POINTWISE ORDER OF GENERALIZED HOFSTADTER FUNCTIONS G, H AND BEYOND

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Abstract. Hofstadter's G function is recursively defined via $G(0) = 0$ and then $G(n) = n - G(G(n-1))$. Following Hofstadter, a family (F_k) of similar functions is obtained by varying the number k of nested recursive calls in this equation. We establish here that this family is ordered pointwise: for all k and n, $F_k(n) \leq F_{k+1}(n)$. For achieving this, a detour is made via infinite morphic words generalizing the Fibonacci word. Various properties of these words are proved, concerning the lengths of substituted prefixes of these words and the counts of some specific letters in these prefixes. We also relate the limits of $\frac{1}{n}F_k(n)$ to the frequencies of letters in the considered words.

1. Introduction

1.1. The functions. For each integer $k \geq 1$, we define recursively the function[1]

$$F_k : \mathbb{N} \to \mathbb{N}, \quad n \mapsto \begin{cases} 0 & \text{if } n = 0, \\ n - F_k^k(n-1) & \text{otherwise.} \end{cases}$$

The function F_k is well defined since one may prove alongside that $0 \leq F_k(n) \leq n$ for all $n \geq 0$. This family of functions is due to Hofstadter [12, Chapter 5]. In particular, F_2 is Hofstadter's function G, see OEIS entry A5206 [1, 12, 4, 9], known to satisfy $G(n) = \lfloor (n+1)/\varphi \rfloor$ where φ is the golden ratio $(1+\sqrt{5})/2$. Similarly, F_3 is Hofstadter's function H, see OEIS A5374, and the generalization to higher degrees of recursive nesting in the definition of F_k was already suggested by Hofstadter [12]. To be complete, the OEIS database already includes F_4 as A5375, F_5 as A5376, and F_6 as A100721. On the other hand, we choose to start this sequence with F_1, where only one recursive call is done, leading to a function that can easily be shown to verify $F_1(n) = \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$. Throughout this paper, we will never consider the case $k = 0$: although the previous definition could be extended and give a non-recursive function F_0, this F_0 has too little in common with the other F_k functions to be of much interest.

1.2. A monotonicity problem. Small values of the functions F_1 to F_5 are displayed in Figure 1.1. From this figure, one may easily hint that F_1 is everywhere below or equal to F_2, similarly that F_2 is below or equal to F_3, and so forth. Indeed, a main contribution of the present article is to prove that for all $k \geq 1$ and $n \geq 0$, we have $F_k(n) \leq F_{k+1}(n)$, see Theorem 7.4. This property seems to defy any attempt to prove it directly via induction on the functions F_k. Therefore, a different approach is used. We relate each function F_k to a morphic word x_k and turn the function comparison into an equivalent statement on these words.

[1] In all this work, the n-th exponent of a function denotes its n-th iterate, for instance $F_2^2(n)$ is $F_2(F_2(n))$, not the square of $F_2(n)$.

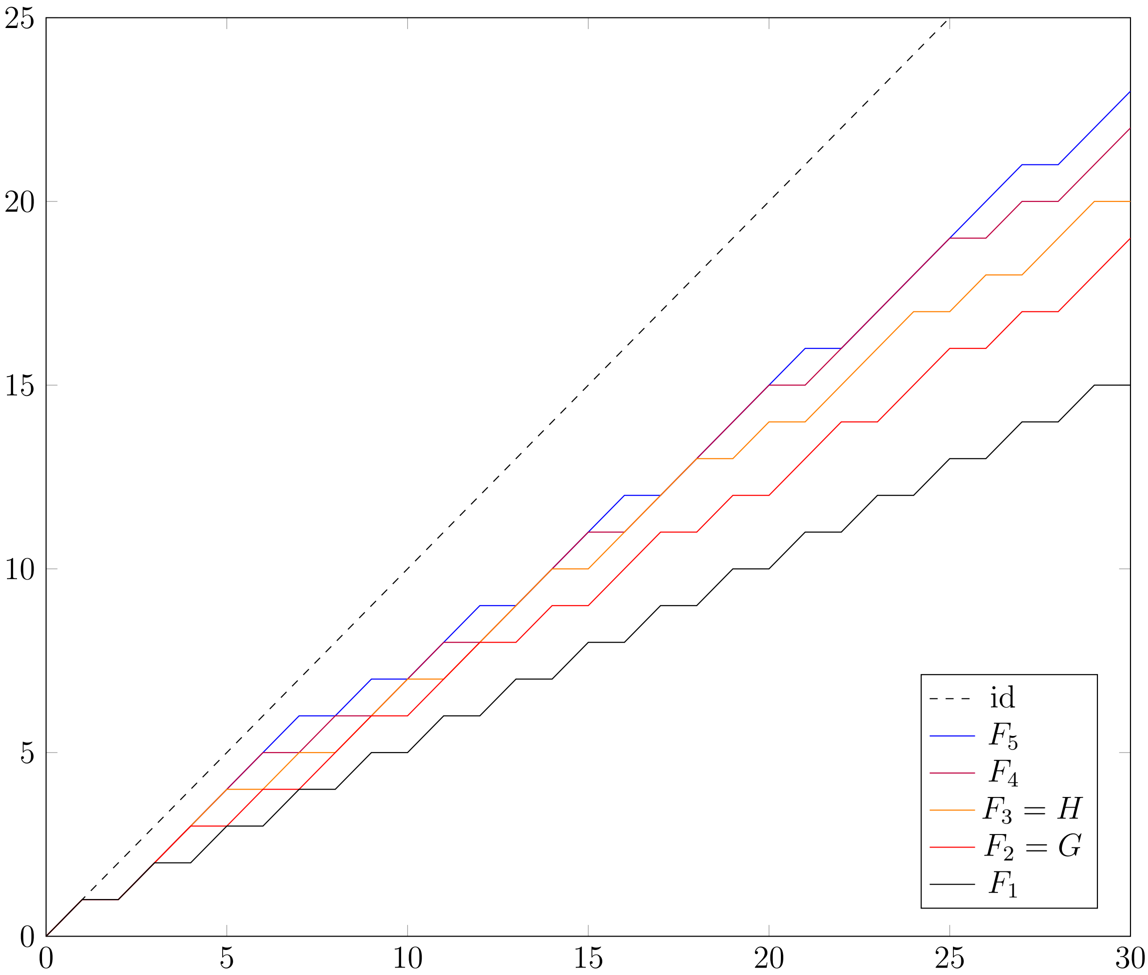


FIGURE 1.1. Plotting $F_1, F_2, \ldots, F_5$

1.3. **Substitutions and morphic words.** For $k \geq 1$, let τ_k be the substitution[2] on the alphabet $\{1, 2, \ldots, k\}$ defined by

$$\begin{aligned} \tau_k : \; k &\mapsto k1, \\ i &\mapsto i{+}1 \quad \text{for } 1 \leq i < k. \end{aligned}$$

Let $x_k = x_k[0]x_k[1] \cdots \in \{1, 2, \ldots, k\}^\infty$ be the fixed point of τ_k.

$$\begin{aligned} x_2 &= 2122121221221212212122122121221221212212 \cdots \\ x_3 &= 3123313123123312331312331312312331312312 \cdots \\ x_4 &= 4123441412412341234412344141234414124123 \cdots \\ x_5 &= 5123455151251235123451234551234551512345 \cdots \end{aligned}$$

FIGURE 1.2. Infinite words $x_2, \ldots, x_5$ (with highest letter in red)

[2]A substitution (or morphism) on an alphabet A is a map $\tau : A^* \to A^*$ satisfying $\tau(uv) = \tau(u)\tau(v)$ for all $u, v \in A^*$, where A^* denotes the set of finite words with letters in A. The map τ is therefore defined by its value on the letters of A, and it is extended in a natural way to infinite words (or sequences) $w = w[0]w[1] \cdots \in A^\infty$ by setting $\tau(x) = \tau(x[0])\tau(x[1]) \cdots$. For more on substitutions, see e.g. [6].

These substitutions and words are not novel. For instance, τ_1 operates on the singleton alphabet $\{1\}$ in such a way that $\tau_1(1) = 11$, and hence $x_1 = (1)^\infty$. It will be the only case where x_k is ultimately periodic. For $k = 2$, we retrieve the well-known Fibonacci substitution and word.[3] The substitution τ_3 already appears (up to letter renaming) in [6] as an example of a *modified Jacobi-Perron substitution*; see $\sigma(1, 0)$ in [6, Exercise 8.1.2]. More generally, the substitutions x_k and words x_k can be associated with the Rényi expansion of 1 in base β_k, where β_k is the positive root of $X^k - X^{k-1} - 1$, see [7, 8]. When $k \geq 2$, this expansion can be written $1(0)^{k-2}1$. As a consequence, the factor complexity of x_k is known to be $n \mapsto (k-1)n+1$; for instance, x_2 is indeed Sturmian.

1.4. **New results.** In this article, in addition to the aforementioned monotonicity of F_k (Theorem 7.4), we exhibit several relations between the functions F_k and the words x_k. For instance, for $k \geq 1$ and a position $n \geq 0$, the value of $x_k[n]$ is either the first index $j \leq k$ such that $F_k^j(n+1) = F_k^j(n)$, or k if there is no such small index; see Proposition 4.4 below. In particular, $x_k[n]$ is the letter 1 exactly when $F_k(n+1) = F_k(n)$, and this also leads to the fact that $F_k(n)$ is equal to the number of non-1 letters in the first n letters of x_k; see Proposition 4.3. Such properties were already known in the case $k = 2$, where $F_2 = G = \lfloor (n+1)/\varphi \rfloor$ and x_2 amounts to the Fibonacci word; see, for instance, [3, Example 8.1.3]. The extended results to an arbitrary parameter k seem new.

All these relations between the functions F_k and the words x_k stem from the key Theorem 3.1 and its Corollary 3.2: F_k and their iterates admit Galois connections (i.e., almost inverses) that can be simply expressed in terms of the length of τ_k-substituted prefixes of x_k. These lengths will be named L_k here, see Proposition 2.3.

The words x_k were primarily considered here in order to study the monotonicity of the functions F_k. Nonetheless, several results on x_k are also worth of interest by themselves. For example, for $k \geq 1$ and $n \geq 0$, we prove that there are more occurrences of the letter k among the first n letters of x_k than occurrences of the letter $k+1$ among the first n letters of x_{k+1} (Theorem 7.2). Similarly, the letter 1 occurs more often among the first n letters of x_k than among the first n letters of x_{k+1} (Proposition 8.1). Such counts of letters i among prefixes of x_k of length n will be denoted $C_k^{(=i)}(n)$, see Section 4.

1.5. **The Coq artifact.** All the proofs presented in this article have been formally certified using the Coq proof assistant [21]. The files of this Coq development are freely available [14], the recommended entry point to read alongside this article being

https://github.com/letouzey/hofstadter_g/blob/main/Article1.v

This formal development ensures precise definitions and statements and rules out any reasoning errors during the proofs. It can hence serve as a reference for the interested reader. The current article tries to be faithful to this formal work while staying readable by a large audience, at the cost of possible remaining mistakes during the transcription.

This Coq development can be machine-checked again by any recent installation of Coq, see the joint `README` file (the authors used Coq version 8.16). The parts corresponding to Sections 5 and 6 involve real numbers and hence rely on some Coq standard libraries that declare four logical axioms, in particular the axiom of excluded middle. All the rest of the development (including Theorem 7.4) have been formalized within Coq core logic, without any extra axioms, as may be checked via the command `Print Assumptions` on our theorems.

[3]Usually, the letter k is replaced by the letter 0 in the definition of τ_k and x_k. We choose the letter k in order to simplify some formulas and to extend the definition without modification to the case $k = 1$.

1.6. **Summary.** After some notations and basic properties in Section 2, we establish a key link between the functions F_k and words x_k in Section 3. Section 4 studies some counts of letters in prefixes of x_k. Section 5 presents some roots of polynomials that are used in Section 6 when stating the infinitary behavior of our objects (i.e. when $n \to \infty$). In particular, we present linear equivalents of F_k and a weak form of monotonicity for the family (F_k). Section 7 contains the promised full monotonicity proof for all $n \geq 0$, at first for words then functions, and various extensions are made or conjectured. Section 8 revisits the previous results, this time in terms of letter counts.

1.7. **Related works.** The functions F_k can be studied under various aspects. We mention some of them, even if they do not seem to be connected to the monotonicity, which is the main focus of this article.

First, the case $k = 2$ has been explored abundantly, see for instance [4, 9]. But the fact that F_2 has an exact expression $\lfloor (n+1)/\varphi \rfloor$ is quite particular and too specific to draw general lessons about all the other F_k.

Now, a nice general fact about functions F_k is that they can be presented as right shifts of digits when numbers n are written with the appropriate numerical representations. In particular, for $k = 2$, the corresponding numerical representation is the Zeckendorf decomposition, writing numbers as sums of distinct Fibonacci numbers. This can be extended to other values of k by changing the sequence used as the base for the decomposition, from Fibonacci numbers to a similar linearly recurrent sequence. For more on this topic, see [16, 4, 13, 5, 17, 19]. Note in particular that [16] focus on a slightly different family of functions, which can be written here as $F_k(n+1)-1$. If we denote by $\tilde{F}_k$ these "shifted" functions, they verify the equations $\tilde{F}_k(0) = 0$ and $\tilde{F}_k(n) = n-1-\tilde{F}_k^k(n-1)$, note the extra -1 when compared to the equation of F_k. Anyhow, having a description of $F_k(n)$ (or $\tilde{F}_k(n)$) in terms of a k-decomposition of n does not directly help for our monotonicity problem, since here the k-decomposition and $(k+1)$-decomposition differ too much, just like a number tends to have deeply different base-2 and base-3 digits. Obviously, this idea of k-decomposition is also related with the notions of substitutions and words we exploited here, but these substitutions and words proved more insightful here.

As a final remark about F_k, note that F_k can also be presented as a companion infinite tree where the same pattern is continuously repeated: a binary node is followed on its right by $k-1$ unary nodes. When the nodes of this tree are labelled by increasing numbers, in a breadth-first manner, left to right, then the node number n has $F_k(n)$ as parent node. This was already described by Hofstadter [12], see also the general introduction to the problem [15]. Here again, these alternative presentations of F_k do not seem to help concerning the monotonicity, with no direct relations between trees for k and for $k+1$.

The sequences x_k defined in Section 1.3 can be considered as generalizations of the Fibonacci word. However, one can find many different "generalized Fibonacci words" in the literature, for example the k-bonacci words, which can be defined as fixed points of the morphism satisfying $1 \mapsto k$, $i \mapsto k(i-1)$ for $2 \leq i \leq k$ (see e.g. [20]), the fixed points of the morphisms $0 \mapsto 0^m1^n$, $1 \mapsto 0$ for positive integers m, n (see [3, Exercise 10.11.18]), and other examples in [18, 10, 11].

2. Notation and basic properties

We abbreviate by $\partial F_k^j(n)$ the difference $F_k^j(n+1)-F_k^j(n)$.

Proposition 2.1. *For all $j, k \geq 1$, the function F_k^j satisfies the following basic properties:*

(a) $F_k^j(1) = F_k^j(2) = 1$,

(b) *$F_k^j(n) \geq 1$ whenever $n \geq 1$,*

(c) *$F_k^j(n) < n$ whenever $n \geq 2$,*

(d) *for all $n > 0$, $\partial F_k(n) = 1 - \partial F_k^k(n{-}1)$,*

(e) *hence $\partial F_k^j(n) \in \{0, 1\}$ for all $n \geq 0$,*

(f) *the function F_k^j is monotonically increasing and onto (but not one-to-one).*

Proof. Direct use of the definition or easy induction on n. □

For $n \geq 0$, we denote by $x_k[0{:}n) = x_k[0]x_k[2]\cdots x_k[n{-}1]$ the prefix of x_k of length n. In particular, $x_k[0{:}0)$ is the empty word, and $\left|x_k[0{:}n)\right| = n$.

Note that x_k can also be seen as the fixed point of the substitution τ_k^{k-1}. Since $\tau_k^{k-1}(i) = k12\cdots(i{-}1)$ for all $1 \leq i \leq k$, this provides a decomposition of x_k into blocks delimited by the letter k (in red in Figure 1.2). The lengths of these blocks are the successive letters of x_k, and this gives us a convenient way to compute x_k. For instance $\tau_3^2(3) = 312$ and then $\tau_3^2(312) = 312\,3\,31$, etc.

Similarly, x_k is also the fixed point of τ_k^k. Since $\tau_k^k(i) = k12\cdots i$ for all $1 \leq i \leq k$, this other decomposition will be interesting for counting occurrences of the letter 1 in x_k, since it occurs only in the second position of these blocks.

As an alternative way of computing x_k, note that its prefixes of the form $\tau_k^j(k)$ follow the following base cases and recursive rule.

Proposition 2.2. *Let $k \geq 1$. Then*

$$\tau_k^j(k) = \begin{cases} k12\cdots j & \text{when } 0 \leq j \leq k, \\ \tau_k^{j-1}(k)\,\tau_k^{j-k}(k) & \text{when } j \geq k. \end{cases}$$

Proof. By induction over j. For $j = 0$, $\tau_k^0(k) = k$. Otherwise, $\tau_k^j(k) = \tau_k(\tau_k^{j-1}(k))$ and we use the induction hypothesis in $j{-}1$. Either $j \leq k$ and

$$\tau_k^j(k) = \tau_k(\tau_k^{j-1}(k)) = \tau_k(k12\cdots(j{-}1)) = k12\cdots j$$

(and that also gives the recursive rule when $j = k$ since $k12\cdots(k{-}1)k = \tau_k^{k-1}(k)\tau_k^0(k)$), or $j > k$, hence $j{-}1 \geq k$ and

$$\tau_k^j(k) = \tau_k(\tau_k^{j-1}(k)) = \tau_k(\tau_k^{j-1-1}(k)\tau_k^{j-1-k}(k)) = \tau_k^{j-1}(k)\tau_k^{j-k}(k).$$ □

For $k \geq 1$, an important counterpart to F_k will be the function

$$L_k : \mathbb{N} \to \mathbb{N}, \quad n \mapsto \left|\tau_k(x_k[0{:}n))\right|.$$

Proposition 2.3. *The lengths $L_k(n) = \left|\tau_k(x_k[0{:}n))\right|$ satisfy the following basic properties for any $k \geq 1$ and $j, n \geq 0$:*

(a) *the j-th iterate of L_k satisfies $L_k^j(n) = \left|\tau_k^j(x_k[0{:}n))\right|$,*

(b) *$L_k^j(0) = 0$ and $L_k^j(n{+}1) = L_k^j(n) + \left|\tau_k^j(x_k[n])\right|$,*

(c) *$L_k^j(1) = j{+}1$ when $j \leq k$ and $L_k^j(1) = L_k^{j-1}(1) + L_k^{j-k}(1)$ when $j \geq k$,*

(d) *L_k^j is strictly monotonically increasing over n,*

(e) *$L_k^j(n) \geq n$, with equality only when $j = 0$ or $n = 0$,*

(f) *for $n > 0$, $L_k^j(n)$ is strictly monotonically increasing over j.*

Proof. First, $\tau_k(x_k[0{:}n))$ is the prefix of x_k of length $L_k(n)$, it is hence equal to $x_k[0{:}L_k(n))$. We claim that, more generally, $\tau_k^j(x_k[0{:}n))$ is the prefix of x_k of length $L_k^j(n)$ for all $j \geq 0$.

Since $\tau_k^j(x_k[0{:}n))$ is a prefix of x_k, we only have to prove that its length is equal to $L_k^j(n)$, and this is follows inductively for $j \geq 1$ from

$$\left|\tau_k^j(x_k[0{:}n))\right| = \left|\tau_k(\tau_k^{j-1}(x_k[0{:}n)))\right| = \left|\tau_k(x_k[0{:}L_k^{j-1}(n)))\right| = L_k(L_k^{j-1}(n)) = L_k^j(n).$$

Point (b) is a direct consequence of this more general claim. Point (c) comes directly from Proposition 2.2.

Now, a key fact: for all finite words w, $|w| \leq |\tau_k(w)|$. This inequality is even strict when w is a non-empty prefix of x_k, since $x_k[0] = k$ and $|\tau_k(k)| = 2$. The remaining points are direct consequences of this key fact. □

Proposition 2.4. *For $j \geq 0$ and $k, m \geq 1$, there exists a unique $n \geq 1$ such that $L_k^j(n{-}1) < m \leq L_k^j(n)$.*

Proof. Standard consequence of $L_k^j(0) = 0$ and the strict monotonicity of L_k^j. □

3. Relating functions and word lengths

We now establish that L_k^j allows to express the antecedents of the function F_k^j.

Theorem 3.1. *Let $j \geq 0$ and $k \geq 1$. For all $n \geq 1$, we have*

$$F_k^{-j}(\{n\}) = \big(L_k^j(n{-}1), L_k^j(n)\big] \cap \mathbb{N}.$$

Equivalently, for all $m \geq 1$ we have

$$L_k^j\big(F_k^j(m){-}1\big) < m \leq L_k^j\big(F_k^j(m)\big). \tag{3.1}$$

Proof. We first prove the equivalence between the two statements. The first one implies directly the second one, by instantiating n with $F_k^j(m)$. Now, we assume the second statement and prove the first one by double inclusion: if $m \in F_k^{-j}(\{n\})$, then $F_k^j(m) = n$ and the inequalities (3.1) become $L_k^j(n{-}1) < m \leq L_k^j(n)$. In the other direction, both n and $F_k^j(m)$ satisfy the conditions of Proposition 2.4, hence they are equal.

We now prove the second statement. The case $k = 1$ must be handled separately, since our general proof below requires the letters 1 and k to differ. Fortunately, when $k = 1$ we have $L_1^j(n) = 2^j n$ and $F_1^j(m) = \lceil m/2^j \rceil$, allowing to easily conclude this case. We can also handle separately the case $j = 0$, which is obvious since both L_k^0 and F_k^0 are the identity functions. For now on, we consider $k \geq 2$ and prove by strong induction over $m \geq 1$ the inequalities (3.1) for all $j \geq 1$.

For $m \in \{1, 2\}$ and $j \geq 1$, we have $F_k^j(m) = 1$ (see Proposition 2.1) while $L_k^j(0) = 0$ and $L_k^j(1) > L_k^0(1) = 1$ (see Proposition 2.3), hence the desired inequalities hold.

Let $n \geq 2$, and assume that (3.1) holds for all $1 \leq m \leq n$, $j \geq 1$. We now prove it for $m = n{+}1$, first in the case $j = 1$, and then for $j \geq 2$.

To show that (3.1) holds for $m = n{+}1$, $j = 1$, we use the following instances of the induction hypothesis:

$$L_k(F_k(n){-}1) < n \leq L_k(F_k(n)), \tag{3.2}$$

$$L_k^k(F_k^k(n){-}1) < n \leq L_k^k(F_k^k(n)), \tag{3.3}$$

$$L_k^k(F_k^k(n{-}1){-}1) < n{-}1 \leq L_k^k(F_k^k(n{-}1)). \tag{3.4}$$

Also recall from Proposition 2.1 that

$$F_k(n{+}1) - F_k(n) = 1 - \big(F_k^k(n) - F_k^k(n{-}1)\big)$$

and $F_k^k(n) - F_k^k(n{-}1) \in \{0, 1\}$.

Assume first that $F_k^k(n) = F_k^k(n{-}1)$. Then $F_k(n{+}1) = F_k(n) + 1$. From (3.2) and the strict monotonicity of L_k (Proposition 2.3) we get

$$n \le L_k(F_k(n)) < L_k(F_k(n){+}1) = L_k(F_k(n{+}1)).$$

Hence the right inequality in (3.1) holds for $m = n{+}1$, $j = 1$. We claim that $L_k(F_k(n)) = n$. If this were not the case, then (3.2) would give $L_k(F_k(n){-}1) \le n{-}1 < n < L_k(F_k(n))$. On the words corresponding to these lengths, this would imply that $x_k[n{-}1]x_k[n]$ is a subword of $\tau_k(x_k[F_k(n){-}1])$, in particular that $|\tau_k(x_k[F_k(n){-}1])| \ge 2$. This could only happen when $x_k[F_k(n){-}1] = k$, and thus $x_k[n] = 1$. From $F_k^k(n) = F_k^k(n{-}1)$, (3.3) and (3.4), we get $L_k^k(F_k^k(n){-}1) \le n{-}2 < n{-}1 < L_k^k(F_k^k(n))$, hence $x_k[n{-}2]x_k[n{-}1]$ is a subword of the block $\tau_k^k(x_k[F_k^k(n){-}1])$. Recall that the letter 1 occurs in this kind of block only at the second position. But here, either $x_k[n]$ is still in the block, but at least in third position, or it starts the next block, and hence $x_k[n] = k$. Anyway, this contradicts that $x_k[n] = 1$. Therefore, we have $L_k(F_k(n)) = n$, hence

$$L_k(F_k(n{+}1){-}1) = L_k(F_k(n)) = n < n{+}1,$$

and also the left inequality in (3.1) holds for $m = n{+}1$, $j = 1$.

Assume now that $F_k^k(n) - F_k^k(n{-}1) = 1$, i.e. $F_k(n{+}1) = F_k(n)$. In this case, the desired left inequality is clear from (3.2):

$$L_k(F_k(n{+}1){-}1) = L_k(F_k(n){-}1) < n < n{+}1.$$

From (3.3) and (3.4), we deduce here $L_k^k(F_k^k(n{-}1)) = n{-}1$. Hence, $x_k[n{-}1]x_k[n]$ starts a new τ_k^k block, thus $x_k[n{-}1]x_k[n] = k1$. This means that the left inequality of (3.2) is actually strict here since the letter 1 cannot start the word $\tau_k(x_k[F_k(n)])$. So finally

$$n < L_k(F_k(n)) = L_k(F_k(n{+}1)),$$

and we have finished the proof of (3.1) for $m = n{+}1$, $j = 1$, in all the possible cases.

Now let $j \ge 2$ and let us prove (3.1) for $m = n{+}1$ and this j. Since $1 \le F_k(n{+}1) \le n$ by Proposition 2.1, the inequalities (3.1) hold for $m = F_k(n{+}1)$ and $j{-}1$, hence

$$L_k^{j-1}(F_k^j(n{+}1){-}1) \le F_k(n{+}1) - 1 \le F_k(n{+}1) \le L_k^{j-1}(F_k^j(n{+}1)).$$

Let us apply L_k on these inequalities, since it is monotonic by Proposition 2.3, and regroup $L_k \circ L_k^{j-1}$ as L_k^j:

$$L_k^j(F_k^j(n{+}1){-}1) \le L_k(F_k(n{+}1) - 1) \le L_k(F_k(n{+}1)) \le L_k^j(F_k^j(n{+}1)).$$

Now, the inequalities (3.1) we proved earlier for $m = n{+}1$ and $j = 1$ can be reused in the middle:

$$L_k^j(F_k^j(n{+}1){-}1) \le L_k(F_k(n{+}1){-}1) < n{+}1 \le L_k(F_k(n{+}1)) \le L_k^j(F_k^j(n{+}1)),$$

hence (3.1) holds indeed for $m = n{+}1$ and j.

By induction, we can now conclude that (3.1) holds for all $j, m \ge 1$. □

In particular, $F_k^j(L_k^j(n)) = n$ for all $k \ge 1$ and $j, n \ge 0$. Moreover $L_k^j(n)$ is the largest antecedent of n by F_k^j while for $n > 0$ the smallest antecedent is $L_k^j(n{-}1){+}1$, and these extrema may coincide. In particular, this is always the case when $j = 0$ and quite frequent when $j = 1$ and $k > 1$, see Proposition 6.5 for a study of this ratio.

The relationship between F_k^j and L_k^j can also be formulated as follows.

Corollary 3.2. *For all $j \ge 0$ and $k \ge 1$, the functions F_k^j and L_k^j form a* Galois connection *between $\mathbb{N}$ and itself (with F_k^j as* left adjoint *and L_k^j as* right adjoint*). Indeed, for all $m, n \ge 0$ we have $F_k^j(n) \le m$ if and only if $n \le L_k^j(m)$. Moreover, this Galois connection is said to be a* Galois insertion *since $F_k^j \circ L_k^j = \mathrm{id}$.*

Proof. The case $n = 0$ is obvious. Now, for $n > 0$, if $F_k^j(n) \leq m$, then $n \leq L_k^j(F_k^j(n)) \leq L_k^j(m)$ by Theorem 3.1 then monotonicity of L_k^j. Conversely, if $n \leq L_k^j(m)$, then $F_k^j(n) \leq F_k^j(L_k^j(m))$ by monotonicity of L_k^j and $F_k^j(L_k^j(m)) = m$ as seen above. □

In the case $j = 1$, note that Proposition 4.2 will give a nice expression of $L_k(n)$ as $n{+}F_k^{k-1}(n)$ and hence

$$L_k(F_k(n)) = F_k(n) + F_k^k(n) = n + 1 - \partial F_k(n) \in \{n, n{+}1\}.$$

4. Counting letters

We express here the number of occurrences for letters $1, \ldots, k$ in prefixes of x_k. Thanks to Theorem 3.1, this will relate them to functions F_k in various ways.

Let us denote by $C_k^{(P)}(n)$ the count of letters satisfying the predicate P in the prefix $x_k[0{:}n)$. More formally

$$C_k^{(P)}(n) = \#\{0 \leq j < n \; : \; P(x_k[j])\}$$

In particular we will use:

- $C_k^{(=i)}(n)$ for counting the occurrences of a specific letter i
- $C_k^{(>i)}(n) = C_k^{(=i+1)}(n) + \cdots + C_k^{(=k)}(n)$ for counting all letters strictly above i.

Proposition 4.1. *For all $k \geq 1$ and $n \geq 0$, we have*

$$F_k^{k-1}(n) = C_k^{(=k)}(n), \tag{4.1}$$

$$F_k^j(n) = C_k^{(>j)}(n) \qquad \text{for all } 0 \leq j < k, \tag{4.2}$$

$$F_k^{k+i-1}(n) = C_k^{(=i)}(n{+}i) \qquad \text{for all } 1 \leq i < k. \tag{4.3}$$

Proof. These three equations are provable via a similar counting technique.

For Equation (4.1), we already mentioned that x_k can be seen as a succession of "blocks" $\tau_k^{k-1}(i) = k1 \cdots (i{-}1)$, each one containing k only as first letter. For a given $m \geq 0$, $x_k[m]$ belongs to one of these blocks, say the p-th one (counting from $p = 1$ for the first block). We hence have p occurrences of the letter k in $x_k[0] \cdots x_k[m]$, so $C_k^{(=k)}(m{+}1) = p$. Note also that the first p blocks have a total length of $\left|\tau_k^{k-1}(x_k[0{:}p))\right| = L_k^{k-1}(p)$. So this quantity is also the first index in the next block, hence strictly more than m since $x_k[m]$ cannot be there. Similarly $L_k^{k-1}(p{-}1)$ is the first index of the p-th block, so all in all

$$L_k^{k-1}(p-1) \leq m < L_k^{k-1}(p).$$

After substituting p and posing $n = m{+}1$, we obtain that for all $n > 0$

$$L_k^{k-1}(C_k^{(=k)}(n){-}1) < n \leq L_k^{k-1}(C_k^{(=k)}(n)).$$

Thanks to Theorem 3.1 and Proposition 2.4, this implies $F_k^{k-1}(n) = C_k^{(=k)}(n)$ for all $k \geq 1$ and $n > 0$. Moreover, this identity trivially holds as well for $n = 0$.

For Equation (4.2), we generalize the previous counting technique. For $0 \leq j < k$ and a letter $1 \leq i \leq k$, $\tau_k^j(i)$ starts with exactly one letter strictly above j, the rest of this word is made of letters less or equal to j. Indeed, either $i + j \leq k$ and $\tau_k^j(i) = i + j > j$, or $i + j \geq k$ and $\tau_k^j(i) = \tau_k^{i+j-k}(k) = k1 \cdots (i{+}j{-}k)$ with $i{+}j{-}k \leq j$. Just as before, we deduce

$$L_k^j(C_k^{(>j)}(n){-}1) < n \leq L_k^j(C_k^{(>j)}(n))$$

for all $0 \leq j < k$ and $n > 0$. As earlier, this allows to establish $F_k^j(n) = C_k^{(>j)}(n)$ for all $0 \leq j < k$ and $n > 0$. Once again, this identity also holds for $n \geq 0$.

Finally, we use yet another instance of the same technique for proving Equation (4.3). Consider $1 \le i < k$. The words $\tau_k^{k+i-1}(j)$ for all letters $1 \le j \le k$ contain the letter i only at the $(i{+}1)$-st position. Indeed, these words can also be written $\tau_k^{j+i-1}(k)$ (since $\tau_k^{k-j}(j) = k$), so they all admit $\tau_k^i(k) = k1\cdots i$ as common prefix, possibly followed first by letters greater than i and then by new blocks no larger than $k1\cdots(i{-}1)$. For similar reasons as before, we hence have

$$L_k^{k+i-1}(C_k^{(=i)}(n){-}1) < n{-}i \le L_k^{k+i-1}(C_k^{(=i)}(n))$$

for all $n \ge i{+}1$. So for all $1 \le i < k$ and $n \ge 0$, we have $F_k^{k+i-1}(n) = C_k^{(=i)}(n{+}i)$. □

Actually, Equation (4.1) can also be deduced from Equation (4.2) in the particular case $j = k{-}1$, since $C_k^{(>k-1)} = C_k^{(=k)}$.

Also note that Equation (4.3) could be extended to the case $i = 0$ if we replace the letter k by 0 in the substitution τ_k and its fixed point x_k.

We can now give an interesting alternative expression for $L_k(n)$ (which we recall is the largest antecedent of n by F_k).

Proposition 4.2. *For all $k \ge 1$ and $n \ge 0$,*

$$L_k(n) = n + C_k^{(=k)}(n) = n + F_k^{k-1}(n).$$

Proof. We have $|\tau_k(k)| = 2$ while $|\tau_k(i)| = 1$ for the other letters $i \ne k$. Hence

$$L_k(n) = 2\,C_k^{(=k)}(n) + C_k^{(\ne k)}(n) = n + C_k^{(=k)}.$$

Finally, Equation (4.1) leads to the last equality. □

Thanks to Equation (4.2), we can also express the count of a specific letter $1 \le j < k$ via the difference between $C_k^{(>j-1)}$ and $C_k^{(>j)}$. Hence for $1 \le j < k$ and $n \ge 0$

$$F_k^{j-1}(n) - F_k^j(n) = C_k^{(>j-1)}(n) - C_k^{(>j)}(n) = C_k^{(=j)}(n). \tag{4.4}$$

In particular, we obtain the following proposition.

Proposition 4.3. *For $k > 1$ and $n \ge 0$,*

$$F_k(n) = n - C_k^{(=1)}(n) = C_k^{(\ne 1)}(n).$$

Hence $\partial F_k(n)$ is 0 if and only if $x_k[n] = 1$ and 1 otherwise.

Proof. Direct use of the previous equation in the particular case $j = 1$. Alternatively, one may use Equation (4.3) for $i = 1$, and then the recursive definition of F_k.

Afterwards, we get $F_k(n{+}1){-}F_k(n) = 1 - (C_k^{(=1)}(n{+}1){-}C_k^{(=1)}(n))$, which is 0 if and only if $x_k[n] = 1$ and 1 otherwise. □

This important link between ∂F_k and x_k helps transferring many properties of one to the other. In particular, ∂F_k cannot have two consecutive zeros, and it admits up to k consecutive ones but not $k{+}1$.

Now, let us describe the letters of x_k in terms of differences $\partial F_k^j(n)$. Recall from Proposition 2.1 that these differences are always either 0 or 1. Of course, $\partial F_k^0(n) = 1$ and $\partial F_k^j(0) = 1$. Now, for a non-zero n, $F_k^j(n) = 1$ when j is large enough, in particular for $j \ge n{-}1$. Indeed, on a non-zero argument, F_k either returns 1 or removes at least one from its argument, and this is iterated here j times. As a consequence, we always have $\partial F_k^n(n) = 0$ when $n > 0$. Moreover, if for some j, $\partial F_k^j(n) = 0$, then $\partial F_k^{j+1}(n) = 0$ as well. Reciprocally, if $\partial F_k^j(n) = 1$, then $\partial F_k^{j-1}(n) = 1$. Hence for any $k, n \ge 1$, the sequence $(\partial F_k^j(n))_{j\in\mathbb{N}}$ consists of a block of ones followed by an infinity of zeros. Actually, the letters of x_k indicate how deep to dive in these differences to find a first zero (or give up after $k-1$ attempts).

Proposition 4.4. *Consider $1 \leq j < k$ and $n \geq 0$. We have $x_k[n] = j$ if and only if both $\partial F_k^{j-1}(n) = 1$ and $\partial F_k^j(n) = 0$. Moreover for $k \geq 1$ we have $x_k[n] = k$ if and only if $\partial F_k^{k-1}(n) = 1$ (in this case $\partial F_k^k(n)$ could be either 0 or 1).*

Proof. First, this statement is obvious for $k = 1$. We now assume $k > 1$. By subtracting Equation (4.4) at $n+1$ and n, we obtain for any $1 \leq j < k$ that

$$\partial F_k^{j-1}(n) - \partial F_k^j(n) = C_k^{(=j)}(n+1) - C_k^{(=j)}(n).$$

This amounts to 1 if and only if $x_k[n] = j$, and 0 otherwise. So in particular for any $i < x_k[n]$ we have $\partial F_k^{i-1}(n) = \partial F_k^i(n)$. Meanwhile, we just noticed in Proposition 4.3 that $\partial F_k(n) = 0$ if and only if $x_k[n] = 1$. Write $\ell = x_k[n]$. Three situations may occur:

- Either $\ell = 1$, and we directly have $\partial F_k(n) = 0$ and $\partial F_k^0(n) = 1$.
- Or $1 < \ell < k$ and $\partial F_k(n) = 1$, and we can propagate $\partial F_k^i(n) = 1$ for all $i < \ell$ and finish with $\partial F_k^{\ell-1}(n) - \partial F_k^\ell(n) = 1$ hence $\partial F_k^\ell(n) = 0$.
- Lastly, if $\ell = k$ the propagation $\partial F_k^i(n) = 1$ goes up to $i \leq k-1$, hence the desired statement. □

5. Related polynomials and algebraic integers

We introduce here two families of polynomials whose positive roots will appear as average slopes for F_k and L_k and letter frequencies for x_k (Section 6).

Definition 5.1. *For $k \geq 1$, we name $P_k(X) = X^k + X - 1$ and $Q_k(X) = X^k - X^{k-1} - 1$. We name α_k (resp. β_k) the unique positive root of P_k (resp. Q_k). Note that $\beta_k = 1/\alpha_k$.*

First, a basic study[4] of the polynomials P_k and Q_k for $k \geq 1$ ensures indeed that they both admit exactly one root each in $\mathbb{R}_+$, named respectively α_k and β_k here, and moreover that $\frac{1}{2} \leq \alpha_k < 1 < \beta_k \leq 2$.

Also note that the polynomials P_k and Q_k are strongly related: each one is the opposite of the *reciprocal polynomial* of the other. Said otherwise, $P_k(X) = -X^k\,Q_k(1/X)$ and vice versa. As a consequence, the roots of P_k are the inverse of the roots of Q_k and vice versa. In particular $\beta_k = 1/\alpha_k$.

Figure 5.1 gives approximate values for the first α_k and β_k. Note in particular that $\beta_1 = 2$ and β_2 is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. Thanks to the rational root theorem, one can easily show that α_k and β_k are irrational for $k \geq 2$.

$\alpha_1 = 0.5$	$\beta_1 = 2$
$\alpha_2 = 0.6180339887498948\dots$	$\beta_2 = 1.618033988749895\dots$
$\alpha_3 = 0.6823278038280193\dots$	$\beta_3 = 1.465571231876768\dots$
$\alpha_4 = 0.7244919590005157\dots$	$\beta_4 = 1.380277569097614\dots$
$\alpha_5 = 0.7548776662466925\dots$	$\beta_5 = 1.324717957244746\dots$
$\alpha_6 = 0.7780895986786012\dots$	$\beta_6 = 1.285199033245349\dots$

Figure 5.1. The first α_k and β_k, positive roots of X^k+X-1 and $X^k-X^{k-1}-1$.

[4]See for instance Descartes' rule of signs.

Proposition 5.2. *$(\alpha_k)_{k\in\mathbb{N}_+}$ is a strictly increasing sequence in $[\frac{1}{2},1)$ while $(\beta_k)_{k\in\mathbb{N}_+}$ is a strictly decreasing sequence in $(1,2]$. Moreover for $k\geq 1$, $1+\frac{1}{k}\leq\beta_k\leq 1+\frac{1}{\sqrt{k}}$ or equivalently $\sqrt{k}\leq\beta_k^{k-1}\leq k$, hence β_k and α_k converge to 1 when $k\to\infty$.*

Proof. Let $k\geq 1$. Suppose $\alpha_{k+1}\leq\alpha_k$. Since α_k and α_{k+1} are in $(0,1)$, we would have $\alpha_{k+1}^{k+1}=\alpha_{k+1}^{k}\alpha_{k+1}<\alpha_{k+1}^{k}\leq\alpha_k^k$, leading to $0=P_{k+1}(\alpha_{k+1})<P_k(\alpha_k)=0$, a contradiction. Hence (α_k) is strictly increasing and $(\beta_k)=(\alpha_k^{-1})$ is strictly decreasing.

Now, $Q_k(\beta_k)=0$ can be reformulated as $\beta_k^{k-1}(\beta_k-1)=1$ and hence

$$\alpha_k^{k-1}=\beta_k^{-(k-1)}=\beta_k-1. \tag{5.1}$$

This provides the equivalence between $\sqrt{k}\leq\beta_k^{k-1}\leq k$ and $1+\frac{1}{k}\leq\beta_k\leq 1+\frac{1}{\sqrt{k}}$.

For proving the lower bound $\sqrt{k}\leq\beta_k^{k-1}$, it is sufficient to note that

$$1+\beta_k+\cdots+\beta_k^{k-1}=\frac{\beta_k^k-1}{\beta_k-1}=\frac{\beta_k^{k-1}}{\beta_k^{-(k-1)}}=\beta_k^{2(k-1)}. \tag{5.2}$$

Each term on the left of Equation (5.2) is 1 or more, so $k\leq\beta_k^{2(k-1)}$, hence the desired lower bound. For the upper bound $\beta_k^{k-1}\leq k$, we divide by β_k^{k-1} in Equation (5.2):

$$\beta_k^{k-1}=\frac{\beta_k^{2(k-1)}}{\beta_k^{k-1}}=\frac{1}{\beta_k^{k-1}}+\cdots+\frac{\beta_k^{k-1}}{\beta_k^{k-1}}=\alpha_k^{k-1}+\cdots+\alpha_k+1\leq k.$$ □

6. Infinitary behavior

For all $k\geq 1$, the substitution τ_k is a *primitive* morphism [2], i.e., there exists an exponent $p\geq 1$ such that for all letters $1\leq i,j\leq k$, the letter i occurs in $\tau_k^p(j)$. Here, the first adequate exponent is $p=2k-2$. Indeed, $\tau_k^{2k-2}(j)$ admits $\tau_k^{k-1}(k)=k1\cdots(k-1)$ as prefix since $\tau_k^{k-1}(j)$ has k as first letter. The word x_k is called *primitive morphic* since it is the fixed point of a primitive morphism. A well-known property of primitive morphic words is that all their letters have a *frequency*, i.e., for all $1\leq i\leq k$, the limit $\lim_{n\to\infty}\frac{1}{n}C_k^{(=i)}(n)$ exists; see e.g. [3, Theorem 8.4.7]. Let us call $\mathrm{freq}_k(i)$ this limit. The same theorem of existence also states $0<\mathrm{freq}_k(i)\leq 1$, and even gives a formula for computing it. But here, an easier approach for computing $\mathrm{freq}_k(i)$ is to consider F_k.

Theorem 6.1. *For $j\geq 0$ and $k\geq 1$, the following limits exist and have the given values, where α_k and β_k come from Definition 5.1:*

$$\begin{aligned}
\lim_{n\to\infty}\tfrac{1}{n}F_k^j(n)&=\alpha_k^j,\\
\lim_{n\to\infty}\tfrac{1}{n}L_k^j(n)&=\beta_k^j,\\
\mathrm{freq}_k(i)&=\alpha_k^{k+i-1} && \text{for } 1\leq i<k,\\
\mathrm{freq}_k(k)&=\alpha_k^{k-1}=\beta_k-1.
\end{aligned}$$

Proof. Recall from Proposition 4.3 that for $k>1$ and $n\geq 0$, $F_k(n)=n-C_k^{(=1)}(n)$. Hence the limit $\lim_{n\to\infty}\frac{1}{n}F_k(n)$ does exist as well[5] and is $1-\mathrm{freq}_k(1)$. In the case $k=1$, $F_1(n)=\lceil n/2\rceil$, hence $\lim_{n\to\infty}\frac{1}{n}F_1(n)=\frac{1}{2}$.

[5]Surprisingly, we have not found any obvious methods for proving this convergence of $\frac{1}{n}F_k(n)$ directly from the recursive definition of F_k, without using this "detour" via words.

Now that $\frac{1}{n}F_k(n)$ is known to converge to some finite limit ℓ, computing this limit ℓ is quite straightforward, since the recursive equation of F_k can be reformulated as

$$\frac{F_k(n)}{n} = 1 - \frac{F_k(F_k^{k-1}(n-1))}{F_k^{k-1}(n-1)} \cdots \frac{F_k(n-1)}{n-1} \cdot \frac{n-1}{n}$$

for $n > 1$, where each fraction can be shown to converge to ℓ, except $\frac{n-1}{n}$, which tends to 1. Hence $\ell = 1-\ell^k$ and obviously ℓ is a nonnegative real number, hence $\ell = \alpha_k$. As a consequence, for $k > 1$ the frequency $\text{freq}_k(1)$ is $1-\alpha_k = \alpha_k^k$ (or 1 when $k = 1$). And the same telescope technique gives $\lim_{n\to\infty} \frac{1}{n}F_k^j(n) = \alpha_k^j$.

Concerning L_k^j, a consequence of Theorem 3.1 is $F_k^j(L_k^j(n)) = n$ hence

$$\frac{L_k^j(n)}{n} = \left(\frac{F_k^j(L_k^j(n))}{L_k^j(n)}\right)^{-1}.$$

Since $L_k^j(n) \geq n$, this implies that $\frac{1}{n}L_k^j(n)$ converges and its limit is $\alpha_k^{-j} = \beta_k^j$.

For the frequency of the letter k, Equation (4.1) implies that $\text{freq}_k(k) = \alpha_k^{k-1}$, which is also $\beta_k - 1$ by Equation (5.1). And for the frequencies of the other letters $1 \leq i < k$, one may exploit either Equation (4.4) or Equation (4.3). For instance, the former leads to

$$\text{freq}_k(i) = \alpha_k^{i-1} - \alpha_k^i = \alpha_k^{i-1}(1-\alpha_k) = \alpha_k^{k+i-1}.$$

In particular, this subsumes the case $i = 1$ seen earlier. Finally, one may check that the sum of all these frequencies, from α_k^{k-1} (letter k) to α_k^{2k-2} (letter $k-1$), is of course 1. □

Definition 6.2. *For two functions $f, g : \mathbb{N} \to \mathbb{N}$, we will say that f is ultimately smaller than g and note $f <_\infty g$ when there exists N such that $f(n) < g(n)$ whenever $n \geq N$.*

Corollary 6.3. *When $\alpha_k^j < \alpha_{k'}^{j'}$ for some $k, k' \geq 1$ and $j, j' \geq 0$, then $F_k^j <_\infty F_{k'}^{j'}$.*

Proof. Theorem 6.1 gives $\lim_{n\to\infty} \frac{1}{n}F_k^j(n) = \alpha_k^j < \alpha_{k'}^{j'} = \lim_{n\to\infty} \frac{1}{n}F_{k'}^{j'}(n)$. For n large enough, both sides will become close enough to their limits, so there must exists $N \geq 1$ such that $n \geq N$ implies $\frac{1}{n}F_k^j(n) < \frac{1}{n}F_{k'}^{j'}(n)$ and hence $F_k^j(n) < F_{k'}^{j'}(n)$. □

Corollary 6.4. *For $k \geq 1$, we have $F_k <_\infty F_{k+1}$, $F_k^k >_\infty F_{k+1}^{k+1}$ and $F_k^{k+1} >_\infty F_{k+1}^{k+2}$. More generally, for a given $j \geq 0$, we have $F_k^{k+j} >_\infty F_{k+1}^{k+j+1}$ for all $k \geq 1$ such that $\alpha_k \geq \frac{j}{j+1}$ (this happens in particular when $k \geq j^2$). On the opposite, $F_k^{k+j} <_\infty F_{k+1}^{k+j+1}$ for all $k \geq 1$ such that $\alpha_{k+1} \leq \frac{j}{j+1}$ (this happens in particular when $1 \leq k < j$).*

Proof. All these facts are obtained by the previous corollary, we just have to compare the corresponding average slopes. For $F_k <_\infty F_{k+1}$, Proposition 5.2 directly gives $\alpha_k < \alpha_{k+1}$. For $F_k^k >_\infty F_{k+1}^{k+1}$, $\alpha_k^k = 1-\alpha_k > 1-\alpha_{k+1} = \alpha_{k+1}^{k+1}$. Now, for $F_k^{k+1} >_\infty F_{k+1}^{k+2}$ we have

$$\alpha_k^{k+1} - \alpha_{k+1}^{k+2} = (1-\alpha_k)\alpha_k - (1-\alpha_{k+1})\alpha_{k+1} = (\alpha_{k+1}-\alpha_k)(\alpha_k+\alpha_{k+1}-1) > 0$$

since $\frac{1}{2} \leq \alpha_k < \alpha_{k+1}$. More generally, let $j \geq 0$. As before,

$$\alpha_k^{k+j} - \alpha_{k+1}^{k+j+1} = (1-\alpha_k)\alpha_k^j - (1-\alpha_{k+1})\alpha_{k+1}^j.$$

The function $(1-X)X^j$ is strictly increasing between 0 and $\frac{j}{j+1}$ and strictly decreasing afterwards. Since we always have $\alpha_k < \alpha_{k+1}$, then $\alpha_k^{k+j} - \alpha_{k+1}^{k+j+1} > 0$ at least when $\alpha_k \geq \frac{j}{j+1}$ or equivalently when $\beta_k \leq 1 + \frac{1}{j}$. Thanks to the bounds in Proposition 5.2, this happens at least when $\sqrt{k} \geq j$, i.e., $k \geq j^2$. Conversely, $\alpha_k^{k+j} - \alpha_{k+1}^{k+j+1} < 0$ at least when $\alpha_{k+1} \leq \frac{j}{j+1}$, for which a sufficient condition is $1 \leq k < j$, still thanks to the bounds in Proposition 5.2. □

This Corollary 6.4 is to be compared with the results of the next section, for instance Theorem 7.4: the latter will prove large inequalities only, but as early as $n \geq 0$, while here we proved strict inequalities, but only when n is above some bounds. Moreover we do not have explicit estimate for these bounds for the moment.

As a related matter, we can now estimate the ratio of numbers having a unique antecedent by F_k.

Proposition 6.5. *For $k \geq 1$, $n \geq 0$, let us call $U_k(n) = \#\{0 \leq j < n : |F_k^{-1}(\{j\})| = 1\}$. Then for $n > 0$*

$$U_k(n) = 2n - 1 - L_k(n-1) = n - F_k^{k-1}(n-1)$$

and hence

$$\lim_{n\to\infty} \tfrac{1}{n} U_k(n) = 1 - \alpha_k^{k-1} = 2 - \beta_k.$$

Proof. Let $n > 0$. We already know from Theorem 3.1 and Proposition 4.2 that the largest antecedent of $n-1$ by F_k is $L_k(n-1) = n-1 + F_k^{k-1}(n-1)$. There are $U_k(n)$ numbers between 0 and $n-1$ with a unique antecedent, and the other $n-U_k(n)$ numbers have exactly two antecedents (see the discussion after Proposition 4.3). By counting all the antecedents of $\{0, \dots, n-1\}$ by F_k, we obtain

$$U_k(n) + 2(n-U_k(n)) = 1 + L_k(n-1) = n + F_k^{k-1}(n-1).$$

Combined with Theorem 6.1, this leads to the desired equations and limits. □

In particular, for $k = 1$ and $n > 0$ we get $U_1(n) = 1$, indeed 0 is the only number with one antecedent by $F_1(n) = \lceil \frac{n}{2} \rceil$. Then $\frac{1}{n} U_k(n)$ tends respectively to $0.3819\dots$ for $k = 2$, $0.5344\dots$ for $k = 3$, $0.6197\dots$ for $k = 4$ and these limits tend to 1 when k grows.

7. Monotonicity over the parameter k

This section studies the monotonicity of L_k and F_k when the parameter k varies. In all this section, we compare functions via *pointwise order*: an inequality such as $F_k \leq F_{k+1}$ means that $F_k(n) \leq F_{k+1}(n)$ *for all* points $n \geq 0$. As such, the results here are quite stronger than the ones of the previous section about the infinitary behavior, i.e., when n is large enough.

First, we state a nice duality between functions F_k^j and L_k^j with respect to this pointwise order, extending the results of Section 3. Thanks to this, all technical lemmas that will follow about L_k will have immediate counterparts for F_k.

Proposition 7.1. *For $j, j' \geq 0$ and $k, k' \geq 1$, we have $L_k^j \leq L_{k'}^{j'}$ if and only if $F_{k'}^{j'} \leq F_k^j$. Furthermore, we can be more precise concerning the relative positions where these inequalities occur. For $n \geq 0$:*

- *(a) $L_k^j(n) \leq L_{k'}^{j'}(n)$ if and only if $F_{k'}^{j'}(m) \leq F_k^j(m)$ where $m = L_k^j(n)$.*
- *(b) $L_k^j(n) < L_{k'}^{j'}(n)$ if and only if $F_{k'}^{j'}(m) < F_k^j(m)$ where $m = L_{k'}^{j'}(n)$.*
- *(c) If $L_k^j(m) \leq L_{k'}^{j'}(m)$ where $m = F_k^j(n)$, then $F_{k'}^{j'}(n) \leq F_k^j(n)$.*
- *(d) If $F_k^j(n) < F_{k'}^{j'}(n)$, then $L_{k'}^{j'}(m) < L_k^j(m)$ where $m = F_{k'}^{j'}(n)$.*

Proof. First, the fact that $L_k^j \leq L_{k'}^{j'} \implies F_{k'}^{j'} \leq F_k^j$ is a consequence of point (c), while point (a) implies the other direction. Also note that (b) is a contrapositive version of (a), and the same for (d) and (c).

For proving point (a), let $n \geq 0$ and $m = L_k^j(n)$. Then $L_k^j(n) \leq L_{k'}^{j'}(n)$ is equivalent to $F_{k'}^{j'}(L_k^j(n)) \leq n$ by Corollary 3.2, i.e. $F_{k'}^{j'}(m) \leq n$. Moreover $n = F_k^j(L_k^j(n)) = F_k^j(m)$ by Theorem 3.1.

We now prove the point (c). Let $n \geq 0$ and $m = F_k^j(n)$ and assume $L_k^j(m) \leq L_{k'}^{j'}(m)$. By Corollary 3.2, proving $F_{k'}^{j'}(n) \leq F_k^j(n)$ is equivalent to proving $n \leq L_{k'}^{j'}(F_k^j(n))$. Here indeed, $n \leq L_k^j(F_k^j(n)) \leq L_{k'}^{j'}(F_k^j(n))$, respectively by Theorem 3.1 (or trivially when $n = 0$) and by the current assumption. □

Let us now study the ordering of functions L_k^j. For $k \geq 1$, note first that

$$|\tau_k^k(i)| = i{+}1 = |\tau_k^{k-1}(i)| + 1 \quad \text{for all } 1 \leq i \leq k,$$

thus $L_k^k(n) = L_k^{k-1}(n){+}n$ for all $n \geq 0$, hence

$$L_{k+1}^{k+1}(n) - L_k^k(n) = L_{k+1}^k(n) - L_k^{k-1}(n) \tag{7.1}$$

for all $n \geq 0$.

Let us now establish that the (L_k) sequence of functions is monotonic. This crucial property is proved here by mutual induction with another property comparing some iterations of L_k and L_{k+1}.

Theorem 7.2. *For all $k, n \geq 1$, $0 \leq j \leq k$, we have*

$$L_k(n) \geq L_{k+1}(n), \quad \textit{i.e.}, \; C_k^{(=k)}(n) \geq C_{k+1}^{(=k+1)}(n), \tag{7.2}$$

$$L_k^j(n) < L_{k+1}^{j+1}(n). \tag{7.3}$$

Proof. Let $k \geq 1$. Since $x_k[0] = k$, $x_{k+1}[0] = k{+}1$ and $|\tau_k^j(k)| = j{+}1 < j{+}2 = |\tau_{k+1}^{j+1}(k{+}1)|$ for $0 \leq j \leq k$, the inequalities (7.2) and (7.3) hold for $n = 1$, $0 \leq j \leq k$.

Let $m \geq 2$ and assume that (7.2) and (7.3) hold for all $1 \leq n < m$, $0 \leq j \leq k$.

We first prove (7.2) at m, i.e. $L_k(m) \geq L_{k+1}(m)$, or equivalently $F_k^{k-1}(m) \geq F_{k+1}^k(m)$ (see Proposition 4.2). Let us abbreviate $F_k^{k-1}(m)$ as c and $F_{k+1}^k(m)$ as c' and prove $c' \leq c$ i.e. $c'{-}1 < c$. By Proposition 2.3, L_{k+1}^k is strictly increasing, hence it is sufficient to prove $L_{k+1}^k(c'{-}1) < L_{k+1}^k(c)$. And indeed:

$$L_{k+1}^k(c'{-}1) < m \leq L_k^{k-1}(c) \leq L_{k+1}^k(c),$$

where the left and middle inequalities come from Theorem 3.1. For the right inequality above, we need to distinguish the case $k = 1$ for which $c = F_1^0(m) = m$ and hence $L_k^{k-1}(c) = L_1^0(m) = m = L_2^0(m) < L_2^1(m) = L_{k+1}^k(c)$ by Proposition 2.3, point (f). Otherwise, when $k \geq 2$, we can use (7.3) for $n = c$, $j = k{-}1$ since in this case $1 \leq c < m$ by Proposition 2.1.

Now, let $0 \leq h \leq k$ and let us prove (7.3) for $n = m$ and $j = h$. Thanks to Equation (7.1), the case $j = k$ is implied by the case $j = k{-}1$, so we can freely assume now $h < k$. If $x_{k+1}[m{-}1] = k{+}1$, then (7.3) holds for $n = m$ because it holds for $n = m{-}1$ and $|\tau_k^h(i)| \leq |\tau_k^h(k)| < |\tau_{k+1}^{h+1}(k{+}1)|$ for all $1 \leq i \leq k$. If $x_{k+1}[m{-}1] \neq k{+}1$, then $x_{k+1}[0{:}m) = \tau_{k+1}(x_{k+1}[0{:}\ell))$ for some $\ell \geq 1$. Proposition 2.3 indicates that $\ell < L_{k+1}(\ell) = m$, hence (7.2) holds for $n = \ell$, hence

$$m = L_{k+1}(\ell) \leq L_k(\ell).$$

We apply L_k^h (which is monotonic by Proposition 2.3) on this inequality, and then (7.3) for $n = \ell$ and $j = h{+}1$:

$$L_k^h(m) \leq L_k^h(L_k(\ell)) = L_k^{h+1}(\ell) < L_{k+1}^{h+2}(\ell) = L_{k+1}^{h+1}(L_{k+1}(\ell)) = L_{k+1}^{h+1}(m).$$

Therefore, (7.3) holds for $n = m$, $0 \leq j \leq k$. By induction, (7.2) and (7.3) hence hold for all $n \geq 1$, $0 \leq j \leq k$. □

Corollary 7.3. *For all $k \geq 1$ and $j \geq 0$, we have $L_k^j \geq L_{k+1}^j$.*

Proof. We proceed by induction on j. The case $j = 0$ is obvious. The case $j = 1$ is given by Equation (7.2) trivially extended to $n = 0$. Assume now $L_k^j \geq L_{k+1}^j$ for some $j \geq 0$. For $n \geq 0$ we hence have

$$L_k^{j+1}(n) = L_k^j(L_k(n)) \geq L_{k+1}^j(L_k(n)) \geq L_{k+1}^j(L_{k+1}(n)) = L_{k+1}^{j+2}(n)$$

thanks to the induction hypothesis for j and then the monotonicity of L_{k+1}^j combined with the statement for $j = 1$. We can hence conclude by induction. □

We obtain at last the monotonicity of the function sequence (F_k) over parameter k.

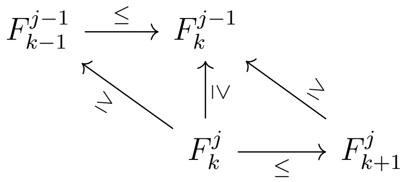
Theorem 7.4. *For all $j \geq 0$ and $k \geq 1$, we have $F_k^j \leq F_{k+1}^j$ and in particular $F_k \leq F_{k+1}$.*

Proof. Consequence of Corollary 7.3 and Proposition 7.1. □

Theorem 7.5. *For all $k \geq 1$ and $0 \leq j \leq k$, we have $F_k^j \geq F_{k+1}^{j+1}$.*

Proof. Let $k \geq 1$ and $0 \leq j \leq k$. Thanks to Proposition 7.1, the statement to prove is equivalent to $L_{k+1}^{j+1} \leq L_k^j$, which is a direct consequence of Equation (7.3) and of $L_{k+1}^{j+1}(0) = 0 = L_k^j(0)$. □

To sum up, the functions $(F_k^j, \leq)$ with their pointwise ordering form (at least) a nice lattice generated by the following basic cells for $1 \leq j \leq k$:



In such a cell, the vertical edge $F_k^j \leq F_k^{j-1}$ is obvious for sub-linear functions such as F_k. Moreover it is also a double consequence of the transitivities $F_k^j \leq F_{k-1}^{j-1} \leq F_k^{j-1}$ and $F_k^j \leq F_{k+1}^j \leq F_k^{j-1}$. Also note that the remaining unrelated functions F_{k-1}^{j-1} and F_{k+1}^j may actually be uncomparable. For instance $F_3^3(5) = 2 > 1 = F_5^4(5)$ while $F_3^3(9) = 3 < 4 = F_5^4(5)$. Even if we only retain the infinitary behavior (as in Section 6), the ordering of these functions may vary. The currently known situation is presented in Figure 7.1, where the edges such as $G \longrightarrow H$ mean $G \leq H$. The row $j = 0$ of identity functions has been omitted here, id being trivially above all other F_k^j. The numbers displayed in blue alongside the nodes F_k^j are approximations of their average slopes $\lim \frac{1}{n} F_k^j(n) = \alpha_k^j$. If one of the functions is pointwise below another, their average slopes will be ordered accordingly.[6] Also note that the diagonal $j = k$ and first row $j = 1$ (in red in the figure) form an interesting chain of inequalities, with slopes ranging symmetrically between 0 and 1 (since $1 - \alpha_k^k = \alpha_k$).

In Figure 7.1, some dotted edges with question mark indicate conjectured inequalities. Indeed, we conjecture that $F_k^{k+1} \geq F_{k+1}^{k+2}$ for all $k \geq 1$, i.e., that Theorem 7.5 may be extended to the case $j = k+1$. Actually, Corollary 6.4 already proved that $F_k^{k+1}(n) > F_{k+1}^{k+2}(n)$ for sufficiently large n. Thanks to Proposition 7.1, this conjecture can also be equivalently formulated as $L_{k+1}^{k+2} \geq L_k^{k+1}$. Said otherwise, the inequality (7.3) appears to still hold as a large inequality in the case $j = k+1$. Note in this case that we may indeed reach an equality: we can prove $L_k^{k+1}(k+1) = L_{k+1}^{k+2}(k+1)$.

After that, for $j \geq k+2$, we prove below that $F_k^j \not\geq F_{k+1}^{j+1}$. More precisely, when $j \geq k+2$ two behaviors seem possible: either $k+2 \leq j \leq 2k$ and we prove below that F_k^j and F_{k+1}^{j+1}

[6] Conversely, strictly ordered average slopes only give clues about infinitary behavior, the functions may well be pointwise uncomparable due to early values.

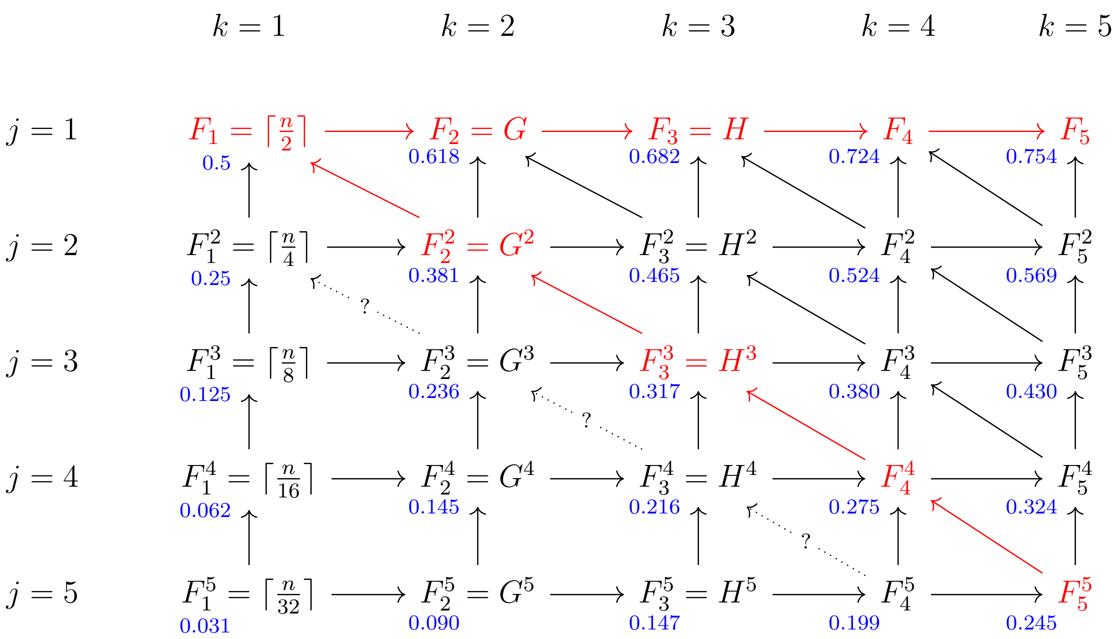


FIGURE 7.1. The known $(F_k^j, \leq)$ lattice, displayed here for $1 \leq k, j \leq 5$.

are uncomparable in this case; or $j > 2k$ in which case we conjecture that $F_k^j \leq F_{k+1}^{j+1}$. For studying these questions, we focus now on $L_{k+1}^{j+1}(1) - L_k^j(1)$.

Lemma 7.6. *For all $k \geq 1$, we have*

$$L_{k+1}^{j+1}(1) - L_k^j(1) = \begin{cases} 1 & \text{if } 0 \leq j \leq 2k, \\ -\frac{(j-2k-1)(j-2k+2)}{2} & \text{if } 2k \leq j \leq 3k. \end{cases}$$

In particular $L_{k+1}^{2k+2}(1) = L_k^{2k+1}(1)$. Moreover for all $j \geq 2k+2$, $L_{k+1}^{j+1}(1) < L_k^j(1)$.

Proof. From Proposition 2.3, we have

$$L_k^j(1) = \begin{cases} j+1 & \text{if } 0 \leq j \leq k, \\ L_k^{j-1}(1) + L_k^{j-k}(1) & \text{if } j \geq k. \end{cases}$$

For $0 \leq j \leq k$, we have $L_{k+1}^j(1) = j+1 = L_k^j(1)$ and $L_{k+1}^{j+1}(1) = j+2 = L_k^j(1)+1$. For $k < j \leq 2k$, we have

$$L_{k+1}^{j+1}(1) - L_k^j(1) = L_{k+1}^j(1) - L_k^{j-1}(1) = 1,$$
$$L_{k+1}^j(1) - L_k^j(1) = L_{k+1}^{j-1}(1) - L_k^{j-1}(1) - 1 = k - j,$$

by induction on j. Note that $-\frac{(j-2k-1)(j-2k+2)}{2} = 1$ for $j = 2k$. For $2k < j \leq 3k$, we have thus

$$L_{k+1}^{j+1}(1) - L_k^j(1) = -\frac{(j-2k-2)(j-2k+1)}{2} + 2k - j = -\frac{(j-2k-1)(j-2k+2)}{2},$$

by induction on j.

Still for $k \geq 1$, we now prove by induction on j that $L_{k+1}^{j+1}(1) < L_k^j(1)$ for all $j \geq 2k+2$. This is true for $j = 2k+2$: indeed, either $k = 1$ and we directly compute $L_2^5(1) = 13 < 16 = L_1^4(1)$, or $k \geq 2$ and hence $2k+2 = j \leq 3k$ so $L_{k+1}^{j+1}(1) - L_k^j(1)$ is given by the previous formula, which is strictly negative here. Now, for the step case of the induction, let $j > 2k + 2$. Then

$$L_{k+1}^{j+1}(1) - L_k^j(1) = (L_{k+1}^j(1) - L_k^{j-1}(1)) + (L_{k+1}^{j-k}(1) - L_k^{j-k}(1))$$

The induction hypothesis on $j-1$ indicates that the central difference above is strictly negative, while the rightmost difference is nonpositive thanks to Corollary 7.3, allowing us to conclude this induction. □

Lemma 7.7. *Let $k \geq 1$. If $j \geq k+2$, the following value n_j satisfies $L_k^j(n_j) > L_{k+1}^{j+1}(n_j)$:*

$$n_j = \begin{cases} 2k+3-j & \text{if } k+2 \leq j \leq 2k+2, \\ 1 & \text{if } 2k+2 \leq j. \end{cases}$$

Proof. The case $j \geq 2k+2$ where $n_j = 1$ is a direct use of Lemma 7.6. Suppose now $k+2 \leq j \leq 2k+2$. By Proposition 2.3, we know that $L_k^p(1) = p+1$ for all $0 \leq p \leq k$. By considering $p = n_j - 1$, we obtain $n_j = L_k^{n_j-1}(1)$ and hence

$$L_k^j(n_j) = L_k^j(L_k^{n_j-1}(1)) = L_k^{j+n_j-1}(1) = L_k^{2k+2}(1).$$

Similarly, $n_j = L_{k+1}^{n_j-1}(1)$ and $L_{k+1}^{j+1}(n_j) = L_{k+1}^{2k+3}(1)$ which is strictly less that $L_k^{2k+2}(1)$ by Lemma 7.6. □

This last lemma implies in particular that $L_k^j \not\leq L_{k+1}^{j+1}$ when $j \geq k+2$, with n_j as counterexample. Thanks to Proposition 7.1, this means equivalently that $F_k^j \not\geq F_{k+1}^{j+1}$ when $j \geq k+2$, with $n = L_k^j(n_j)$ as counterexample. Moreover, when $k+2 \leq j \leq 2k$, Lemma 7.6 implies that $L_k^j \not\geq L_{k+1}^{j+1}$, with $n = 1$ as counterexample, and hence that L_k^j and L_{k+1}^{j+1} are uncomparable, and equivalently that F_k^j and F_{k+1}^{j+1} are also uncomparable.

We conclude this section with a last conjecture about F_k, as always for $k \geq 1$. From Corollary 6.4, we know that the inequality $F_k(n) \leq F_{k+1}(n)$ becomes strict when n is large enough. Actually, we conjecture an explicit bound $N_k = \frac{1}{2}(k+1)(k+6)$, for which $F_k(n) < F_{k+1}(n)$ as soon as $n > N_k$. At least, it can be proved that $F_k(N_k) = F_{k+1}(N_k)$, so the bound cannot be less than N_k, but it remains to be confirmed that no equality occurs after N_k. These constants N_k can also be expressed as $\frac{1}{2}(k+3)(k+4)-3$ and satisfy $N_{k+1} = N_k + (k+4)$. In particular $N_1 = 7$, $N_2 = 12$, $N_3 = 18$. Interestingly, we also have $L_{k+1}(N_k) = L_{k+2}(N_k)$. Finally, this conjecture implies two other interesting statements:

- For all $n \geq 2$, $F_k(n) < F_{k+1}(n+1)$.
- For all $n > N_k$, $L_{k+1}(n) > L_{k+2}(n)$.

8. More on letter counts

Several results and conjectures of the last section can be rephrased into statements about letter counts for words x_k. In particular, let us consider again the letter 1 and study $C_k^{(=1)}$.

Proposition 8.1. *For all $k \geq 1$ and $n \geq 0$, we have $C_k^{(=1)}(n) \geq C_{k+1}^{(=1)}(n)$.*

Proof. When $k > 1$, this is a consequence of Proposition 4.3 and Theorem 7.4. And for $k = 1$, we have $C_1^{(=1)}(n) = n \geq C_2^{(=1)}(n)$. □

Considering now the letter 2, we conjecture that $C_k^{(=2)}(n) \geq C_{k+1}^{(=2)}(n)$ for all $k > 2$ and $n \geq 0$. In particular, this is a consequence of the conjecture $F_k^{k+1} \geq F_{k+1}^{k+2}$ mentioned in the previous section: when combining it with Equation (4.3), we get

$$C_k^{(=2)}(n) \geq C_{k+1}^{(=2)}(n) \quad \text{for all } k > 2 \text{ and } n \geq 2$$

and we can then relax the condition on n since all these counts are null when n is 0 or 1 (in the latter case, $x_k[0] = k \neq 2$ and similarly for x_{k+1}).

Actually, this property $C_k^{(=2)} \geq C_{k+1}^{(=2)}$ also extends to $k = 2$, and is easy to prove in this case, since for all $n \geq 0$, we have $C_2^{(=2)}(n) \geq C_3^{(=3)}(n)$ (by Equation 7.2 trivially extended to $n = 0$) as well as $C_3^{(=3)}(n) \geq C_3^{(=2)}(n)$ (indeed, in x_3, any occurrence of the letter 2 is in a subword 312). We cannot extend further: for $k = 1$, there is no letter 2 in the word x_1, hence $C_1^{(=2)}(n) = 0$ while $C_2^{(=2)}(n) = F_2(n)$ (by Equation 4.1) and this differs from 0 as soon as $n > 0$.

Now, for the other letters $3 \leq i < k$, there is no pointwise monotonicity anymore between $C_k^{(=i)}$ and $C_{k+1}^{(=i)}$:

Proposition 8.2. *For $3 \leq i < k$, we have*

- $C_k^{(=i)}(n) < C_{k+1}^{(=i)}(n)$ *when* $n = i + L_k^{2k+2}(1)$,
- $C_k^{(=i)}(n) > C_{k+1}^{(=i)}(n)$ *when* $n = i + L_{k+1}^{k+i}(1)$.

Proof. After Lemma 7.7, we noticed that F_k^j and F_{k+1}^{j+1} are uncomparable for all $j \geq k+2$. Some counterexamples are $L_k^j(n_j)$ in one direction and $L_{k+1}^{j+1}(1)$ in the other (thanks to Proposition 7.1). Now, we use Equation (4.3) to express this in terms of letter count, by choosing $j = k+i-1$. When $3 \leq i < k$, we have indeed $j \geq k+2$ (and also $j \leq 2k+2$). Due to the shape of Equation (4.3), the previous counterexamples are now shifted by i. Moreover, here

$$n_j = 2k + 3 - j = k + 4 - i = L_k^{k+3-i}(1)$$

and hence

$$L_k^j(n_j) = L_k^{k+i-1}(L_k^{k+3-i}(1)) = L_k^{2k+2}(1). \qquad \square$$

For instance, for $k = 5$ and $i = 4$, we have $C_5^{(=4)}(49) = 5 < 6 = C_6^{(=4)}(49)$ while $C_5^{(=4)}(20) = 2 > 1 = C_6^{(=4)}(20)$.

Finally, we compare these letter counts when n is large enough. Let $k \geq 1$ and $1 \leq i < k$. Thanks to Corollary 6.4 and Equation (4.3), we obtain that $C_k^{(=i)} >_\infty C_{k+1}^{(=i)}$ at least when $\alpha_k \geq 1 - \frac{1}{i}$, which happens in particular when $(i-1)^2 \leq k$. Note that this condition is always satisfied when $i = 1$ or $i = 2$. Otherwise, for $i \geq 3$, the early values of k may exhibit the opposite infinitary behavior. For example $C_6^{(=5)} <_\infty C_7^{(=5)}$ since

$$\lim_{n\to\infty} \tfrac{1}{n} C_6^{(=5)}(n) = \alpha_6^{10} \approx 0.0813, \quad \lim_{n\to\infty} \tfrac{1}{n} C_7^{(=5)}(n) = \alpha_7^{11} \approx 0.0819.$$

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