Numeration systems: automata, combinatorics, dynamical systems, number theory

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devant le jury composé de

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Donald Knuth
(The art of computer programming, Chapter 4 – Arithmetic):

> *The way we do arithmetic is intimately related to the way we represent the numbers we deal with, so it is appropriate to begin our study of the subject with a discussion of the principal means for representing numbers.*

Representations of integers in positional numeration systems

$$\color{red}{(x_n x_{n-1} \cdots x_0)_\beta} := \sum_{k=0}^{n} x_k \beta^k \qquad \text{(base } \beta \text{, digits } x_k\text{)}$$

$$\begin{aligned}
&(2021)_{10} \\
&= (11111100101)_2 \\
&= (1100000100011)_{-2} \\
&= (100000\bar{1}00101)_2 && \bar{1} = -1 \\
&= (1110000000000111011101001)_{-1+i} && (-1+i)^2 + (-1+i) + 2 = 0 \\
&= \tfrac{1}{2}\,(2120012220012122 01)_{3/2} \\
&= (1001000010100000)_F && = \underbrace{F_{15}}_{1597} + \underbrace{F_{12}}_{377} + \underbrace{F_7}_{34} + \underbrace{F_5}_{13}
\end{aligned}$$

Fibonacci sequence $F_0 = 1, F_1 = 2, F_k = F_{k-1} + F_{k-2}$

$$\color{red}{(x_n x_{n-1} \cdots x_0)_U} := \sum_{k=0}^{n} x_k U_k \quad \text{(base sequence } (U_k)_{k\ge 0}\text{, digits } x_k\text{)}$$

Representations of numbers in $[0, 1)$

$$(0.x_1 x_2 \cdots)_\beta := \sum_{k=1}^{\infty} \frac{x_k}{\beta^k} \qquad \text{(base } \beta\text{, digits } x_k\text{)}$$

$$\frac{96}{365} = \left(0.2(63013698)^\infty\right)_{10}$$

$$= \left(0.(010000110101010011011101101000100000)^\infty\right)_2$$

$$= \left(\begin{array}{l} 0.(0010000100100001010000001010000000101010000000100000010100100010100000010010 \\ 000000001000000001001001000000010101000000010100100010101001001010010100000 \\ 1000100000000100000010101001001001001001001001001010100101010001010100010 01 \\ 0001001000010010000000100101010000000101010000000010010010101010001010101 0 \\ 0101000000101000010000100000010101000000010101000010000010010100100101000 10 \\ 1000100100101001000101001000101010000000101001010100100001010000001010000 10 \\ 1010101000101010010010010101010000000101010000000101000000000100100001000100 \\ 101000101010001010010000000100100100100100100100100000001000000100000000100001 \\ 010000101001001010101000010000010101000000001010100100100100000000100000000 0 \\ 0100010100000010010100101010010000010101000000001001010100001000010010001000)^\infty \end{array}\right)_{\frac{1+\sqrt{5}}{2}}$$

$$= \left(1.(110100011101000011011100110100010001110000011100110011000000110000000001)^\infty\right)_{-1+i}$$

$$= \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{19}}}}$$

Aviezri Fraenkel (The use and usefulness of numeration systems. Inform. and Comput. 81 (1989), 46–61):

> *The proper choice of a counting system may solve mathematical problems or lead to improved algorithms. This is illustrated by a problem in combinatorial group theory, compression of sparse binary strings, encoding of contiguous binary strings of unknown lengths, ranking of permutations and combinations, strategies of games, and other examples. Two abstract counting systems are given from which the concrete ones used for the applications can be derived in an easy and transparent manner.*

Representations with few digits

For many applications, e.g. in elliptic curve cryptography, one has to compute (large) scalar multiples of group elements.

A simple way to compute mP is the double-and-add algorithm (Horner's scheme). If $m = (1x_{n-1} \cdots x_0)_2$, then

$$mP = 2(\cdots 2(2(2P + x_{n-1}P) + x_{n-2}P) + \cdots) + x_0 P\,.$$

n duplications, $\sum_{k=0}^{n-1} x_k$ additions (in average $n/2$), $x_k \in \{0, 1\}$

Non-Adjacent Form (NAF): Every $m \in \mathbb{Z}$ has a (unique) expansion

$$m = (x_n x_{n-1} \cdots x_0)_2 \quad \text{with} \quad x_k \in \{-1, 0, 1\}$$

and $x_k x_{k+1} = 0$ for all k, i.e., no two adjacent digits are non-zero. On average, $\sum_{k=0}^{n-1} |x_k| \approx n/3$.

Applications of the NAF:

- Efficient arithmetic operations (Reitwiesner '60)
- Coding Theory
- Elliptic Curve Cryptography (Morain and Olivos '90)

Expansions of minimal weight

Weight $w(x_n \cdots x_0) = \sum_{k=0}^{n} |x_k|$

$x_n \cdots x_0 \in \mathbb{Z}^*$ is a 2-expansion of minimal weight if

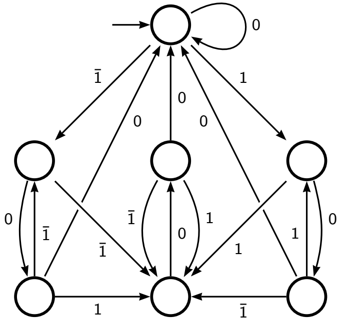
$$w(x_n \cdots x_0) \leq w(y_m \cdots y_0)$$

for all $y_m \cdots y_0 \in \mathbb{Z}^*$ with $(x_n \cdots x_0)_2 = (y_m \cdots y_0)_2$.

Heuberger '04:
$x_n \cdots x_0 \in \{0, \pm 1\}^*$ is a 2-expansion of minimal weight if and only if it contains no factor in

$$11(01)^*1 \cup 1(0\bar{1})^*\bar{1} \cup \bar{1}\bar{1}(0\bar{1})^*\bar{1} \cup \bar{1}(01)^*1.$$

(The set of 2-expansions of minimal weight is therefore recognised by the automaton on the right, where all states are final.)



Fibonacci expansions of minimal weight

Heuberger '04: "Fibonacci-NAF"

Each integer has an F-expansion $x_n \cdots x_0 \in \{0, \pm 1\}^*$ avoiding

$$\{11,\ 1\bar{1},\ 10\bar{1},\ 101,\ 1001,\quad \bar{1}\bar{1},\ \bar{1}1,\ \bar{1}01,\ \bar{1}0\bar{1},\ \bar{1}00\bar{1}\};$$

all such $x_n \cdots x_0$ are F-expansions of minimal weight.

Meloni '07: Fibonacci-and-add algorithm for computing mP

Frougny–St '08:

$x_n \cdots x_0 \in \{0, \pm 1\}^*$ is an F-expansion of minimal weight if and only if it contains no factor in

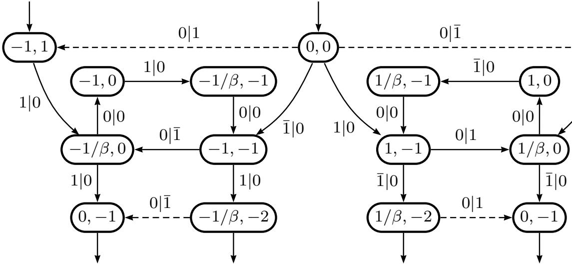
$$1(0100)^*1 \cup 1(0100)^*0101 \cup 1(00\bar{1}0)^*\bar{1} \cup 1(00\bar{1}0)^*0\bar{1}$$
$$\cup\ \bar{1}(0\bar{1}00)^*\bar{1} \cup \bar{1}(0\bar{1}00)^*0\bar{1}0\bar{1} \cup \bar{1}(0010)^*1 \cup \bar{1}(0010)^*01.$$

On average, $w(x_n \cdots x_0) \approx n/5 \approx 0.288 \log_2 (x_n \cdots x_0)_F$.

(Proof strategy: Find a transducer with "heavy" expansions as input and "lighter" expansions as output. Take the complement of the input automaton)

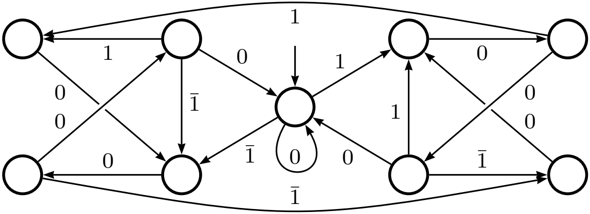
φ-expansions and F-expansions of minimal weight

$\beta = \varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$, $\varphi^2 = \varphi + 1$: φ-heavy words (with no proper φ-heavy factor) are inputs of the transducer



$$(s,\delta) \xrightarrow{x|y} (\beta s + x - y, \delta + |y| - |x|)$$

The φ-expansions (and F-expansions) of minimal weight in $\{0, \pm 1\}^*$ are recognised by the following automaton (all states are final).



β-expansions and U-expansions of minimal weight

$x_n \cdots x_0 \in \mathbb{Z}^*$ is a β-expansion of minimal weight if

$$w(x_n \cdots x_0) \leq w(y_m \cdots y_0)$$

for all $y_m \cdots y_0 \in \mathbb{Z}^*$ s.t. $(x_n \cdots x_0 0^k)_\beta = (y_m \cdots y_0)_\beta$ for some $k \geq 0$.

An algebraic integer $\beta > 1$ is a Pisot number if $|\alpha| < 1$ for all Galois conjugates α of β (with $\alpha \neq \beta$).

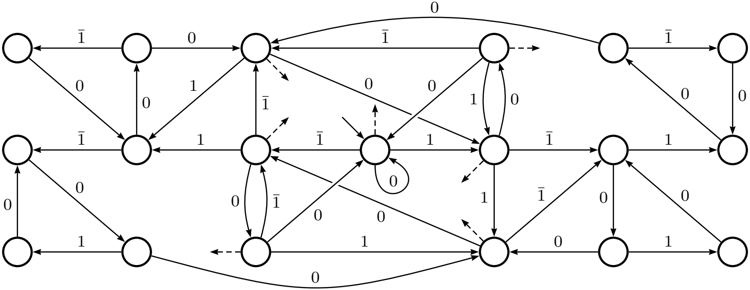
Frougny–St '08:
If β is a Pisot number, then the set of β-expansions of minimal weight is recognised by a finite automaton.

Dubickas '11: also holds when β is an algebraic number with $|\alpha| \neq 1$ for all conjugates α of β (β root of polynomial with dominant term: $\exists C \geq 2$ s.t. $(C0^k)_\beta = (x_n \cdots x_0)_\beta$, $w(x_n \cdots x_0) \leq C$, $|x_j| < C \, \forall j$)

Grabner–St '11:
Let $U = (U_k)_{k \geq 0}$ be an increasing sequence of integers satisfying eventually a linear recurrence with characteristic polynomial equal to the minimal polynomial of a Pisot number. Then the set of U-expansions of minimal weight is recognised by a finite automaton.

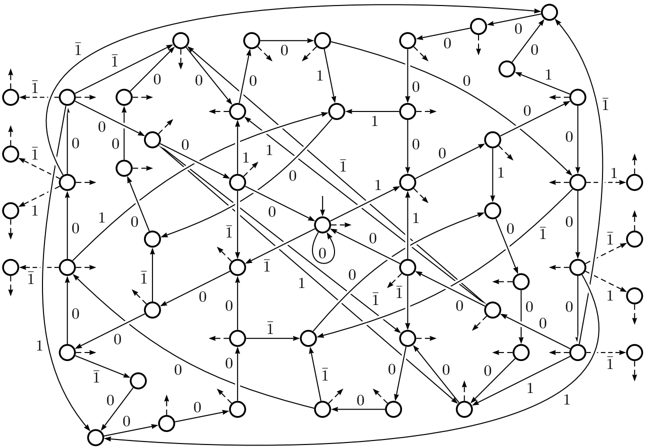
Avanzi–Heuberger–Prodinger '06, '11:
τ-NAF and τ-expansions of minimal weight for $\tau^2 + 2 = \pm\tau$
(scalar multiplication on Koblitz curves using Frobenius endomorphism)

Frougny–St '08:
β-expansions of minimal weight, $\beta^3 = \beta^2 + \beta + 1$ (Tribonacci number)
and U-expansions of minimal weight, $U_0 = 1$, $U_1 = 2$, $U_2 = 4$,
$U_k = U_{k-1} + U_{k-2} + U_{k-3}$ (Tribonacci sequence)
(all states final for β-expansions, dashed arrows final for U-expansions)



"β-NAF", "U-NAF": exclude factors $11, 101, 1\bar{1}$ and their opposites

Frougny–St '08: β-expansions of minimal weight, $\beta^3 = \beta + 1$ (smallest Pisot number) and U-expansions of minimal weight, $U_0 = 1$, $U_1 = 2$, $U_2 = 3$, $U_3 = 4$, $U_k = U_{k-2} + U_{k-3}$
(all states final for β-expansions, dashed arrows final for U-expansions)



"NAF": exclude factors $10^k 1, 10^k \bar{1}, 0 \le k \le 5$, and either $10^6 1$ or $10^6 \bar{1}$

On average, $w(x_n \cdots x_0) \approx \frac{n}{7+2\beta^2} \approx 0.235 \log_2(x_n \cdots x_0)_U$.

Different minimal weight expansions of a number

Let $f_U(m)$ be the number of U-expansions of minimal weight of m.

Grabner–St '11:
We have
$$\sum_{|m|<N} f_U(m) = N^{\log_\beta \alpha}\,\Phi(\log_\beta N) + O(N^\lambda),$$
where α is the dominant eigenvalue of the automaton recognising the set of U-expansions of minimal weight, Φ is a continuous periodic function of period 1, and $\lambda < \log_\beta \alpha$.

Grabner–Heuberger '06:
If $U_k = 2^k$, then $f_U(m)$ is bounded by $F_{\lfloor \log_4 |m| \rfloor + 1}$,
$\log_2 \alpha \approx 1.11775$ ($\alpha^3 = \alpha^2 + 3\alpha - 1$).

Grabner–St '11:
The number of F-expansions of minimal weight of m with digits in $\{0, \pm 1\}$ is bounded by (approximately) $2^{\lfloor (\log_\varphi |m|)/3 \rfloor}$,
$\log_\varphi \alpha \approx 1.15188$ ($\alpha^5 = \alpha^4 + 2\alpha^2 + \alpha - 1$).

How to obtain a β-expansion of a number, $\beta > 1$

(greedy) β-transformation

$$T_\beta : [0,1) \to [0,1), \quad x \mapsto \beta x - \lfloor \beta x \rfloor$$

$$x = \frac{\lfloor \beta x \rfloor}{\beta} + \frac{T_\beta(x)}{\beta} = \frac{\lfloor \beta x \rfloor}{\beta} + \frac{\lfloor \beta T_\beta(x) \rfloor}{\beta^2} + \frac{T_\beta^2(x)}{\beta^2} = \cdots$$

$$= (0.x_1 x_2 \cdots)_\beta \quad \text{with } x_k = \lfloor \beta T_\beta^{k-1}(x) \rfloor \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$$

Equivalently, $x_k \in \mathbb{Z}$ and $(0.x_k x_{k+1} \cdots)_\beta \in [0,1)$ for all $k \geq 1$.

Write numbers in $[0, \infty)$ as

$$\beta^n x = (x_1 x_2 \cdots x_n . x_{n+1} x_{n+2} \cdots)_\beta$$

Example: $\beta = \varphi = \frac{1+\sqrt{5}}{2}$, $x = \frac{2021}{\varphi^{16}}$, $n = 16$

$$2021 = \big(1010010000100010.0010010000100001(0)^\infty\big)_\varphi$$

β-adic expansions, β-adic integers for algebraic β

$a_d\beta^d + \cdots + a_1\beta + a_0 = 0$ (minimal polynomial), $\gcd(a_d, \ldots, a_1, a_0) = 1$

$D_\beta : \mathbb{Z}[\beta] \to \mathbb{Z}[\beta], \quad x \mapsto \frac{x - x_0}{\beta},$ with $x_0 \in \{0, 1, \ldots, |a_0| - 1\}$ s.t. $x - x_0 \in \beta\mathbb{Z}[\beta]$

$$x = \beta D_\beta(x) + x_0 = \beta^2 D_\beta^2(x) + \beta x_1 + x_0 = \cdots$$
$$= (x_n \cdots x_1 x_0)_\beta \qquad \text{if } D_\beta^{n+1}(x) = 0$$

Example ($\frac{3}{2}$-expansions of Akiyama–Frougny–Sakarovitch '08):

$\beta = 3/2$, $x = 2 \cdot 2021 = 4042 = (212001222001212201)_{3/2}$

k	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
$D_\beta^k(x)$	0	2	4	8	12	18	28	44	68	104	156	234	352	530	796	1196	1796	2694	4042
x_k		2	1	2	0	0	1	2	2	2	0	0	1	2	1	2	2	0	1

$$x = -2 = \tfrac{3}{2}(-2) + 1 = \tfrac{9}{4}(-2) + \tfrac{3}{2} + 1 = \cdots = (\cdots 11)_{3/2}$$

β-adic integers $\mathbb{Z}_\beta$: closure of $\mathbb{Z}[\beta]$ w.r.t. to the topology where $x, y \in \mathbb{Z}[\beta]$ are close if $x - y \in \beta^k\mathbb{Z}[\beta]$ for large k

$$\mathbb{Z}_\beta \simeq \varprojlim \mathbb{Z}[\beta]/\beta^k\mathbb{Z}[\beta] \simeq \{0, 1, \ldots, |a_0| - 1\}^\infty, \quad (\cdots x_1 x_0)_\beta \in \mathbb{Z}_\beta$$

D_β can be continously extended to a map $\mathbb{Z}_\beta \to \mathbb{Z}_\beta$

Eventual periodicity

An algebraic integer $\beta > 1$ is a Pisot number if $|\alpha| < 1$ for all Galois conjugates $\alpha \neq \beta$.
An algebraic integer $\beta > 1$ is a Salem number if $|\alpha| = 1$ for some conjugate α and $|\alpha| \leq 1$ for all conjugates $\alpha \neq \beta$.

Bertrand '77, Schmidt '80:
β Pisot $\Rightarrow$ $(T_\beta^k(x))_{k\geq 0}$ eventually periodic for all $x \in \mathbb{Q}(\beta) \cap [0,1)$
$(T_\beta^k(x))_{k\geq 0}$ ev. periodic for all $x \in \mathbb{Q} \cap [0,1)$ $\Rightarrow$ β Pisot or Salem

We don't know for any Salem number β if $(T_\beta^k(x))_{k\geq 0}$ is eventually periodic for all $x \in \mathbb{Q}(\beta) \cap [0,1)$.

An algebraic number β is expansive if $|\alpha| > 1$ for all conjugates α;
β is weakly expansive if $|\alpha| \geq 1$ for all conjugates α.

β expansive $\Rightarrow$ $(D_\beta^k(x))_{k\geq 0}$ eventually periodic for all $x \in \mathbb{Z}[\beta]$
$(D_\beta^k(x))_{k\geq 0}$ eventually periodic for all $x \in \mathbb{N}$ $\Rightarrow$ β weakly expansive

Finiteness

$$(\mathrm{F}_\beta): \quad \forall x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)\ \exists\, n \geq 0 \text{ s.t. } T_\beta^n(x) = 0$$

Frougny–Solomyak '92:
$(\mathrm{F}_\beta) \Rightarrow \beta$ is a Pisot number
$\beta > 1$, $\beta^d = a_{d-1}\beta^{d-1} + \cdots + a_0\beta^0$, $a_{d-1} \geq \cdots \geq a_0 \geq 1 \Rightarrow (\mathrm{F}_\beta)$

$$(\mathrm{F}'_\beta): \quad \forall x \in \mathbb{Z}[\beta]\ \exists\, n \geq 0 \text{ s.t. } D_\beta^n(x) = 0$$

("canonical number system")

Kovács '81, Pethő '91:
$(\mathrm{F}'_\beta) \Rightarrow \beta$ is weakly expansive (*can we remove weakly?*)
(F'_β), β algebraic integer $\Rightarrow \beta$ is expansive
$a_d\beta^d + \cdots + a_1\beta + a_0 = 0$, $2 \leq a_0 \geq a_1 \geq \cdots \geq a_d \geq 1 \Rightarrow (\mathrm{F}'_\beta)$

(F_β) and (F'_β) can be characterised in terms of shift radix systems
Akiyama, Brunotte, Pethő, Thuswaldner, Kirschenhofer, Surer, Weitzer, ...

Krčmáriková–Vávra–St '17:
some necessary or sufficient conditions for $(\mathrm{F}_{-\beta})$

Rauzy fractals and natural extension of T_β, β Pisot

conjugates $\beta_1 = \beta, \beta_2, \ldots, \beta_r \in \mathbb{R}$, $\beta_{r+1}, \ldots, \beta_{r+s} \in \mathbb{C} \setminus \mathbb{R}$

$$\mathbb{K}_\beta := \mathbb{R}^r \times \mathbb{C}^s \times \mathbb{Z}_\beta, \quad \mathbb{K}'_\beta := \mathbb{R}^{r-1} \times \mathbb{C}^s \times \mathbb{Z}_\beta$$

$$\beta \cdot (x_1, \ldots, x_{r+s}, y) = (\beta_1 x_1, \ldots, \beta_{r+s} x_{r+s}, \beta y)$$

$$\delta_\beta : \mathbb{Z}[\beta] \to \mathbb{K}_\beta, \quad p(\beta) \mapsto (p(\beta_1), \ldots, p(\beta_{r+s}), p(\beta))$$

$$\delta'_\beta : \mathbb{Z}[\beta] \to \mathbb{K}'_\beta, \quad p(\beta) \mapsto (p(\beta_2), \ldots, p(\beta_{r+s}), p(\beta))$$

Rauzy fractals

$$\mathcal{R}_\beta(x) := \overline{\bigcup_{k \geq 0} \delta'_\beta\big(x - \beta^k T_\beta^{-k}(x)\big)}$$

natural extension domain

$$\hat{X}_\beta := \bigcup_{x \in [0,1)} \{x\} \times \mathcal{R}_\beta(x) \quad \subset \quad \mathbb{K}_\beta,$$

natural extension map

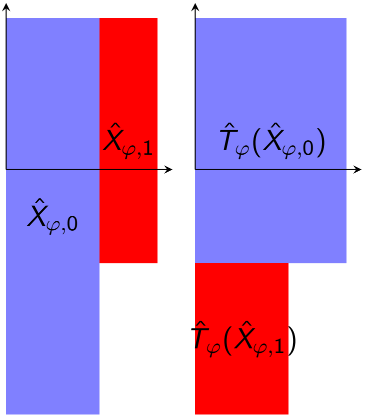
$$\hat{T}_\beta : \hat{X}_\beta \to \hat{X}_\beta, \; (x_1, \ldots, x_{r+s}, y) \mapsto \beta \cdot (x_1, \ldots, x_{r+s}, y) - \delta_\beta(\lfloor \beta x_1 \rfloor)$$

bijective up to a set of Haar measure 0, first coordinate equal to T_β

Natural extensions for Pisot units

$\beta^d = a_{d-1}\beta^{d-1} + \cdots + a_0\beta^0$, $|a_0| = 1$

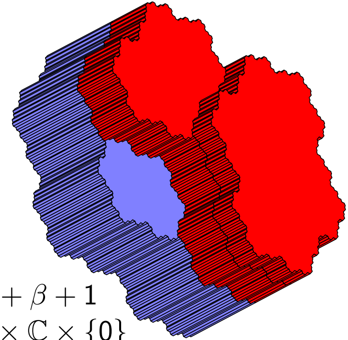
$\Rightarrow \mathbb{Z}_\beta = \{0\}$ (can be omitted)



$\varphi^2 = \varphi + 1$, $\mathbb{K}_\varphi = \mathbb{R}^2 \times \{0\}$

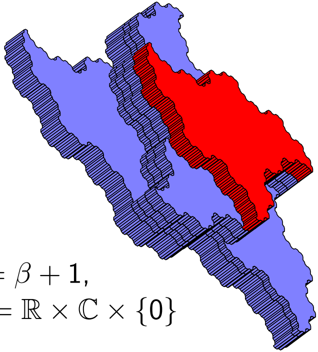
$\hat{X}_{\beta,0} := \bigcup_{x\in[0,1/\beta)} \{x\} \times \mathcal{R}_\beta(x)$

$\hat{X}_{\beta,1} := \bigcup_{x\in[1/\beta,1)} \{x\} \times \mathcal{R}_\beta(x)$



$\beta^3 = \beta^2 + \beta + 1$

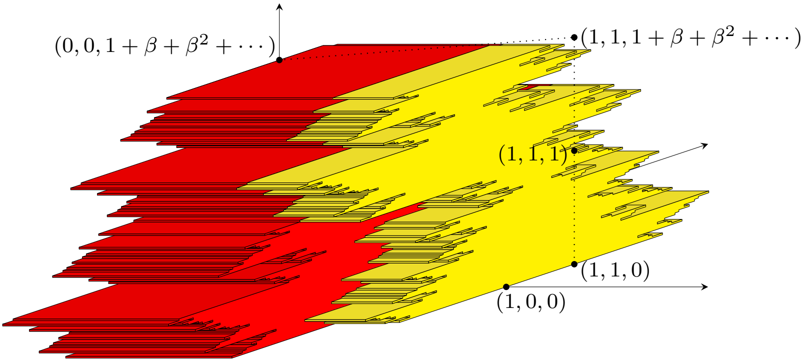
$\mathbb{K}_\beta = \mathbb{R} \times \mathbb{C} \times \{0\}$



$\beta^3 = \beta + 1$,

$\mathbb{K}_\beta = \mathbb{R} \times \mathbb{C} \times \{0\}$

Natural extension of T_β for a Pisot non-unit



$\beta^2 = \beta + 2$, $\mathbb{K}_\beta = \mathbb{R}^2 \times \mathbb{Z}_\beta$, $\mathbb{Z}_\beta \simeq \{0,1\}^\infty$

Berthé–Siegel '07 and Minervino–St '14 use non-Archimedean places of $\mathbb{Q}(\beta)$ instead of $\mathbb{Z}_\beta$, which makes no difference (Hejda–St '17).

Purely periodic β-expansions, β Pisot

$\beta^d = a_{d-1}\beta^{d-1} + \cdots + a_0\beta^0$

Hama–Imahashi '97, Ito–Rao '05, Berthé–Siegel '07: For $x \in [0,1)$,

$$\exists\, n \geq 1 : T_\beta^n(x) = x \iff \exists q \geq 1 : qx \in \mathbb{Z}[\beta],\ \delta_\beta(qx) \in q\hat{X}_\beta$$

(In other words, there exists $q \in \mathbb{Z}$ s.t. $x \in \frac{1}{q}\mathbb{Z}[\beta]$, $\gcd(q, a_0) = 1$ and $\delta_\beta(x) \in \hat{X}_\beta$; $\delta_\beta(x)$ is defined by $q\delta_\beta(x) = \delta_\beta(qx)$ for such x.)

Which rational numbers have purely periodic β-expansions?

$\gamma(\beta) := \inf\left\{\frac{p}{q} \in \mathbb{Q}\cap[0,1) : \gcd(q, a_0) = 1,\ T_\beta^n(\frac{p}{q}) \neq \frac{p}{q}\ \forall n \geq 1\right\} \cup \{1\}$

$\gamma(\beta) = 1$ for integer $\beta \geq 2$

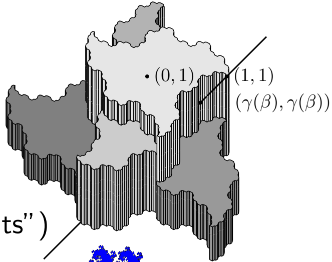
Schmidt '80:
$\gamma(\beta) > 0 \Rightarrow \beta$ Pisot, $\quad \gamma(\beta) = 1$ for $\beta^2 = a\beta + 1$

Hama–Imahashi '97, Akiyama '98:
β has conjugate $0 < \alpha \neq \beta \Rightarrow T_\beta^n(x) \neq x$ for all $x \in \mathbb{Q} \cap (0,1)$, $n \geq 1$
in particular, $\gamma(\beta) = 0$ for $\beta^2 = a\beta - 1$

Akiyama '98: $(\mathrm{F}_\beta) \Rightarrow \gamma(\beta) > 0$

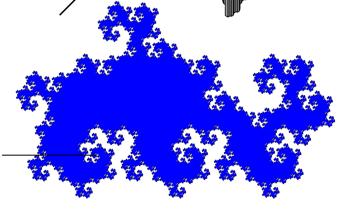
$$\gamma(\beta) = \inf\left\{\tfrac{p}{q} \in \mathbb{Q}\cap[0,1) : \gcd(q,a_0)=1,\ T_\beta^n\left(\tfrac{p}{q}\right) \neq \tfrac{p}{q}\ \forall n \geq 1\right\} \cup \{1\}$$

Akiyama–Scheicher '05:
$\gamma(\beta) = 0.666666666086\cdots$ for $\beta^3 = \beta + 1$



Adamczewski–Frougny–Siegel–St '10:
β cubic Pisot unit satisfying (F_β),
conjugate $\alpha \notin \mathbb{R} \Rightarrow \gamma(\beta) \notin \mathbb{Q}$
(all $\delta_\beta(x) \in \partial\hat{X}_\beta$, $x \in \mathbb{Q}(\beta)$, are "spiral points")

Akiyama–Barat–Berthé–Siegel '08:
$\gamma(\beta) = 0$ for $\beta^2 = 4\beta + 3$,
$\gamma(\beta) = \frac{7-\sqrt{7}}{12}$ for $\beta^2 = 10\beta + 3$



$\mathcal{R}_\beta(0)$ for
$\beta^3 = 3\beta^2 - 2\beta + 1$

Hejda–St '17: $\beta^2 = a\beta + b$, $b \mid a$
$\gamma(\beta) = 1 \iff a \geq b^2$ or $a \in \{24, 30\}$, $b = 6$
$a = b \geq 3 \Rightarrow \gamma(\beta) = 0$

Algorithm for approximating $\gamma(\beta)$, $\beta^2 = a\beta + b$
(using the boundary graph of Rauzy fractals given by Minervino–St '14)

Fundamental domain, natural extension of D_β, β expansive

$a_d\beta^d + \cdots + a_1\beta + a_0 = 0$, $\gcd(a_d, \ldots, a_1, a_0) = 0$
conjugates $\beta_1, \beta_2, \ldots, \beta_r \in \mathbb{R}$, $\beta_{r+1}, \ldots, \beta_{r+s} \in \mathbb{C} \setminus \mathbb{R}$

$$\mathbb{K}_\beta := \mathbb{Z}_\beta \times \mathbb{R}^r \times \mathbb{C}^s \times \mathbb{Z}_{\beta^{-1}}, \quad \mathbb{K}''_\beta := \mathbb{R}^r \times \mathbb{C}^s \times \mathbb{Z}_{\beta^{-1}}$$

$$D_\beta : \mathbb{Z}_\beta \to \mathbb{Z}_\beta, \quad x \mapsto \frac{x - x_0}{\beta}, \qquad \begin{array}{l}\text{with } x_0 \in \{0, 1, \ldots, |a_0| - 1\} \\ \text{s.t. } x - x_0 \in \beta\mathbb{Z}_\beta\end{array}$$

fundamental domain

$$\mathcal{F}_\beta := \overline{\bigcup_{k\geq 1} \delta''_\beta(\beta^{-k} D_\beta^{-k}(0))} = \left\{ \sum_{k=1}^{\infty} \delta''_\beta(x_{-k}\beta^{-k}) : x_{-k} \in \{0, \ldots, |a_0|-1\} \right\}$$

natural extension domain

$$\hat{Y}_\beta := \mathbb{Z}_\beta \times (-\mathcal{F}_\beta) \quad \subset \quad \mathbb{K}_\beta,$$

natural extension map

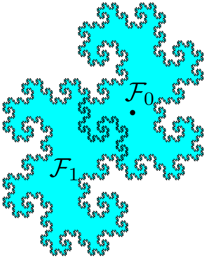
$$\hat{D}_\beta : \hat{Y}_\beta \to \hat{Y}_\beta,\ (x, y_1, \ldots, y_{r+s}, z) \mapsto \beta^{-1} \cdot \big((x, y_1, \ldots, y_{r+s}, z) - \delta_\beta(x_0)\big)$$

bijective up to a set of Haar measure 0, first coordinate equal to D_β

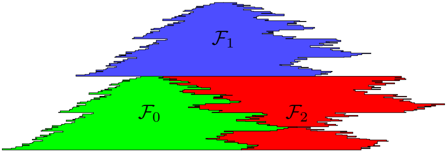
β expansive, $a_d\beta^d + \cdots + a_1\beta + a_0 = 0$, $\gcd(a_d, \ldots, a_1, a_0) = 0$
conjugates $\beta_1, \beta_2, \ldots, \beta_r \in \mathbb{R}$, $\beta_{r+1}, \ldots, \beta_{r+s} \in \mathbb{C} \setminus \mathbb{R}$

$$\mathbb{K}''_\beta = \mathbb{R}^r \times \mathbb{C}^s \times \mathbb{Z}_{\beta^{-1}}, \quad \mathbb{Z}_{\beta^{-1}} \simeq \{0, 1, \ldots, |a_d| - 1\}^\infty$$

$$\mathcal{F}_\beta = \bigcup_{b=0}^{|a_0|-1} \beta^{-1} \cdot \left(\mathcal{F}_\beta + \delta''_\beta(b)\right) \qquad \text{(self-affine set)}$$



$\beta = -1 + i$, $\beta^2 + \beta + 2 = 0$
$\mathbb{K}''_\beta = \mathbb{C} \times \{0\}$
(Knuth's twindragon)

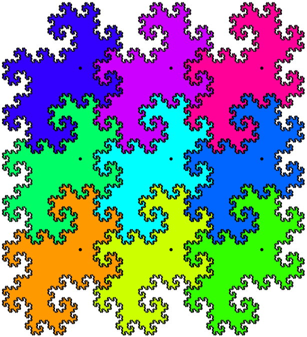


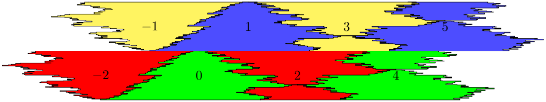
$\beta = 3/2$, $2\beta - 3 = 0$
$\mathbb{K}''_\beta \simeq \mathbb{R} \times \{0, 1\}^\infty$

$$\mathcal{F}_b = \beta^{-1} \cdot \left(\mathcal{F}_\beta + \delta''_\beta(b)\right)$$

Lagarias–Wang '97, St–Thuswaldner '15:
For expansive algebraic β, $\{\delta''_\beta(x) + \mathcal{F}_\beta : x \in \mathbb{Z}[\beta] \cap \mathbb{Z}[\beta^{-1}]\}$ forms a tiling of $\mathbb{K}''_\beta$ (and $\{\delta_\beta(x) + \hat{Y}_\beta : x \in \mathbb{Z}[\beta] \cap \mathbb{Z}[\beta^{-1}]\}$ tiles $\mathbb{K}_\beta$).

This holds not only for the canonical digit set $\{0, 1, \ldots, |a_0| - 1\}^\infty$ but for arbitrary complete residue systems of $\mathbb{Z}[\beta]/\beta\mathbb{Z}[\beta]$, and for expansive integer matrices. (*Open problem for expansive rational matrices with reducible characteristic polynomial*)





Tilings $\{\delta''_\beta(x) + \mathcal{F}_\beta : x \in \mathbb{Z}[\beta] \cap \mathbb{Z}[\beta^{-1}]\}$ of $\mathbb{K}''_\beta$ for $\beta = -1+i$ and $\beta = 3/2$

$$\mathbb{Z}[-1+i] \cap \mathbb{Z}[(-1+i)^{-1}] = \mathbb{Z}[i]$$

$$\mathbb{Z}[3/2] \cap \mathbb{Z}[2/3] = \mathbb{Z}$$

β-expansion tilings, β Pisot

$$\hat{X}_\beta = \bigcup_{x \in [0,1)} \{x\} \times \mathcal{R}_\beta(x), \quad \mathcal{R}_\beta(x) = \overline{\bigcup_{k \ge 0} \delta'_\beta\big(x - \beta^k T_\beta^{-k}(x)\big)}$$

$\{\delta_\beta(x) + \hat{X}_\beta : x \in \mathbb{Z}[\beta]\}$ forms a tiling of $\mathbb{K}_\beta$ if and only if
$\{\mathcal{R}_\beta(x) - \delta'_\beta(x) : x \in \mathbb{Z}[\beta] \cap [0,1)\}$ forms a tiling of $\mathbb{K}'_\beta$

$$\mathcal{R}_\beta(x) = \bigcup_{y \in T_\beta^{-1}(x)} \big(\beta \cdot \mathcal{R}_\beta(y) - \delta'_\beta(\beta y - x)\big) \quad \left(\begin{array}{c}\text{graph-directed}\\ \text{iterated function system}\end{array}\right)$$

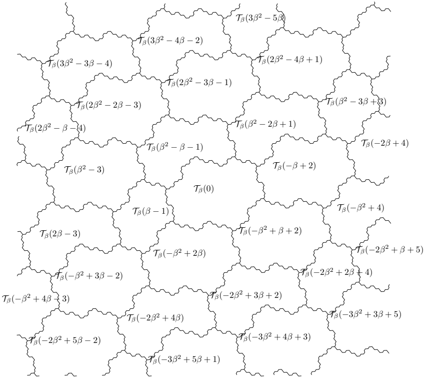
(W_β) : $\forall x \in \mathbb{Z}[\beta] \cap [0,1)$ s.t. $T_\beta^n(x) = x$ for some $n \ge 1$
$\exists\, y \in [0, 1-x), m \ge 0$ s.t. $T_\beta^m(x+y) = T_\beta^m(y) = 0$

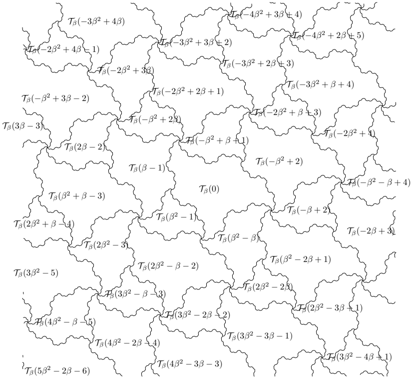
Hollander '96, Akiyama '02, Sidorov '03, Minervino–St '14:
$(\mathrm{F}_\beta) \Rightarrow (\mathrm{W}_\beta) \iff$ tiling

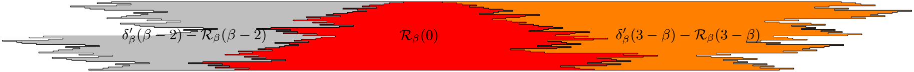
Akiyama–Rao–St '04:
$\beta > 1,\ \beta^d = a_{d-1}\beta^{d-1} + \cdots + a_0\beta^0,\ a_{d-1} > \sum_{k=0}^{d-2} |a_k| \Rightarrow (\mathrm{W}_\beta)$

Barge '18: β Pisot $\Rightarrow$ tiling (Pisot conjecture for β-substitutions)

Tilings $\{\delta'_\beta(x) - \mathcal{R}_\beta(x) : x \in \mathbb{Z}[\beta] \cap [0,1)\}$ of $\mathbb{C}$ for $\beta^3 = \beta^2 + \beta + 1$ and $\beta^3 = \beta + 1$, of $\mathbb{R} \times \mathbb{Z}_\beta$ for $\beta^2 = 2\beta + 2$







Determining a digit in the β-expansion without knowing the other ones, β Pisot

$\beta \geq 2$ integer, $x \in [0, 1)$:

$$\beta^k x = \beta^k (0.x_1 x_2 \cdots)_\beta = (x_1 \cdots x_k . x_{k+1} x_{k+2} \cdots)_\beta \in (0.x_1 x_2 \cdots)_\beta + \mathbb{Z}$$

$$x_{k+1} = b \iff \beta^k x \in \left[\tfrac{b}{\beta}, \tfrac{b+1}{\beta}\right) + \mathbb{Z} \iff (\beta^k x, 0) \in \left[\tfrac{b}{\beta}, \tfrac{b+1}{\beta}\right) \times \mathbb{Z}_2 + \delta_2(\mathbb{Z})$$

β Pisot number:

$$\hat{X}_{\beta,b} := \bigcup_{x \in [\frac{b}{\beta}, \frac{b+1}{\beta}) \cap [0,1)} \{x\} \times \mathcal{R}_\beta(x), \quad \hat{X}_\beta = \bigcup_{b=0}^{\lceil \beta \rceil - 1} \hat{X}_{\beta,b} \quad \text{(disjoint)}$$

$$x \in [0, 1) \Rightarrow (\beta^k x, 0, \ldots, 0) + \delta_\beta\big(T_\beta^k(x) - \beta^k x\big) \in \{T_\beta^k(x)\} \times \mathcal{R}_\beta(T_\beta^k(x))$$

$$\lfloor \beta T_\beta^k(x) \rfloor = b \quad \Rightarrow \quad (\beta^k x, 0, \ldots, 0) \in \hat{X}_{\beta,b} + \delta_\beta(\mathbb{Z}[\beta])$$

Since $\{\delta_\beta(x) + \hat{X}_\beta : x \in \mathbb{Z}[\beta]\}$ forms a tiling (Barge '18):

$$(\beta^k x, 0, \ldots, 0) \in \big(\hat{X}_{\beta,b} \setminus \partial \hat{X}_{\beta,b}\big) + \delta_\beta(\mathbb{Z}[\beta]) \;\Rightarrow\; \lfloor \beta T_\beta^k(x) \rfloor = b$$

(cf. Kalle–St '12, used in Drmota–St '01, St '02)

Generalised β-transformations

Instead of $T_\beta : x \mapsto \beta x - \lfloor \beta x \rfloor$, consider other piecewise affine maps with slope β, e.g., the symmetric β-transformation

$$\tilde{T}_\beta : [-1/2, 1/2) \to [-1/2, 1/2), \quad x \mapsto \beta x - \lfloor \beta x + 1/2 \rfloor$$

or "$\beta x + \alpha$ mod 1", $\alpha \in [0, 1)$,

$$T_{\beta,\alpha} : [0, 1) \to [0, 1), \quad x \mapsto \beta x + \alpha - \lfloor \beta x + \alpha \rfloor$$

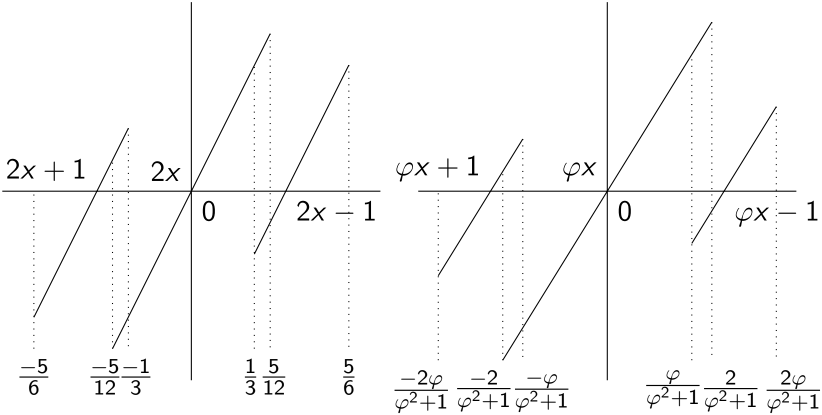
Kalle–St '12:
Many properties of T_β hold for these transformations, but double tilings instead of tilings for $\tilde{T}_\beta$, $\beta^3 = \beta^2 + \beta + 1$ and $\beta^3 = \beta + 1$

Hejda '18:
Tilings with multiplicity $d-1$ for $\tilde{T}_\beta$, $\beta^d = \beta^{d-1} + \cdots + \beta^0$

Frougny–St '08:
for $\beta = 2$, $\beta^2 = \beta + 1$, $\beta^3 = \beta + 1$, $\beta^3 = \beta + 1$, $\beta^3 = \beta^2 + \beta + 1$, (some) β-expansions of minimal weight are given by generalised β-transformations

Frougny–St '08:

2-expansions and φ-expansions of minimal weight are given by the following transformations (in the "switch regions", either branch can be chosen, cf. random β-transformations)



Continued fractions

Gauss map

$$T: [0,1) \to [0,1), \quad x \mapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \quad 0 \mapsto 0$$

(regular) continued fraction with partial quotients $a_k = \left\lfloor \frac{1}{T^{k-1}(x)} \right\rfloor$

$$x = \frac{1}{a_1 + T(x)} = \frac{1}{a_1 + \frac{1}{a_2 + T^2(x)}} = \cdots = \frac{1}{a_1 + \frac{1}{a_2 + \ddots}}$$

$$\frac{96}{365} = \frac{1}{3 + \frac{1}{1 + \frac{1}{4 + \frac{1}{19}}}}$$

$T^n(x) = 0$ for some $n \geq 0 \iff x \in \mathbb{Q} \cap [0,1)$

Euler 1737, Lagrange 1770:
$T^n(x) = T^m(x)$ for some $n > m \geq 0 \iff x \in [0,1)$ quadratic

Galois 1829:
$T^n(x) = x$ for some $n \geq 0 \iff x \in [0,1)$ quadratic, $x' \in [0,1)$

Convergence

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}, \qquad \frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}} \to x$$

$$|p_n - q_n x| \leq \frac{1}{q_{n+1}} \to 0 \qquad \text{(strong convergence)}$$

$$A^{(n)}(x) = \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

$$\frac{96}{365} = \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{19}}}}, \quad \begin{pmatrix} 365 & 19 \\ 96 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 19 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^{(n)}(x) = A(x)\, A(T(x)) \cdots A(T^{n-1}(x)), \quad A(x) = \begin{pmatrix} \lfloor \frac{1}{x} \rfloor & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^{(m+n)}(x) = A^{(m)}(x)\, A^{(n)}(T^m(x)) \quad \text{(transposed linear cocycle)}$$

$$\bigcap_{n=1}^{\infty} A^{(n)}(x)\, \mathbb{R}_+^2 = \mathbb{R}_+ \begin{pmatrix} 1 \\ x \end{pmatrix} \qquad \text{(weak convergence)}$$

Multidimensional continued fraction algorithms

Aim: Given $\mathbf{x} \in [0,1]^d$, construct $\mathbf{p}^{(n)} \in \mathbb{N}^d$, $q^{(n)} \in \mathbb{N}$ such that

- $\mathbf{x} = \lim_{n\to\infty} \frac{\mathbf{p}^{(n)}}{q^{(n)}}$ (weak convergence),
- $\lim_{n\to\infty} \|\mathbf{p}^{(n)} - q^{(n)}\mathbf{x}\| = 0$ (strong convergence).

A MCF algorithm constructs matrices

$$A^{(n)}(\mathbf{x}) = \begin{pmatrix} q_0^{(n)} & q_1^{(n)} & \cdots & q_d^{(n)} \\ p_{0,1}^{(n)} & p_{1,1}^{(n)} & \cdots & p_{1,d}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{0,d}^{(n)} & p_{d,1}^{(n)} & \cdots & p_{d,d}^{(n)} \end{pmatrix} \in \mathbb{N}^{(d+1)\times(d+1)}$$

where each column vector is close to the line $\mathbb{R}\,{}^t(1, x_1, \ldots, x_d)$.
Note that

$$\begin{pmatrix} q \\ p_1 \\ \vdots \\ p_d \end{pmatrix} - q \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} 0 \\ p_1 - qx_1 \\ \vdots \\ p_d - qx_d \end{pmatrix}$$

Jacobi–Perron algorithm 1868, 1907

$$T: [0,1]^d \to [0,1]^d, \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \mapsto \begin{pmatrix} \frac{x_2}{x_1} - \left\lfloor \frac{x_2}{x_1} \right\rfloor \\ \vdots \\ \frac{x_d}{x_1} - \left\lfloor \frac{x_d}{x_1} \right\rfloor \\ \frac{1}{x_1} - \left\lfloor \frac{1}{x_1} \right\rfloor \end{pmatrix}$$

$$A(\mathbf{x}) = \begin{pmatrix} \lfloor \frac{1}{x_1} \rfloor & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \lfloor \frac{x_2}{x_1} \rfloor & 1 & \ddots & & & \vdots \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \lfloor \frac{x_d}{x_1} \rfloor & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{N}^{(d+1)\times(d+1)}$$

$$\begin{pmatrix} 1 \\ T(\mathbf{x}) \end{pmatrix} = \frac{1}{x_1} A(\mathbf{x})^{-1} \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}$$

$A^{(n)}(\mathbf{x}) = A(\mathbf{x}) A(T(\mathbf{x})) \cdots A(T^{n-1}(\mathbf{x}))$ (transposed linear cocycle)

other algorithms: Brun 1919, Selmer 1961, ...

Approximation cocycle

$$D^{(n)}(\mathbf{x}) = \begin{pmatrix} p_{1,1}^{(n)} - q_1^{(n)} x_1 & \cdots & p_{1,d}^{(n)} - q_1^{(n)} x_d \\ \vdots & \ddots & \vdots \\ p_{d,1}^{(n)} - q_d^{(n)} x_1 & \cdots & p_{d,d}^{(n)} - q_d^{(n)} x_d \end{pmatrix}$$

(given by $A^{(n)}(\mathbf{x})$) satisfies

$$D^{(n)}(\mathbf{x}) = D(\mathbf{x}) D(T(\mathbf{x})) \cdots D(T^{n-1}(\mathbf{x}))$$

For Jacobi–Perron,

$$D(\mathbf{x}) = \begin{pmatrix} 0 & \cdots & \cdots & 0 & -x_1 \\ 1 & \ddots & & \vdots & -x_2 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -x_{d-1} \\ 0 & \cdots & 0 & 1 & -x_d \end{pmatrix} \in \mathbb{R}^{d \times d}$$

The second Lyapunov exponent of the transposed cocycle ${}^tA$ is the top Lyapunov exponent of the transposed cocycle ${}^tD$: $\theta_2(A) = \theta_1(D)$.

$$\theta_1(D) = \lim_{n \to \infty} \frac{1}{n} \log \| D^{(n)}(\mathbf{x}) \| \quad \text{for a.e. } \mathbf{x}$$

Upper bounds for $\theta_1(D) = \theta_2(A)$

ergodicity w.r.t. absolutely continuous invariant measure μ, subadditivity $\log \|M\,N\| \le \log \|M\| + \log \|N\| \Rightarrow$

$$\theta_1(D) = \inf_{n\in\mathbb{N}} \frac{1}{n} \int_{[0,1]^d} \log \|D^{(n)}(\mathbf{x})\| \,\mathrm{d}\mu(\mathbf{x})$$

$d = 2$:

$\theta_1(D) < 0$ for several classical MCF algorithms

Berthé–St–Thuswaldner '21: $\theta_1(D) < -0.05243599$ for Selmer ($n = 50$)

$d = 3$:

Schratzberger '01, Hardcastle(–Khanin) '02: $\theta_1(D) < 0$ for Brun, $\theta_1(D) < -0.0053293$ for modified Jacobi–Perron (d-dimensional Gauss)

Berthé–St–Thuswaldner '21: $\theta_1(D) < -0.00043645$ for Selmer ($n = 52$)

Computer simulations (calculating $D^{(2^{30})}(\mathbf{x})$ for randomly chosen $\mathbf{x}$ without accuracy) indicate $\theta_1(D) > 0$ for Selmer ($d \ge 4$), Brun ($d \ge 10$), Jacobi–Perron ($d \ge 10$), Garrity ($2 \le d \le 6$, $d \ge 11$), nearest integer Jacobi–Perron ($d \ge 14$), contradicting conjectures.

Proving lower bounds is much more difficult than proving upper bounds.

Comparison between (simulations of) algorithms

uniform approximation coefficients $1 - \frac{\theta_1(D)}{\theta_1(A)} \leq 1 + \frac{1}{d}$

d	Selmer	Brun	JP	BST	Garrity	NIJP
2	**1.3871**	1.3683	1.3735	1.3606	0.6859	**1.40145**
3	1.1444	1.2203	1.1922	**1.2430**	0.5798	1.22519
4	0.9866	1.1504	1.1114	**1.1817**	0.6286	1.14373
5	0.8577	1.1065	1.0676	**1.1388**	0.7778	1.09786
6	0.7442	1.0746	1.0413	**1.1034**	0.9468	1.06898
7	0.6437	1.0493	1.0243	**1.0729**	1.0225	1.04944
8	0.5561	1.0283	1.0127	**1.0468**	1.0304	1.03551
9	0.4810	1.0102	1.0044	**1.0246**	1.0189	**1.02521**
10	0.4173	0.9943	0.9981	**1.0054**	1.0035	**1.01737**
11	0.3636	0.9799	**0.9933**	0.9886	0.9880	**1.01125**
12						**1.00639**
13						**1.00246**
14						**0.99924**

bold face: best algorithm or best positive algorithm in dimension d (among those that we considered) according to our simulations

S-adic dynamical systems associated with MCF

positive d-dimensional continued fraction algorithm (Δ, T, A):

$$\Delta \subseteq [0,1]^d, \quad A : \Delta \to \mathrm{GL}(d+1, \mathbb{Z}) \cap \mathbb{N}^{(d+1)\times(d+1)}$$

$$T : \Delta \to \Delta \quad \text{defined by} \begin{pmatrix} 1 \\ T(\mathbf{x}) \end{pmatrix} = \frac{1}{x_1} A(\mathbf{x})^{-1} \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}$$

select substitutions $\varphi_{\mathbf{x}}$, $\mathbf{x} \in \Delta$, over alphabet $\mathcal{A} = \{0, 1, \ldots, d\}$
with incidence matrix $A(\mathbf{x})$,

$$\varphi_{\mathbf{x}}^{(n)} := \varphi_{\mathbf{x}} \circ \varphi_{T(\mathbf{x})} \circ \cdots \circ \varphi_{T^{n-1}(\mathbf{x})} \quad (\text{incidence matrix } A^{(n)}(\mathbf{x}))$$

This gives an S-adic shift $(X_{\varphi,\mathbf{x}}, \Sigma)$, with shift map Σ and

$$X_{\varphi,\mathbf{x}} := \{\omega \in \mathcal{A}^{\mathbb{N}} : \text{each factor of } \omega \text{ is factor of } \varphi_{\mathbf{x}}^{(n)}(i) \text{ for some } i \in \mathcal{A}, n \geq 1\}$$

Example: $d = 1$, $\Delta = [0,1] \setminus \mathbb{Q}$

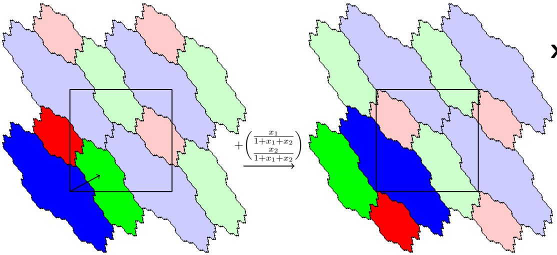
$$A(x) = \begin{pmatrix} \lfloor \frac{1}{x} \rfloor & 1 \\ 1 & 0 \end{pmatrix}, \quad \varphi_x : \{0,1\} \to \{0,1\}^*, \begin{array}{l} 0 \mapsto 0^{\lfloor 1/x \rfloor} 1 \\ 1 \mapsto 0 \end{array}$$

$X_{\varphi,x}$ consists of the Sturmian sequences with slope x

Berthé–St–Thuswaldner '19, Fogg–Noûs, Berthé–St–Thuswaldner:
For a positive d-dimensional continued fraction algorithm (Δ, T, A, μ) satisfying $\theta_2(A) < 0$, under some mild conditions on the invariant measure μ and a substitution selection φ, for μ-almost all $\mathbf{x} \in \Delta$:

$(X_{\varphi,\mathbf{x}}, \Sigma)$ is a natural coding (w.r.t. Rauzy fractals) of the translation by $\frac{\mathbf{x}}{1+\|\mathbf{x}\|_1}$ on $\mathbb{R}^d/\mathbb{Z}^d$; hence $(X_{\varphi,\mathbf{x}}, \Sigma)$ has purely discrete spectrum.

$X_{\varphi,\mathbf{x}}$ is balanced on factors, i.e., for each $v \in \mathcal{A}^*$ there exists C_v s.t. the number of occurrences of v in w and w' differs by at most C_v for all w, w' in the language of $X_{\varphi,\mathbf{x}}$ with $|w| = |w'|$. In other words, each cylinder of $X_{\varphi,\mathbf{x}}$ is a bounded remainder set of $(X_{\varphi,\mathbf{x}}, \Sigma)$.



$$\mathbf{x} = \begin{pmatrix}1/\beta \\ 1/\beta^2\end{pmatrix},\ A(\mathbf{x}) = \begin{pmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

$$\beta^3 = \beta^2 + \beta + 1,\ T(\mathbf{x}) = \mathbf{x}$$

$$\varphi_{\mathbf{x}}: \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 02 \\ 2 \mapsto 0 \end{array}$$ (Rauzy '82)

Other contributions to numeration systems

- $(-\beta)$-expansions, $\beta > 1$: gaps in the invariant measure, kneading invariants for (intermediate) $(-\beta)$-shifts, $(-\beta)$-integers, occurrence of permutations
- unique β-expansions, intermediate β-shifts of finite type, base β van der Corput sequences, odometers
- patterns in base p/q number systems (Champernowne type construction of a normal number in base p)
- abstract numeration systems: preservation of recognisability by multiplication, van der Corput sequences, periodic expansions
- shift radix systems: tilings, discretised rotations
- α-(Rosen-)continued fractions: natural extensions, entropy, matching