover, it suffices to prove for a single $\mathbf{x}$ that we have a tiling to obain the same property for almost all points. The following theorem is a special case of [**39**, Theorem 3.1].

Theorem 4.3. — *Let (Δ, T, A, ν) be one of the following continued multidimensional continued fraction algorithms: Brun with $d \leq 3$, Cassaigne–Selmer or Jacobi–Perron with $d = 2$, with invariant measure ν equivalent to Lebesgue. Let $\boldsymbol{\varphi}$ be the associated substitution selection. Then, for almost all $(x_0, x_1, \ldots, x_d) \in \Delta$, the S-adic dynamical system $(X_{\boldsymbol{\varphi}(x_0,x_1,\ldots,x_d)}, \Sigma)$ is a natural coding by bounded Rauzy fractals of the (minimal) translation by $(x_1, \ldots, x_d)$ on $\mathbb{R}^d/\mathbb{Z}^d$.*

In particular, when $d \leq 3$, we have for almost all vectors $\mathbf{t} \in \mathbb{R}^d$ partitions of the torus $\mathbb{R}^d/\mathbb{Z}^d$ such that the coding of the translation by $\mathbf{t}$ is conjugate to an S-adic Brun shift.

The particularity of the natural codings in Theorem 4.3 is that length n subtiles are also bounded remainder sets, for all $n \geq 1$. (Other natural codings are given e.g. by billiards, where refinements of the partititions do not provide bounded remainder sets.)

Open question 4.3. — Let $\{F_1, \ldots, F_d\}$ be a partitition of $\mathbb{R}^d/\mathbb{Z}^d$ generating a natural coding of a translation such that each set $\bigcap_{k=0}^{n} R_{\mathbf{t}}^{-k} F_{i_k}$, $i_0 \cdots i_n \in \{1, \ldots, h\}^*$, is a bounded remainder set. Is it possible that the sets F_i have smooth boundaries?

We have studied only S-adic systems with unimodular incidence matrices, coming from usual (multidimensional) continued fraction alogirhtms. However, there also exist (onedimensional) continued fraction algorithms with non-unimodular matrices; see e.g. Dajani, Kraaikamp and van der Wekken [**82**] and Dajani, Kraaikamp and Langeveld [**80**]. These algorithms give S-adic words that generalise the Sturmian words. One can associate Rauzy fractals to these shifts, by adding $\varprojlim \mathbb{Z}^d/M_0M_1\cdots M_k\mathbb{Z}^d$ to the representation space (similarly to non-unit Pisot numbers or rational self-affine tiles). Most results from [**34**] remain valid, and one gets tilings and rotations on compact groups, generalising those on the torus and also tilings for non-unit Pisot numbers. What are the combinatorial properties (e.g. pattern complexity, balancedness) of these S-adic words?

4.4. Balancedness of S-adic words

A word is C-balanced if the difference between the number of occurrences of a letter in factors u and v of the same length is bounded by C. The word is balanced if such a C exists and unbalanced otherwise. (Sometimes, balancedness refers only to 1-balancedness, but we use the broader definition here.) Balanced words have letter frequencies and stay in bounded distance to the line of the frequency vector. Sturmian words are exactly the aperiodic 1-balanced words.

Cassaigne, Ferenczi and Zamboni [**77**] gave examples of unbalanced Arnoux–Rauzy sequences. On the other hand, we showed with Berthé and Cassaigne [**31**] that many Arnoux–Rauzy sequences on three letters are balanced, in particular those with "bounded partial quotients". We also showed that a large class of Arnoux–Rauzy sequences is 2-balanced. Using results of Avila and Delecroix [**61**], we showed with Delecroix and Hejda [**32**] that almost all Arnoux–Rauzy sequences (on arbitrary alphabets) as well as Brun sequences on 3 letters are balanced. Moreover, we gave the following result for sequences with strong partial quotients.

Theorem 4.4. — *There is a constant $C(h)$ such that each Arnoux–Rauzy word (on $d \geq 3$ letters) and each Brun word on 3 letters with strong partial quotients bounded by h is $C(h)$-balanced.*

Here, the strong partial quotients are bounded by h if each block of h substitutions contains all Arnoux–Rauzy and Brun substitutions respectively.

Open question 4.4. — Are there 2-balanced Brun words?

4.5. Recognisability for sequences of substitutions

Given a substitution σ and a long enough word w in the language generated by σ, *recognisability* is a form of injectivity of σ that allows one to uniquely *desubstitute* most of w to another word v, i.e., express w as a concatenation of substitution words dictated by the letters in v, with v traditionally required to be in the substitution's language. With Berthé, Thuswaldner and Yassawi [**35**], we investigate different notions of recognisability. Full recognisability occurs when each (aperiodic) point admits at most one tiling with words in the image of σ. This is stronger than the classical notion of recognisability of a substitution, where the tiling must be compatible with the language of the substitution. We show that if the substitution is on two letters, or if its incidence matrix has full rank, or if σ is permutative, then σ is fully recognisable. Next we investigate the classical notion of recognisability and improve earlier results of Mossé [**119**] and Bezuglyi, Kwiatkowski and Medynets [**71**], by showing that any substitution is recognisable for aperiodic points in its substitutive shift. Finally we define recognisability and also eventual recognisability for sequences of morphisms which define an S-adic shift. We prove that a sequence of morphisms on alphabets of bounded size, such that compositions of consecutive morphisms are growing on all letters, is eventually recognisable for aperiodic points. We provide examples of eventually recognisable, but not recognisable, sequences of morphisms, and sequences of morphisms which are not eventually recognisable. As an application, for a recognisable sequence of morphisms, we obtain an almost everywhere bijective correspondence between the S-adic shift it generates, and the measurable Bratteli–Vershik dynamical system that it defines.

Theorem 4.5. — *If a substitution σ is on two letters or left or right permutative, then σ is fully recognisable on aperiodic points. If a sequence of substitutions $\boldsymbol{\sigma}$ is eventually growing, then $\boldsymbol{\sigma}$ is eventually recognisable on aperiodic points.*

Here, left permutative means that the first letters of $\sigma(a)$ and $\sigma(b)$ are different for all distinct letters a, b; right permutative refers to the last letters. A sequence of substitutions $(\sigma_n)_{n\geq 0}$ is everywhere growing if the length of the words $\sigma_0\sigma_1\cdots\sigma_n(a)$ tends to infinity for all letters a.

Open question 4.5. — Which sequences of substitutions (without aperiodic points) are recognisable?

4.6. Sequence related to π and $\sqrt{2}$

With Bosma and Dekking [**49**], we prove that the following five ways to define entry **A086377** in the OEIS do lead to the same integer sequence.

(a_n) defined by $a_1 = 1$ and for $n \geq 2$:

$$a_n = \begin{cases} a_{n-1} + 2 & \text{if } n \text{ is in the sequence,} \\ a_{n-1} + 2 & \text{if } n \text{ and } n-1 \text{ are not in the sequence,} \\ a_{n-1} + 3 & \text{if } n \text{ is not in the sequence, but } n-1 \text{ is in the sequence;} \end{cases}$$

(b_n) defined by $b_1 = 1$ and for $n \geq 2$:

$$b_n = \begin{cases} b_{n-1} + 2 & \text{if } n-1 \text{ is not in the sequence,} \\ b_{n-1} + 3 & \text{if } n-1 \text{ is in the sequence;} \end{cases}$$

(c_n) for $n \geq 1$ defined as the position of the n-th zero in the fixed point of the morphism

$$\phi:\ 0 \mapsto 011, \quad 1 \mapsto 01;$$

(d_n) defined by $d_n = \lfloor (1+\sqrt{2})\, n - 1/\sqrt{2} \rfloor$ for $n \geq 1$;

(e_n) defined by $e_n = \lceil r_n \rfloor = \lfloor r_n + \frac{1}{2} \rfloor$, with $r_1 = \frac{4}{\pi}$ and $r_{n+1} = \frac{n^2}{r_n - (2n-1)}$, for $n \geq 1$.

The paper shows therefore links between recursions, morphic sequences, Beatty sequences (and Sturmian sequences) related to $\sqrt{2}$ and a continued fraction expansion of $4/\pi$.

Open question 4.6. — Are there similar relations between other quadratic numbers and continued fraction expansions?

CHAPTER 5

OTHER NUMERATION SYSTEMS

In Chapters 2 and 3, we have studied one-dimensional positional numeration systems, in Chapter 4 we have considered higher-dimensional non-positional numeration systems. Now we turn first to higher-dimensional positional numeration systems, then to shift radix systems, which encompass other systems, finally to abstract numeration systems for integers. We discuss the publications [**40**]–[**45**].

5.1. Canonical and rational base number systems

A generalisation of integer base numeration systems consists in taking an expanding matrix $M \in \mathbb{Z}^{d\times d}$ as base and a "digit set" $A \subset \mathbb{Z}^d$ (with $0 \in A$). This is more general than taking an algebraic integer α (with all conjugates outside the unit circle) and a digit set $A \subset \mathbb{Z}[\alpha]$, which was first considered by Knuth [**104**] for $\alpha = 2i$, $A = \{0,1,2,3\}$, and by Penney [**123**] for $\alpha = -1+i$, $A = \{0,1\}$. Here, our main object of study is the fundamental domain $F = \{\sum_{k=1}^\infty M^{-k} a_k : a_k \in A\}$. Each point in $\mathbb{R}^d$ has an expansion $\sum_{k=\ell}^\infty M^{-k} a_k$ if and only if 0 is an inner point of F. A general characterisation of pairs (M, A) satisfying this property seems to be difficult, even for simple digit sets; see also the next section. Another problem is to know whether almost all points in $\mathbb{R}^d$ have at most one expansion in this system, i.e., whether F induces a lattice tiling with the lattice generated by $\bigcup_{k=0}^\infty M^k A$. Note that F is the solution of the set equation $MF = \bigcup_{a\in A}(F+a)$, i.e., an integral self-affine tile. Lagarias and Wang [**112**] showed that the tiling property holds for all M with irreducible characteristic polynomial and for all complete residue systems A of $\mathbb{Z}^d/M\mathbb{Z}^d$.

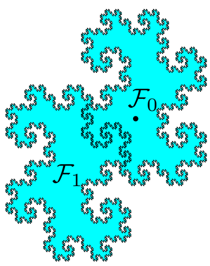

FIGURE 5.1. Knuth's twindragon $F = \sum_{k=1}^\infty \{0,1\}(-1+i)^{-k}$ and its subtiles $(-1+i)^{-1}F$ and $(-1+i)^{-1}(F+1)$.

With Thuswaldner [**45**], we have extended this theory to matrices $M \in \mathbb{Q}^{d\times d}$, thus in particular to rational bases. We suppose that M has irreducible characteristic polynomial, and can thus be identified with one of the eigenvalues α, i.e., we have $F = \{\sum_{k=1}^{\infty} \alpha^{-k} a_k : a_k \in A\}$ with $A \subset \mathbb{Z}[\alpha]$. Similarly to the representation space for natural extensions of non-unit Pisot numbers in Section 2.2, we extend the fundamental domain to $\mathbb{R}^d \times \mathbb{Q}_{\alpha^{-1}}$; here we use $\mathbb{Q}_{\alpha^{-1}} = \bigcup_{n=0}^{\infty} \alpha^n \mathbb{Z}_{\alpha^{-1}}$. Employing methods from classical algebraic number theory, Fourier analysis in number fields, and results on zero sets of transfer operators, we establish a general tiling theorem for these tiles. We also associate a second kind of tiles with a rational matrix. These tiles are defined as the intersection of a (translation of a) rational self-affine tile with $\mathbb{R}^d \times \{0\}$. Although these intersection tiles have a complicated structure and are no longer self-affine, we are able to prove a tiling theorem for these tiles as well. For particular choices of the digit set A, intersection tiles are instances of tiles defined in terms of shift radix systems and canonical number systems. This enables us to gain new results for tilings associated with numeration systems.

Theorem 5.1. — *Let α be an expanding algebraic number and $A \subset \mathbb{Z}[\alpha]$ a complete residue system of $\mathbb{Z}[\alpha]/\alpha\mathbb{Z}[\alpha]$. Then $\{F + (x, x) : x \in \mathbb{Z}[\alpha]\}$ forms a tiling of $\mathbb{R}^d \times \mathbb{Q}_{\alpha^{-1}}$.*

Open question 5.1. — Can Theorem 5.1 be extended to fundamental domains defined by matrices $M \in \mathbb{Q}^{d\times d}$ (with reducible characteristic polynomial)?

In the special case of A being the canonical digit set $\{0, 1, \ldots, |p_0| - 1\}$, where $p_d x^d + \cdots + p_0 x^0$ is the minimal polynomial of α, we obtain SRS tilings associated with the parameter $\mathbf{r} = \left(\frac{p_d}{p_0}, \ldots, \frac{p_1}{p_0}\right)$; here SRS stands for shift radix systems and we refer to the next section for details. When $d = 1$, i.e., $\alpha = -p_0/p_d$, we have $\sum_{i=1}^{\infty} a_i \alpha^{-i} = 0$ in $\mathbb{Z}_{\alpha^{-1}}$ if and only if $\frac{1}{p_d} \sum_{i=1}^{\infty} a_i \alpha^{-i}$ is the expansion of a real number in the sense of Akiyama, Frougny and Sakarovitch [**56**]. They showed that the set of these expansions has a complicated structure, in particular it does not come from a context-free language. However, we have seen that it is the intersection of a simple structure (the self-affine set F) with a line. This point of view gives us results on rational base number systems. With Morgenbesser and Thuswaldner [**14**], we proved that all $|p_0|^k$ blocks of digits of length k appear in the expansions of integers with probability $|p_0|^{-k}$.

5.2. Shift radix systems

Akiyama and his coauthors [**55**] showed that two important classes of numeration systems, β-expansions and canonical number systems (CNS), can be placed under the same umbrella: shift radix systems (SRS). For a real vector $\mathbf{r} = (r_0, \ldots, r_{d-1})$, the SRS $(\mathbb{Z}^d, \tau_{\mathbf{r}})$ is defined by the transformation

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \quad (z_0, z_1, \ldots, z_{d-1}) \mapsto (z_1, \ldots, z_{d-1}, -\lfloor z_0 r_0 + \cdots + z_{d-1} r_{d-1} \rfloor).$$

The parameter $\mathbf{r}$ associated to β-expansions with an algebraic integer β is obtained by decomposing the minimal polynomial of β as $(x-\beta)(x^d + r_{d-1}x^{d-1} + \cdots + r_0 x^0)$. Then CNS associated to the polynomial $p_d x^d + \cdots + p_0 x^0$ is given by the parameter $\mathbf{r} = \left(\frac{p_d}{p_0}, \ldots, \frac{p_1}{p_0}\right)$.

Open question 5.2. — Can we give an interpretation to shift radix systems with $(x - \beta)(x - \overline{\beta})(x^d + r_{d-1}x^{d-1} + \cdots + r_0 x^0) \in \mathbb{Z}[x]$ for some $\beta \in \mathbb{C} \setminus \mathbb{R}$?

[UPDATE: Such an interpretation has been provided very recently by Surer [**131**].]

When the companion matrix $M_{\mathbf{r}}$ is contracting (in particular when β is a Pisot number or the CNS polynomial is expanding), one can define compact tiles in $\mathbb{R}^d$ by

$$\mathcal{R}(\mathbf{x}) = \lim_{k\to\infty} M_{\mathbf{r}}^k \tau_{\mathbf{r}}^{-k}(\mathbf{x}) \qquad (\mathbf{x} \in \mathbb{Z}^d),$$

where the limit is taken with respect to the Hausdorff distance. With Berthé, Siegel, Surer and Thuswaldner [**43**], we established connections between these tiles and Rauzy fractals as well as self-affine lattice tiles, and proved the following.

Theorem 5.2. — *Let $\mathbf{r} \in \mathbb{R}^d$ be such that $M_{\mathbf{r}}$ is contracting. The collection $\{\mathcal{R}(\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^d\}$ forms a multiple tiling of $\mathbb{R}^d$ if $\mathbf{r} \in \mathbb{Q}^d$ or $(x-\beta)(x^d + r_{d-1}x^{d-1} + \cdots + r_0 x^0) \in \mathbb{Z}[x]$ for some $\beta > 1$, or $r_0, \ldots, r_{d-1}$ are algebraically independent over $\mathbb{Q}$.*

We have also seen in [**16**] and [**45**] that the boundary has zero measure for all SRS tiles related to β-expansions and CNS, since otherwise the corresponding Rauzy fractals and self-affine lattice tiles would have a boundary of positive measure. We do not know whether the measure of the boundary of SRS tiles is always zero.

With Akiyama, Brunotte, Pethő and Thuswaldner [**44**], we have compiled a list of open questions on SRS; see also the survey by Kirschenhofer and Thuswaldner [**103**]. In particular, generalising problems for β-expansions with Salem numbers, it is difficult to determine for those parameters $\mathbf{r}$ where $M_{\mathbf{r}}$ has an eigenvalue on the unit circle whether all $\tau_{\mathbf{r}}$-orbits are eventually periodic. For $\mathbf{r} = (1, \lambda)$ with $|\lambda| < 2$, we conjecture that this is always true, but we have been able to prove this conjecture only for certain quadratic values of λ, with Akiyama, Brunotte and Pethő [**40, 41**].

5.3. Abstract numeration systems

In most of the considered numeration systems, the language of the digital expansions is regular, i.e., recognised by a finite automaton. Lecomte and Rigo [**113**] took the opposite approach and defined numeration systems by finite automata: A non-negative integer n is represented by the $(n+1)$-st word in the recognised language, ordered by the shortlex order. With Charlier and Rigo [**42**], we studied S-recognisable sets in abstract numeration systems S defined by slender languages $a_1^* a_2^* \cdots a_\ell^*$, i.e., sets of integers with S-representations that are recognised by finite automata. We showed that usually S-recognisability is not preserved by multiplication by integers.

Theorem 5.3. — *For a numeration system S with slender language $a_1^* a_2^* \cdots a_\ell^*$, multipication by $m \geq 2$ preserves S-recognisability if and only if $\ell = 1$, or $\ell = 2$ and m is an odd square.*

Open question 5.3. — For which languages is S-recognisability preserved by addition?

When the language grows exponentially, one can define representations of real numbers that are close to β-expansions. With Rigo [**6**], we have extended well known results on eventually periodic β-expansions to these numeration systems. In [**10**], we proved that certain abstract van der Corput sequences are low discrepancy sequences and determined bounded remainder sets for them (cf. Section 2.8).

CHAPTER 6

OUTLOOK

We have already stated several open questions in the preceding chapters. In the final chapter, we describe several questions that are common to most numeration systems, in view of a unified approach to these questions.

6.1. Finite and periodic expansions, natural extensions

A common problem for all numeration systems is the characterisation of finite and periodic expansions. Each numeration system requires its own method, but some procedures work in several systems, such as the use of natural extensions. Our goal is on the one hand to bring together many results of this kind, on the other hand to extend them to non-standard numeration systems (negative or complex bases, canonical numeration systems, continued fraction variants, continued logarithms, Möbius numeration sytems, ...).

Most numeration systems have a natural candidate for the set of finite expansions. For β-expansions, this is the set $\mathbb{Z}[\beta^{-1}]$, and it is sufficient to check whether each element of $\mathbb{Z}[\beta]$ has a finite β-expansion; the properties (F_β) or $(\mathrm{F}_{-\beta})$ can be decided for each β, but the domains of the β's satisfying these properties have a complicated structure. For α-continued fractions, all rational numbers have finite expansions. For Rosen continued fractions, the natural candidate for finite expansions is $\mathbb{Q}(\lambda)$, but it is true only for $q = 3$ and $q = 5$ that each number in $\mathbb{Q}(\lambda)$ has a finite expansion. For multidimensional continued fractions algorithms, the question is whether a rational dependency between the coordinates of a vector implies that the algorithm terminates. This is true for many 2-dimensional algorithms but fails for 3-dimensional Farey algorithms; see Grabiner [**96**].

The natural candidate for the set of eventually periodic β-expansions is $\mathbb{Q}(\beta)$. We know that each element of $\mathbb{Q}(\beta)$ has a periodic β-expansion if β is a Pisot nunber, while this is not true when β has a conjugate strictly outside the unit circle. Determining the structure of the eventually periodic expansions for Salem numbers is a difficult problem. The numbers with eventually periodic continued fractions are the quadratic numbers. For Rosen continued fractions, the eventually periodic elements must be quadratic over $\mathbb{Q}(\lambda)$, but we do not know if the converse is true. It is a notoriously difficult problem to characterise the eventually periodic expansions for multidimensional continued fractions, for example for the Jacobi–Perron algorithm.

Once that we know the eventually periodic expansions, we can ask which of them are purely periodic. For β-expansions with Pisot numbers, we have seen the connection between natural extensions and periodic expansions. A similar relation exists for regular continued fractions; its extension to α-continued fraction expansions is rather straightforward but not written up yet. Since we have no characterisation of the eventually periodic expansions for Rosen fractions of multidimensional continued fractions, we also have no appropriate version of the natural extension detecting purely periodic expansions.

6.2. Normal numbers

Somewhat opposite to the countable set of finite and periodic expansions is the set of *normal* expansions. A number is called normal to an integer base $\beta \geq 2$ if its expansions contains each of the β^n blocks of length n with frequency β^{-n}. For real bases $\beta > 1$, a normal β-expansion contains each block of digits with the frequency given by the invariant measure of the β-transformation. Similarly, a number is continued fraction normal if each block of partial quotients occurs with the frequency given by the Gauss measure. Ergodic theory tells us that almost all numbers are normal in this sense. The problem becomes more difficult e.g. for rational base number systems described in Section 5.1. Also a notion of normality for S-adic systems as in Section 4.3, generalising Cantor series expansions, would be interesting.

While ergodic theory implies that almost all numbers are normal (with respect to any countable number of bases), it is sometimes not easy to construct such numbers. In integer bases, there are many variations of Champernowne's construction, and it is possible to construct absolutely normal numbers, i.e., numbers that are normal w.r.t. all integer bases (and even w.r.t. all Pisot bases and continued fractions). For rational base number systems, which have a complicated language, the construction of normal numbers seems to be much harder.

A wide open question is to know whether a particular number is normal. It is conjectured that the binary expansion of $\sqrt{2}$ or any other algebraic number or π is normal, but we have only much weaker partial results in this direction.

Another problem is to find normal numbers with small discrepancy. For integer bases, we do not know if a normal expansion can be a low-discrepancy sequence, with the best known discrepancy being $\mathcal{O}((\log N)^2/N)$ by Levin [**114**]; see also Becher and Carton [**65**]. For absolutely normal numbers, all the known constructions provide sequences with rather larger discrepancy; see Scheerer [**127**].

Finally, we would like to know which operations preserve normality. Wall [**135**] proved that $qx + r$ is normal for each normal number x and rational numbers q, r (with $q \neq 0$). Vandehey [**134**] showed that Möbius transformations preserve normality for continued fractions. Other operations like the prefix selection by regular languages also preserve normality (in integer bases); see Agafonov [**51**] and the chapter by Becher and Carton [**64**]. The corresponding problems for β-expansions or multidimensional continued fractions are open.

6.3. Matching properties

A numeration system can often be modified to give a family of numeration systems depending on a parameter α, for example the intermediate β-transformation $\beta x+\alpha \bmod 1$ or the α-continued fractions. For at most countably parameters α, the orbits of the discontinuity points are finite and hence the associated subshifts satisfy a Markov property. However, we have seen above that we also get nice properties when the limits from the left and the right at the discontinuity points meet again in the future. This "matching" property holds for almost all α in the case of α-continued fractions and for the intermediate β-transformation with certain (Pisot) bases β; it also holds for α-Rosen continued fractions at least in a certain range.

We conjecture that the intermediate β-transformations satisfy almost everywhere matching for all Pisot numbers β (but not for other bases $\beta > 1$, except possibly for Salem numbers); this would also give a positive answer to the first part of the Open Question 2.9. Note that continued fractions satisfy the Pisot condition of Section 4.3. For the moment, the only known method for checking the matching property is to make a combinatorial analysis of each underlying system. It would be very interesting to find a general argument. Another problem is to find matching properties in higher dimensions (e.g., for β-expansions with complex bases, or for multidimensional continued fraction algorithms with the Pisot condition, such as the Jacobi–Perron algorithm).

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Career

- Since 2005: CNRS researcher (chargé de recherche) at the IRIF (UMR 8243), previously LIAFA (UMR 7089), Université de Paris, previously Paris Diderot University
- 2012 and 2014 (1 month each): Guest professor at the Institute of Analysis and Number Theory of the Graz University of Technology, Austria, teaching courses on *Automata, languages and numeration systems*
- 2011–2012: Visiting academic at the Department of Computing of Macquarie University, Sydney, Australia
- 2003–2004: Assistant at the Institute of Mathematics of the University of Vienna
- 2000–2005: Research assistant of the Austrian Science Fund (FWF) at the Institute of Discrete Mathematics and Geometry of the Vienna University of Technology
- 2000–2002: Ph.D. at the Vienna University of Technology, Ph.D. thesis *The Distribution of Digital Expansions on Polynomial Sequences*, supervisor Michael Drmota
- 1994–2000: Studies of Technical Mathematics at the Vienna University of Technology, Ingénieur École Centrale Paris (within the double degree program TIME)

Grants

- Coordinator of the ANR young researchers project *DyCoNum – Études diophantiennes, dynamiques et combinatoires de différentes numérations* (2006–2010)
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Organisation of events

Organiser of the seminars

- *One World Numeration Seminar* (online, started in May 2020),
- *Automata seminar* of the LIAFA/IRIF (2012–2017).

Main organiser of the conferences

- *Numeration 2018*,
- *Numération : Mathématiques et Informatique* (2009),

of the workshops

- *Approximation and numeration* (2013),
- *Digital expansions, dynamics and tilings* (2010),
- *Dynamical aspects of numeration* (2006),

participation in the organisation of the

- *Journées nationales 2014 du GDR Informatique Mathématique*,
- *Journées d'Informatique Fondamentale de Paris Diderot* (2013),
- *Journées Numération 2008*,
- *Journées SDA2 2007*,
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Refereeing, committees and editorial activities

- Referee and member of the thesis committee of Charlene Kalle's Ph.D. thesis at Utrecht University (2009)
- Member of the selection comittee for a maître de conférences position at the University of the Mediterranean Aix–Marseille II (2009)
- Member of the conseil scientifique de l'UFR Informatique of the Paris Diderot University (2010–2011)
- Expert for the ANR (2018), NWO (2020) and Île-de-France (2021)
- Member of the scientific committee for the conference *Numeration* (2015, 2017, 2019) and the *16th Mons Theoretical Computer Science Days* (2016)
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