Ramanujan and asymptotic formulas

Kathrin Bringmann

University of Cologne

This research was supported by the Alfried Krupp Prize

April 19, 2013

1. Modular forms

Outline

1. Modular forms

2. Mock modular forms

Outline

1. Modular forms
2. Mock modular forms
3. Mixed mock modular forms

Outline

1. Modular forms

2. Mock modular forms

3. Mixed mock modular forms

4. Non-modular objects

Outline

1. Modular forms
2. Mock modular forms
3. Mixed mock modular forms
4. Non-modular objects

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \#\text{ of partitions of } n$$

Examples:

$n = 1$
1

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad$ $p(1) = 1$

1

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\quad p(1) = 1$

1

$n = 2$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad$ $p(1) = 1$

1

$n = 2$

2,

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad\qquad$ $p(1) = 1$

1

$n = 2$

$2, \quad 1 + 1$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad$ $p(1) = 1$

1

$n = 2$ $\qquad$ $p(2) = 2$

$2, \quad 1 + 1$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad$ $p(1) = 1$

1

$n = 2$ $\qquad$ $p(2) = 2$

$2, \quad 1 + 1$

$n = 3$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad p(1) = 1$

1

$n = 2$ $\qquad p(2) = 2$

2, $\quad 1 + 1$

$n = 3$

3,

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad$ $p(1) = 1$

1

$n = 2$ $\qquad$ $p(2) = 2$

$2, \quad 1 + 1$

$n = 3$

$3, \quad 2 + 1,$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad\qquad$ $p(1) = 1$

1

$n = 2$ $\qquad\qquad$ $p(2) = 2$

$2, \quad 1 + 1$

$n = 3$

$3, \quad 2 + 1, \quad 1 + 1 + 1$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad$ $p(1) = 1$

1

$n = 2$ $\qquad$ $p(2) = 2$

$2, \quad 1 + 1$

$n = 3$ $\qquad$ $p(3) = 3$

$3, \quad 2 + 1, \quad 1 + 1 + 1$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\quad p(1) = 1$

1

$n = 2$ $\quad p(2) = 2$

$2, \quad 1+1$

$n = 3$ $\quad p(3) = 3$

$3, \quad 2+1, \quad 1+1+1$

$n = 4$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad$ $p(1) = 1$

1

$n = 2$ $\qquad$ $p(2) = 2$

$2, \quad 1 + 1$

$n = 3$ $\qquad$ $p(3) = 3$

$3, \quad 2 + 1, \quad 1 + 1 + 1$

$n = 4$

$4,$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad$ $p(1) = 1$

1

$n = 2$ $\qquad$ $p(2) = 2$

$2, \quad 1 + 1$

$n = 3$ $\qquad$ $p(3) = 3$

$3, \quad 2 + 1, \quad 1 + 1 + 1$

$n = 4$

$4, \quad 3 + 1,$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad p(1) = 1$

1

$n = 2$ $\qquad p(2) = 2$

$2, \quad 1 + 1$

$n = 3$ $\qquad p(3) = 3$

$3, \quad 2 + 1, \quad 1 + 1 + 1$

$n = 4$

$4, \quad 3 + 1, \quad 2 + 2,$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad$ $p(1) = 1$

1

$n = 2$ $\qquad$ $p(2) = 2$

$2, \quad 1 + 1$

$n = 3$ $\qquad$ $p(3) = 3$

$3, \quad 2 + 1, \quad 1 + 1 + 1$

$n = 4$

$4, \quad 3 + 1, \quad 2 + 2, \quad 2 + 1 + 1,$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\qquad p(1) = 1$

1

$n = 2$ $\qquad p(2) = 2$

$2, \quad 1 + 1$

$n = 3$ $\qquad p(3) = 3$

$3, \quad 2 + 1, \quad 1 + 1 + 1$

$n = 4$

$4, \quad 3 + 1, \quad 2 + 2, \quad 2 + 1 + 1, \quad 1 + 1 + 1 + 1$

Partitions

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n.

$$p(n) := \# \text{ of partitions of } n$$

Examples:

$n = 1$ $\quad p(1) = 1$

1

$n = 2$ $\quad p(2) = 2$

$2, \quad 1+1$

$n = 3$ $\quad p(3) = 3$

$3, \quad 2+1, \quad 1+1+1$

$n = 4$ $\quad p(4) = 5$

$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1$

Fibonacci numbers

Fibonacci numbers:

$$F_n = F_{n-1} + F_{n-2}$$
$$F_0 = 0 \qquad F_1 = 1$$

Fibonacci numbers

Fibonacci numbers:

$$F_n = F_{n-1} + F_{n-2}$$
$$F_0 = 0 \qquad F_1 = 1$$

First elements:

$$1 \quad 1 \quad 2 \quad 3 \quad 5$$

Fibonacci numbers

Fibonacci numbers:

$$F_n = F_{n-1} + F_{n-2}$$
$$F_0 = 0 \qquad F_1 = 1$$

First elements:

$$1 \quad 1 \quad 2 \quad 3 \quad 5$$

but

Fibonacci numbers

Fibonacci numbers:

$$F_n = F_{n-1} + F_{n-2}$$
$$F_0 = 0 \qquad F_1 = 1$$

First elements:

$$1 \quad 1 \quad 2 \quad 3 \quad 5$$

but

$$p(5) = 7$$
$$F_6 = 8$$

Generating functions

Euler:

$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

Generating functions

Euler:

$$P(q) := \sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

Pentagonal Number Theorem:

$$\prod_{n=1}^{\infty} (1-q^n) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n(3n-1)}{2}}$$

Generating functions

Euler:

$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

Pentagonal Number Theorem:

$$\prod_{n=1}^{\infty} (1-q^n) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n(3n-1)}{2}}$$

Recursion:

$$p(k) = p(k-1) + p(k-2) - p(k-5) - p(k-7) \\ + p(k-12) + p(k-15) - p(k-22) - \dots$$

Modularity

$f : \mathbb{H} \to \mathbb{C}$ holomorphic is modular of weight k if for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

Modularity

$f : \mathbb{H} \to \mathbb{C}$ holomorphic is modular of weight k if for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{\overset{\text{weight}}{\downarrow}k} f(\tau)$$

Modularity

$f : \mathbb{H} \to \mathbb{C}$ holomorphic is modular of weight k if for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{\overset{\substack{\textit{weight}\\ \downarrow}}{k}} f(\tau)$$

plus growth condition

Modularity

$f : \mathbb{H} \to \mathbb{C}$ holomorphic is modular of weight k if for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{\overset{\overset{\textit{weight}}{\downarrow}}{k}} f(\tau)$$

plus growth condition

Fourier expansion $\left(q = e^{2\pi i\tau}\right)$

$$f(\tau) = \sum_{n\in\mathbb{Z}} a(n)q^n$$

Examples

1) Dedekind η-function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

Examples

1) Dedekind η-function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

Modularity:

$$\eta\left(\tau + 1\right) = e^{\frac{\pi i}{12}} \eta(\tau),$$

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau).$$

Examples

1) Dedekind η-function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

Modularity:

$$\eta\left(\tau + 1\right) = e^{\frac{\pi i}{12}} \eta(\tau),$$

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau).$$

2) Theta function

$$\Theta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2}.$$

Growth of $p(n)$

$$p(10) = 42$$

Growth of $p(n)$

$$p(10) = 42$$

$$p(50) = 204226$$

Growth of $p(n)$

$$p(10) = 42$$

$$p(50) = 204226$$

$$p(100) = 190569292$$

Growth of $p(n)$

$$p(10) = 42$$

$$p(50) = 204226$$

$$p(100) = 190569292$$

Asymptotic behavior (Hardy–Ramanujan)

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}} \qquad (n \to \infty)$$

Exact formula

Kloosterman sum:

$$A_k(n) = \sum_{h \pmod{k}^*} \omega_{h,k} e^{-\frac{2\pi i h n}{k}}$$

Exact formula

Kloosterman sum:

$$A_k(n) = \sum_{h \pmod{k}^*} \underset{\substack{\uparrow \\ \text{some multiplier}}}{}\omega_{h,k} e^{-\frac{2\pi i h n}{k}}$$

Exact formula

Kloosterman sum:

$$A_k(n) = \sum_{h \pmod{k}^*} \overset{\substack{\text{some multiplier} \\ \downarrow}}{\omega_{h,k}} e^{-\frac{2\pi i h n}{k}}$$

Bessel function of order α:

$$I_\alpha(x) := \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

Exact formula

Kloosterman sum:

$$A_k(n) = \sum_{h \pmod{k}^*} \overset{\substack{\text{some multiplier} \\ \downarrow}}{\omega_{h,k}} e^{-\frac{2\pi i h n}{k}}$$

Bessel function of order α:

$$I_\alpha(x) := \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

Rademacher formula:

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right)$$

1. Modular forms

2. Mock modular forms

3. Mixed mock modular forms

4. Non-modular objects

Ramanujan's last letter

"I am extremely sorry for not writing you a single letter up to now. I recently discovered very interesting functions which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions they enter into mathematics as beautifully as the theta functions. I am sending you with this letter some examples."

Ramanujan's $f(q)$

Partition function:

$$P(q) = \sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n^2}$$

Ramanujan's $f(q)$

Partition function:

$$P(q) = \sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n^2}$$

with

$$(a)_n = (a;q)_n := \prod_{j=0}^{n-1} \left(1 - aq^j\right)$$

Ramanujan's $f(q)$

Partition function:

$$P(q) = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2}$$

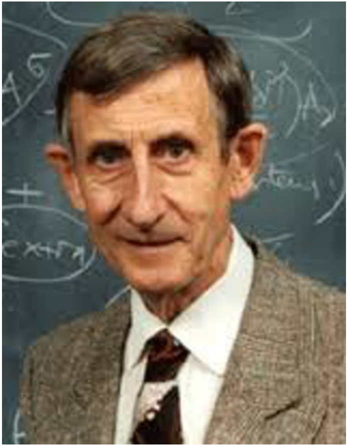
with

$$(a)_n = (a;q)_n := \prod_{j=0}^{n-1} \left(1 - aq^j\right)$$

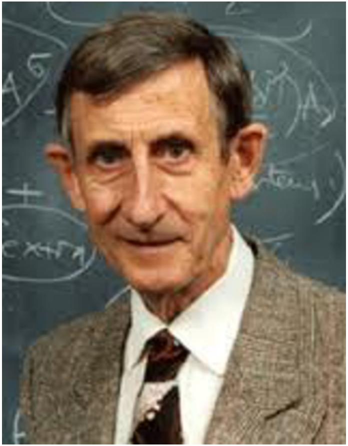
Ramanujan's mock theta function:

$$f(q) = \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2} = \sum_{n\geq 0} \alpha(n) q^n.$$

Dyson's challenge for the future



Dyson's challenge for the future



"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta functions of Jacobi. This remains a challenge for the future..."

Harmonic Maass forms

Definition:

$\mathcal{F} : \mathbb{H} \to \mathbb{C}$ is a harmonic weak Maass form if it is modular of weight k and

Harmonic Maass forms

Definition:

$\mathcal{F} : \mathbb{H} \to \mathbb{C}$ is a harmonic weak Maass form if it is modular of weight k and

$$\Delta_k(\mathcal{F}) = 0$$

with $(\tau = x + iy)$

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Harmonic Maass forms

Definition:

$\mathcal{F} : \mathbb{H} \to \mathbb{C}$ is a harmonic weak Maass form if it is modular of weight k and

$$\Delta_k(\mathcal{F}) = 0$$

with $(\tau = x + iy)$

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Moreover growth condition.

Examples

- weight 2 Eisenstein series

$$\widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi y}$$

Examples

- weight 2 Eisenstein series

$$\widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi y}$$

where

$$E_2(\tau) := 1 - 24 \sum_{n \geq 1} \sigma(n) q^n$$

with $\sigma(n) := \sum_{d|n} d$.

Examples

- weight 2 Eisenstein series

$$\widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi y}$$

where

$$E_2(\tau) := 1 - 24 \sum_{n \geq 1} \sigma(n) q^n$$

with $\sigma(n) := \sum_{d|n} d$.

- Class number generating function

$$\widehat{h}(\tau) := \sum_{\substack{n \geq 0 \\ n \equiv 0,3 \pmod 4}} H(n) q^n + \frac{i}{8\sqrt{2}\pi} \int_{-\overline{\tau}}^{i\infty} \frac{\Theta(w)}{(-i(\tau + w))^{\frac{3}{2}}} dw$$

where $H(n)$ is the Hurwitz class number.

Natural splitting

$\mathcal{F}$ harmonic Maass form

$$\mathcal{F} \quad = \quad \mathcal{F}^{+} \quad + \quad \mathcal{F}^{-}$$

.

Natural splitting

$\mathcal{F}$ harmonic Maass form

$$\mathcal{F} \quad = \quad \underset{\substack{\uparrow \\ \text{holomorphic} \\ \text{part}}}{\mathcal{F}^+} \quad + \quad \mathcal{F}^-$$

.

Natural splitting

$\mathcal{F}$ harmonic Maass form

$$\mathcal{F} = \underset{\substack{\uparrow \\ \text{holomorphic} \\ \text{part}}}{\mathcal{F}^+} + \underset{\substack{\uparrow \\ \text{non-holomorphic} \\ \text{part}}}{\mathcal{F}^-}$$

.

Natural splitting

$\mathcal{F}$ harmonic Maass form

$$\mathcal{F} \quad = \quad \underset{\substack{\uparrow \\ \text{holomorphic} \\ \text{part}}}{\mathcal{F}^{+}} \quad + \quad \underset{\substack{\uparrow \\ \text{non-holomorphic} \\ \text{part}}}{\mathcal{F}^{-}}$$

with

$$\mathcal{F}^{+}(\tau) := \sum_{n \gg -\infty} a^{+}(n) q^{n},$$

$$\mathcal{F}^{-}(\tau) := \sum_{n>0} a^{-}(n) \Gamma(k-1; 4\pi|n|y) q^{n}.$$

Natural splitting

$\mathcal{F}$ harmonic Maass form

$$\mathcal{F} \quad = \quad \underset{\substack{\uparrow \\ \text{holomorphic} \\ \text{part}}}{\mathcal{F}^+} \quad + \quad \underset{\substack{\uparrow \\ \text{non-holomorphic} \\ \text{part}}}{\mathcal{F}^-}$$

with

$$\mathcal{F}^+(\tau) := \sum_{n \gg -\infty} a^+(n) q^n,$$

$$\mathcal{F}^-(\tau) := \sum_{n>0} a^-(n) \underset{\substack{\uparrow \\ \text{incomplete gamma} \\ \text{function}}}{\Gamma(k-1; 4\pi|n|y)} q^n.$$

Returning to $f(q)$

Ramanujan's claim:

$$\alpha(n) \sim \frac{(-1)^{n+1}}{2\sqrt{n}} e^{\pi\sqrt{\frac{n}{6}}} \qquad (n \to \infty)$$

Returning to $f(q)$

Ramanujan's claim:

$$\alpha(n) \sim \frac{(-1)^{n+1}}{2\sqrt{n}} e^{\pi\sqrt{\frac{n}{6}}} \qquad (n \to \infty)$$

Andrews-Dragonette Conjecture:

$$\alpha(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{k \geq 1} \frac{(-1)^{\left\lfloor \frac{k+1}{2} \right\rfloor}}{k} A_{2k}\left(n - \frac{k\left(1+(-1)^k\right)}{4}\right) \times I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12n}\right)$$

Returning to $f(q)$

Ramanujan's claim:

$$\alpha(n) \sim \frac{(-1)^{n+1}}{2\sqrt{n}} e^{\pi\sqrt{\frac{n}{6}}} \qquad (n \to \infty)$$

Andrews-Dragonette Conjecture:

$$\alpha(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{k \geq 1} \frac{(-1)^{\left\lfloor \frac{k+1}{2} \right\rfloor}}{k} A_{2k}\left(n - \frac{k\left(1+(-1)^k\right)}{4}\right) \times I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12n}\right)$$

Theorem (B-Ono)

The Andrews-Dragonette Conjecture is true.

Key ingredient 1: (mock) modularity of $f(q)$

Build the vector-valued function

$$F(\tau) = (F_0(\tau), F_1(\tau), F_2(\tau))^T$$
$$:= \left(q^{-\frac{1}{24}} f(q), 2q^{\frac{1}{3}}\omega\left(q^{\frac{1}{2}}\right), 2q^{\frac{1}{3}}\omega\left(-q^{\frac{1}{2}}\right)\right)^T$$

Key ingredient 1: (mock) modularity of $f(q)$

Build the vector-valued function

$$F(\tau) = (F_0(\tau), F_1(\tau), F_2(\tau))^T$$
$$:= \left(q^{-\frac{1}{24}} f(q), 2q^{\frac{1}{3}}\omega\left(q^{\frac{1}{2}}\right), 2q^{\frac{1}{3}}\omega\left(-q^{\frac{1}{2}}\right)\right)^T$$

with

$$\omega(q) := \sum_{n\geq 0} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}.$$

Non-holomorphic part

Period integral:

$$G(\tau) :\doteq \int_{-\overline{\tau}}^{i\infty} \frac{(g_1(z), g_0(z), -g_2(z))^T}{\sqrt{-i(\tau+z)}} dz,$$

Non-holomorphic part

Period integral:

$$G(\tau) :\doteq \int_{-\overline{\tau}}^{i\infty} \frac{\left(g_1(z), g_0(z), -g_2(z)\right)^T}{\sqrt{-i(\tau+z)}} dz,$$

where

Non-holomorphic part

Period integral:

$$G(\tau) :\doteq \int_{-\overline{\tau}}^{i\infty} \frac{(g_1(z), g_0(z), -g_2(z))^T}{\sqrt{-i(\tau+z)}} dz,$$

where

$$g_0(z) := \sum_{n\in\mathbb{Z}} (-1)^n \left(n+\frac{1}{3}\right) e^{3\pi i \left(n+\frac{1}{3}\right)^2 z},$$

Non-holomorphic part

Period integral:

$$G(\tau) :\doteq \int_{-\overline{\tau}}^{i\infty} \frac{\left(g_1(z), g_0(z), -g_2(z)\right)^T}{\sqrt{-i(\tau + z)}} dz,$$

where

$$g_0(z) := \sum_{n\in\mathbb{Z}} (-1)^n \left(n + \frac{1}{3}\right) e^{3\pi i \left(n+\frac{1}{3}\right)^2 z},$$

$$g_1(z) := -\sum_{n\in\mathbb{Z}} \left(n + \frac{1}{6}\right) e^{3\pi i \left(n+\frac{1}{6}\right)^2 z},$$

Non-holomorphic part

Period integral:

$$G(\tau) :\doteq \int_{-\overline{\tau}}^{i\infty} \frac{\left(g_1(z), g_0(z), -g_2(z)\right)^T}{\sqrt{-i(\tau+z)}} dz,$$

where

$$g_0(z) := \sum_{n\in\mathbb{Z}} (-1)^n \left(n+\frac{1}{3}\right) e^{3\pi i\left(n+\frac{1}{3}\right)^2 z},$$

$$g_1(z) := -\sum_{n\in\mathbb{Z}} \left(n+\frac{1}{6}\right) e^{3\pi i\left(n+\frac{1}{6}\right)^2 z},$$

$$g_2(z) := \sum_{n\in\mathbb{Z}} \left(n+\frac{1}{3}\right) e^{3\pi i\left(n+\frac{1}{3}\right)^2 z}.$$

Modularity of $f(q)$

Completion:

$$\widehat{F}(\tau) := F(\tau) - G(\tau)$$

Completion:

$$\widehat{F}(\tau) := F(\tau) - G(\tau)$$

Theorem (Zwegers)

We have

$$\widehat{F}(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} \widehat{F}(\tau),$$

$$\widehat{F}\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \widehat{F}(\tau),$$

with $\zeta_n := e^{\frac{2\pi i}{n}}$.

Modularity of $f(q)$

Completion:

$$\widehat{F}(\tau) := F(\tau) - G(\tau)$$

Theorem (Zwegers)

We have

$$\widehat{F}(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} \widehat{F}(\tau),$$

$$\widehat{F}\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \widehat{F}(\tau),$$

with $\zeta_n := e^{\frac{2\pi i}{n}}$. *Moreover* $\Delta_{\frac{1}{2}}\left(\widehat{F}\right) = 0$.

Key ingredient 2: Maass Poincaré series

General shape:

$$\sum_{M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} (c\tau+d)^{-k}\phi\left(\frac{a\tau+b}{c\tau+d}\right)$$

Key ingredient 2: Maass Poincaré series

General shape:

$$\sum_{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} (c\tau+d)^{-k}\phi\left(\frac{a\tau+b}{c\tau+d}\right)$$

with

$$\Gamma_\infty := \left\{\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}; n\in\mathbb{Z}\right\}.$$

Maass-Poincaré series

Average

$$\varphi_s(\tau) := \mathcal{M}_s\left(-\frac{\pi y}{6}\right) e^{-\frac{\pi i x}{12}}$$

Maass-Poincaré series

Average

$$\varphi_s(\tau) := \mathcal{M}_s\left(-\frac{\pi y}{6}\right) e^{-\frac{\pi i x}{12}}$$

with

$$\mathcal{M}_s(u) := |u|^{-\frac{1}{2}} M_{\frac{1}{4}\operatorname{sgn}(u), s-\frac{1}{2}}(|u|)$$

Maass-Poincaré series

Average

$$\varphi_s(\tau) := \mathcal{M}_s\left(-\frac{\pi y}{6}\right) e^{-\frac{\pi i x}{12}}$$

with

$$\mathcal{M}_s(u) := |u|^{-\frac{1}{2}} M_{\frac{1}{4}\operatorname{sgn}(u), s-\frac{1}{2}}(|u|)$$

↑ M-Whittaker function

Maass-Poincaré series

Average

$$\varphi_s(\tau) := \mathcal{M}_s\left(-\frac{\pi y}{6}\right) e^{-\frac{\pi i x}{12}}$$

with

$$\mathcal{M}_s(u) := |u|^{-\frac{1}{2}} \underset{\substack{\uparrow \\ M\text{-Whittaker} \\ \text{function}}}{M_{\frac{1}{4}\operatorname{sgn}(u), s-\frac{1}{2}}}(|u|)$$

Poincaré series $(\mathrm{Re}(s) > 1)$:

$$\mathcal{P}_{\frac{1}{2}}(s;\tau) := \sum_{M \in \Gamma_\infty \backslash \Gamma_0(2)} \chi(M)^{-1} (c\tau + d)^{-\frac{1}{2}} \varphi_s(M\tau).$$

Average

$$\varphi_s(\tau) := \mathcal{M}_s\left(-\frac{\pi y}{6}\right) e^{-\frac{\pi i x}{12}}$$

with

$$\mathcal{M}_s(u) := |u|^{-\frac{1}{2}} \underset{\substack{\uparrow \\ M\text{-Whittaker} \\ \text{function}}}{M_{\frac{1}{4}\operatorname{sgn}(u), s-\frac{1}{2}}}(|u|)$$

Poincaré series ($\mathrm{Re}(s) > 1$):

$$\mathcal{P}_{\frac{1}{2}}(s;\tau) := \sum_{M \in \Gamma_\infty \backslash \Gamma_0(2)} \underset{\substack{\uparrow \\ \chi \text{ appropriate} \\ \text{multiplier}}}{\chi(M)^{-1}} (c\tau + d)^{-\frac{1}{2}} \varphi_s(M\tau).$$

Sketch of the proof

Idea:

Sketch of the proof

Idea:

- Analytically continue to $s = \frac{3}{4}$

Sketch of the proof

Idea:

- Analytically continue to $s = \frac{3}{4}$ (via Fourier expansion)

Sketch of the proof

Idea:

- Analytically continue to $s = \frac{3}{4}$ (via Fourier expansion)
- Show
$$\widehat{F}_0(\tau) = \mathcal{P}_{\frac{1}{2}}\left(\frac{3}{4};\tau\right)$$
where
$$\widehat{F}_0(\tau) := F_0(24\tau) - G_0(24\tau).$$

1. Modular forms

2. Mock modular forms

3. Mixed mock modular forms

4. Non-modular objects

Combinatorics

Joint with K. Mahlburg

Combinatorics

Joint with K. Mahlburg

Definition:

A partition without sequences is a partition with no adjacent parts

Joint with K. Mahlburg

Definition:

A partition without sequences is a partition with no adjacent parts

$$s(n) := \#\ \text{partitions of } n \text{ without sequences}$$

Combinatorics

Joint with K. Mahlburg

Definition:

A partition without sequences is a partition with no adjacent parts

$$s(n) := \#\text{ partitions of } n \text{ without sequences}$$

Example:

$6+6+4+1+1+1$ is a partition without sequences of 19

Generating function

Andrews:

$$\sum_{n\geq 0} s(n)q^n = \frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty}\chi(q)$$

Generating function

Andrews:

$$\sum_{n\geq 0} s(n)q^n = \frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty}\chi(q)$$

$$\uparrow$$
modular

Generating function

Andrews:

$$\sum_{n\geq 0} s(n)q^n = \frac{(-q^3;q^3)_\infty}{(q^2;q^2)_\infty}\chi(q)$$

$$\underset{\text{modular}}{\uparrow} \qquad \underset{\text{mock}}{\uparrow}$$

with

$$\chi(q) := \sum_{n\geq 0} \frac{(-q;q)_n}{(-q^3;q^3)_n} q^{n^2}$$

Algebraic geometry

Joint with J. Manschot

Algebraic geometry

Joint with J. Manschot

Generating function for Euler numbers:

For $j \in \{0, 1\}$, we have

$$f_j(\tau) = \sum_{n=0}^{\infty} \alpha_j(n) q^n := \frac{1}{\eta^6(\tau)} \sum_{n=0}^{\infty} H(4n+3j) q^{n+\frac{3j}{4}}$$

Algebraic geometry

Joint with J. Manschot

Generating function for Euler numbers:

For $j \in \{0, 1\}$, we have

$$f_j(\tau) = \sum_{n=0}^{\infty} \alpha_j(n) q^n := \underset{\substack{\uparrow \\ \text{modular}}}{\frac{1}{\eta^6(\tau)}} \sum_{n=0}^{\infty} H(4n+3j) q^{n+\frac{3j}{4}}$$

Algebraic geometry

Joint with J. Manschot

Generating function for Euler numbers:

For $j \in \{0, 1\}$, we have

$$f_j(\tau) = \sum_{n=0}^{\infty} \alpha_j(n) q^n := \underset{\substack{\uparrow \\ \text{modular}}}{\frac{1}{\eta^6(\tau)}} \sum_{n=0}^{\infty} \underset{\substack{\uparrow \\ \text{mock}}}{H(4n+3j)} q^{n+\frac{3j}{4}}$$

Lie superalgebras

Joint with K. Ono, K. Mahlburg

Lie superalgebras

Joint with K. Ono, K. Mahlburg

Character formula for $s\ell(m|1)^{\wedge}$-modules ($s \in \mathbb{Z}$)

$$2q^{-\frac{s}{2}} \frac{\left(q^2;q^2\right)_\infty^2}{(q;q)_\infty^{m+2}} \sum_{k=(k_1,\ldots,k_{m-1})\in\mathbb{Z}^{m-1}} \frac{q^{\frac{1}{2}\sum_{i=1}^{m-1} k_i(k_i+1)}}{1+q^{\sum_{i=1}^{m-1} k_i - s}}$$

An exact formula

Recall generating function for Euler numbers:

$$f_j(\tau) = \sum_{n=0}^{\infty} \alpha_j(n) q^n$$

An exact formula

Recall generating function for Euler numbers:

$$f_j(\tau) = \sum_{n=0}^{\infty} \alpha_j(n) q^n$$

Notation:

For $k \in \mathbb{N}$, $g \in \mathbb{Z}$, $u \in \mathbb{R}$

$$f_{k,g}(u) := \begin{cases} \frac{\pi^2}{\sinh^2\left(\frac{\pi u}{k} - \frac{\pi i g}{2k}\right)} & \text{if } g \not\equiv 0 \pmod{2k}, \\ \frac{\pi^2}{\sinh^2\left(\frac{\pi u}{k}\right)} - \frac{k^2}{u^2} & \text{if } g \equiv 0 \pmod{2k}. \end{cases}$$

An exact formula (cont.)

Kloosterman sums:

$$K_{j,\ell}(n,m;k) := \sum_{\substack{0\leq h<k \\ (h,k)=1}} \psi_{j\ell}\left(h,h',k\right) e^{-\frac{2\pi i}{k}\left(hn+\frac{h'n}{4}\right)}$$

An exact formula (cont.)

Kloosterman sums:

$$K_{j,\ell}(n,m;k) := \sum_{\substack{0\le h<k \\ (h,k)=1}} \underset{\substack{\uparrow \\ \text{multiplier}}}{\psi_{j\ell}(h,h',k)} e^{-\frac{2\pi i}{k}\left(hn+\frac{h'n}{4}\right)}$$

An exact formula (cont.)

Kloosterman sums:

$$K_{j,\ell}(n, m; k) := \sum_{\substack{0 \le h < k \\ (h,k)=1}} \underset{\substack{\uparrow \\ \text{multiplier}}}{\psi_{j\ell}(h, h', k)} e^{-\frac{2\pi i}{k}\left(hn + \frac{h'n}{4}\right)}$$

Bessel function integral:

$$\mathcal{I}_{k,g}(n) := \int_{-1}^{1} f_{k,g}\left(\frac{u}{2}\right) I_{\frac{7}{2}}\left(\frac{\pi}{k}\sqrt{(4n-(j+1))(1-u^2)}\right) \times \left(1-u^2\right)^{\frac{7}{4}} du$$

An exact formula (cont.)

Theorem (B-Manschot)

The coefficients $\alpha_j(n)$ *equal*

An exact formula (cont.)

Theorem (B-Manschot)

The coefficients $\alpha_j(n)$ *equal*

$$-\frac{\pi}{6\left(4n-(j+1)\right)^{\frac{5}{4}}}\sum_{k\geq 1}\frac{K_{j,0}\left(n,0;k\right)}{k}I_{\frac{5}{2}}\left(\frac{\pi}{k}\sqrt{4n-(j+1)}\right)$$

An exact formula (cont.)

Theorem (B-Manschot)

The coefficients $\alpha_j(n)$ *equal*

$$- \frac{\pi}{6\left(4n-(j+1)\right)^{\frac{5}{4}}} \sum_{k\geq 1} \frac{K_{j,0}\left(n,0;k\right)}{k} I_{\frac{5}{2}}\left(\frac{\pi}{k}\sqrt{4n-(j+1)}\right)$$

$$+ \frac{1}{\sqrt{2}\left(4n-(j+1)\right)^{\frac{3}{2}}} \sum_{k\geq 1} \frac{K_{j,0}\left(n,0;k\right)}{\sqrt{k}} I_{3}\left(\frac{\pi}{k}\sqrt{4n-(j+1)}\right)$$

An exact formula (cont.)

Theorem (B-Manschot)

The coefficients $\alpha_j(n)$ *equal*

$$-\frac{\pi}{6\left(4n-(j+1)\right)^{\frac{5}{4}}}\sum_{k\geq 1}\frac{K_{j,0}\left(n,0;k\right)}{k}I_{\frac{5}{2}}\left(\frac{\pi}{k}\sqrt{4n-(j+1)}\right)$$

$$+\frac{1}{\sqrt{2}\left(4n-(j+1)\right)^{\frac{3}{2}}}\sum_{k\geq 1}\frac{K_{j,0}\left(n,0;k\right)}{\sqrt{k}}I_{3}\left(\frac{\pi}{k}\sqrt{4n-(j+1)}\right)$$

$$-\frac{1}{8\pi\left(4n-(j+1)\right)^{\frac{7}{4}}}\sum_{k\geq 1}\sum_{\substack{\ell\in\{0,1\}\\ -k<g\leq k\\ g\equiv\ell\pmod 2}}\frac{K_{j,\ell}\left(n,g^2;k\right)}{k^2}\mathcal{I}_{k,g}(n).$$

Main term

Corollary

We have as $n \to \infty$

Main term

Corollary

We have as $n \to \infty$

$$\alpha_j(n) = \left(\frac{1}{96} n^{-\frac{3}{2}} - \frac{1}{32\pi} n^{-\frac{7}{4}} + O\left(n^{-2}\right) \right) e^{2\pi\sqrt{n}}.$$

1. Modular forms

2. Mock modular forms

3. Mixed mock modular forms

4. Non-modular objects

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

such that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_r \leq c < b_s \geq \cdots \geq b_1 > 0.$$

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

such that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_r \leq c < b_s \geq \cdots \geq b_1 > 0.$$

$$s(n) := \# \text{ stacks of size } n$$

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

such that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_r \leq c < b_s \geq \cdots \geq b_1 > 0.$$

$$s(n) := \# \text{ stacks of size } n$$

Example:

$n = 4$:

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

such that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_r \leq c < b_s \geq \cdots \geq b_1 > 0.$$

$$s(n) := \# \text{ stacks of size } n$$

Example:

$n = 4$: 4

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

such that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_r \leq c < b_s \geq \cdots \geq b_1 > 0.$$

$$s(n) := \# \text{ stacks of size } n$$

Example:

$n = 4$: 4 3+1

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

such that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_r \leq c < b_s \geq \cdots \geq b_1 > 0.$$

$$s(n) := \# \text{ stacks of size } n$$

Example:

$n = 4$: 4 3+1 1+3

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

such that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_r \leq c < b_s \geq \cdots \geq b_1 > 0.$$

$$s(n) := \# \text{ stacks of size } n$$

Example:

$n = 4$: 4 3+1 1+3 2+2

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

such that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_r \leq c < b_s \geq \cdots \geq b_1 > 0.$$

$$s(n) := \# \text{ stacks of size } n$$

Example:

$n = 4$: 4 3+1 1+3 2+2 2+1+1

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

such that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_r \leq c < b_s \geq \cdots \geq b_1 > 0.$$

$$s(n) := \# \text{ stacks of size } n$$

Example:

$n = 4$:

4	3+1	1+3	2+2
2+1+1	1+2+1		

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

such that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_r \leq c < b_s \geq \cdots \geq b_1 > 0.$$

$$s(n) := \# \text{ stacks of size } n$$

Example:

$n = 4$:

4	3+1	1+3	2+2
2+1+1	1+2+1	1+1+2	

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

such that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_r \leq c < b_s \geq \cdots \geq b_1 > 0.$$

$$s(n) := \# \text{ stacks of size } n$$

Example:

$n = 4$:

4	3+1	1+3	2+2
2+1+1	1+2+1	1+1+2	1+1+1+1

Stacks

Definition:

A stack of size $n \in \mathbb{N}_0$ is a decomposition of n into positive integers

$$n = a_1 + \cdots + a_r + c + b_s + \cdots + b_1$$

such that

$$0 < a_1 \leq a_2 \leq \cdots \leq a_r \leq c < b_s \geq \cdots \geq b_1 > 0.$$

$$s(n) := \#\ \text{stacks of size } n$$

Example:

$n = 4$:

4	3+1	1+3	2+2
2+1+1	1+2+1	1+1+2	1+1+1+1

Thus $s(4) = 8$.

Generating function

Auluck:

$$\mathcal{S}(q) := \sum_{n\geq 0} s(n)q^n = 1 + \sum_{n\geq 1} \frac{q^n}{(q)_{n-1}^2(1-q^n)}$$

Generating function

Auluck:

$$\mathcal{S}(q) := \sum_{n\geq 0} s(n)q^n = 1 + \sum_{n\geq 1} \frac{q^n}{(q)_{n-1}^2(1-q^n)}$$

(Non)-modularity:

$$\mathcal{S}(q) = 1 + \frac{1}{(q)_\infty^2}\vartheta(q)$$

with the false theta function

$$\vartheta(q) := \sum_{r\geq 1} (-1)^{r+1} q^{\frac{r(r+1)}{2}}$$

Asymptotic behavior

Wright:

$$s(n) \sim 2^{-3} 3^{-\frac{3}{4}} n^{-\frac{5}{4}} e^{2\pi\sqrt{\frac{n}{3}}}$$

Asymptotic behavior

Wright:

$$s(n) \sim 2^{-3}3^{-\frac{3}{4}}n^{-\frac{5}{4}}e^{2\pi\sqrt{\frac{n}{3}}}$$

Key ingredients:

Asymptotic behavior

Wright:

$$s(n) \sim 2^{-3} 3^{-\frac{3}{4}} n^{-\frac{5}{4}} e^{2\pi\sqrt{\frac{n}{3}}}$$

Key ingredients:

- Modularity of the infinite product

Asymptotic behavior

Wright:

$$s(n) \sim 2^{-3} 3^{-\frac{3}{4}} n^{-\frac{5}{4}} e^{2\pi\sqrt{\frac{n}{3}}}$$

Key ingredients:

- Modularity of the infinite product
- Asymptotic behavior as $q \to 1$ of the false theta function

$$\vartheta(q) \sim \frac{1}{2} \qquad \text{as } q \to 1$$

Asymptotic behavior

Wright:

$$s(n) \sim 2^{-3} 3^{-\frac{3}{4}} n^{-\frac{5}{4}} e^{2\pi\sqrt{\frac{n}{3}}}$$

Key ingredients:

- Modularity of the infinite product
- Asymptotic behavior as $q \to 1$ of the false theta function

$$\vartheta(q) \sim \frac{1}{2} \qquad \text{as } q \to 1$$

- Tauberian Theorem

Shifted stacks

Definition:

A shifted stack of size $n \in \mathbb{N}_0$ is a stack of size n with the extra condition that

Shifted stacks

Definition:

A shifted stack of size $n \in \mathbb{N}_0$ is a stack of size n with the extra condition that

$$
\begin{aligned}
a_j &\geq a_{j+1} - 1 \qquad && \text{for } 1 \leq j \leq r-1, \\
b_j &\leq b_{j+1} + 1 \qquad && \text{for } 1 \leq j \leq s-1.
\end{aligned}
$$

Shifted stacks

Definition:

A shifted stack of size $n \in \mathbb{N}_0$ is a stack of size n with the extra condition that

$$\begin{aligned} a_j &\geq a_{j+1} - 1 && \text{for } 1 \leq j \leq r-1, \\ b_j &\leq b_{j+1} + 1 && \text{for } 1 \leq j \leq s-1. \end{aligned}$$

$$ss(n) := \#\text{shifted stacks of size } n$$

Shifted stacks

Definition:

A shifted stack of size $n \in \mathbb{N}_0$ is a stack of size n with the extra condition that

$$\begin{aligned} a_j &\geq a_{j+1} - 1 && \text{for } 1 \leq j \leq r-1, \\ b_j &\leq b_{j+1} + 1 && \text{for } 1 \leq j \leq s-1. \end{aligned}$$

$$ss(n) := \#\text{shifted stacks of size } n$$

Example:

The shifted stacks of size 4 are

Shifted stacks

Definition:

A shifted stack of size $n \in \mathbb{N}_0$ is a stack of size n with the extra condition that

$$a_j \geq a_{j+1} - 1 \qquad \text{for } 1 \leq j \leq r-1,$$
$$b_j \leq b_{j+1} + 1 \qquad \text{for } 1 \leq j \leq s-1.$$

$$ss(n) := \#\text{shifted stacks of size } n$$

Example:

The shifted stacks of size 4 are

$$1 + 2 + 1$$

Shifted stacks

Definition:

A shifted stack of size $n \in \mathbb{N}_0$ is a stack of size n with the extra condition that

$$a_j \geq a_{j+1} - 1 \qquad \text{for } 1 \leq j \leq r-1,$$
$$b_j \leq b_{j+1} + 1 \qquad \text{for } 1 \leq j \leq s-1.$$

$$ss(n) := \#\text{shifted stacks of size } n$$

Example:

The shifted stacks of size 4 are

$$1+2+1 \qquad 1+1+2$$

Shifted stacks

Definition:

A shifted stack of size $n \in \mathbb{N}_0$ is a stack of size n with the extra condition that

$$a_j \geq a_{j+1} - 1 \qquad \text{for } 1 \leq j \leq r-1,$$
$$b_j \leq b_{j+1} + 1 \qquad \text{for } 1 \leq j \leq s-1.$$

$$ss(n) := \#\text{shifted stacks of size } n$$

Example:

The shifted stacks of size 4 are

$$1+2+1 \qquad 1+1+2 \qquad 1+1+1+1$$

Shifted stacks

Definition:

A shifted stack of size $n \in \mathbb{N}_0$ is a stack of size n with the extra condition that

$$\begin{aligned} a_j &\geq a_{j+1} - 1 && \text{for } 1 \leq j \leq r-1, \\ b_j &\leq b_{j+1} + 1 && \text{for } 1 \leq j \leq s-1. \end{aligned}$$

$$ss(n) := \#\text{shifted stacks of size } n$$

Example:

The shifted stacks of size 4 are

$$1+2+1 \qquad 1+1+2 \qquad 1+1+1+1$$

Thus $ss(4) = 3$.

Generating function

Auluck:

$$\mathcal{S}_s(q) := \sum_{n \geq 0} ss(n) q^n = 1 + \sum_{n \geq 1} \frac{q^{\frac{n(n+1)}{2}}}{(q)_{n-1}^2 (1 - q^n)}$$

Generating function

Auluck:

$$\mathcal{S}_s(q) := \sum_{n\geq 0} ss(n)q^n = 1 + \sum_{n\geq 1} \frac{q^{\frac{n(n+1)}{2}}}{(q)_{n-1}^2(1-q^n)}$$

(log) asymptotic behavior (Wright)

$$\log(ss(n)) \sim 2\pi\sqrt{\frac{n}{5}} \qquad (n\to\infty)$$

Generating function

Auluck:

$$\mathcal{S}_s(q) := \sum_{n\geq 0} ss(n)q^n = 1 + \sum_{n\geq 1} \frac{q^{\frac{n(n+1)}{2}}}{(q)_{n-1}^2(1-q^n)}$$

(log) asymptotic behavior (Wright)

$$\log(ss(n)) \sim 2\pi\sqrt{\frac{n}{5}} \qquad (n\to\infty)$$

Theorem (B-Mahlburg)

We have

$$ss(n) \sim \frac{\phi^{-1}}{2\sqrt{2}5^{\frac{3}{4}}n} e^{2\pi\sqrt{\frac{n}{5}}} \qquad (n\to\infty)$$

with ϕ *the Golden Ratio.*

Key idea of the proof

- Embedding into the modular world

$$\mathcal{S}_s(q) = 1 + \text{coeff}\left[x^0\right]\left(\sum_{r\geq 0}\frac{x^{-r}q^{\frac{r^2-r}{2}}}{(q)_r}\sum_{m\geq 1}\frac{x^m q^m}{(q)_{m-1}}\right)$$

Key idea of the proof

- Embedding into the modular world

$$\mathcal{S}_s(q) = 1 + \text{coeff}\left[x^0\right]\left(\sum_{r\geq 0}\frac{x^{-r}q^{\frac{r^2-r}{2}}}{(q)_r}\sum_{m\geq 1}\frac{x^m q^m}{(q)_{m-1}}\right)$$
$$= 1 + \text{coeff}\left[x^0\right]\left(qx\left(-x^{-1}\right)_\infty (xq)_\infty^{-1}\right)$$

Key idea of the proof

- Embedding into the modular world

$$\mathcal{S}_s(q) = 1 + \text{coeff}\left[x^0\right]\left(\sum_{r\geq 0}\frac{x^{-r}q^{\frac{r^2-r}{2}}}{(q)_r}\sum_{m\geq 1}\frac{x^m q^m}{(q)_{m-1}}\right)$$
$$= 1 + \text{coeff}\left[x^0\right]\left(qx\left(-x^{-1}\right)_\infty (xq)_\infty^{-1}\right)$$

- Use Jacobi triple product formula

$$\left(-x^{-1}\right)\infty\left(-xq\right)_\infty (q)_\infty = -q^{-\frac{1}{8}}x^{-\frac{1}{2}}\vartheta\left(u+\frac{1}{2};\frac{i\varepsilon}{2\pi}\right)$$

with the Jacobi theta function

$$\vartheta(u;\tau) := \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n\left(u+\frac{1}{2}\right)}$$

Proof continued . . .

- Also use quantum dilogarithm

$$\mathrm{Li}_2(x;q) := -\log(x)_\infty$$

Proof continued ...

- Also use quantum dilogarithm

$$\mathrm{Li}_2(x;q) := -\log(x)_\infty$$

- Modular inversion

$$\vartheta\left(u+\frac{1}{2};\frac{i\varepsilon}{2\pi}\right) = i\sqrt{\frac{2\pi}{\varepsilon}}e^{-\frac{2\pi^2}{\varepsilon}\left(u+\frac{1}{2}\right)^2}\vartheta\left(\frac{2\pi\left(u+\frac{1}{2}\right)}{i\varepsilon};\frac{2\pi i}{\varepsilon}\right)$$

Proof continued ...

- Also use quantum dilogarithm

$$\mathrm{Li}_2(x;q) := -\log(x)_\infty$$

- Modular inversion

$$\vartheta\left(u+\frac{1}{2};\frac{i\varepsilon}{2\pi}\right) = i\sqrt{\frac{2\pi}{\varepsilon}}e^{-\frac{2\pi^2}{\varepsilon}\left(u+\frac{1}{2}\right)^2}\vartheta\left(\frac{2\pi\left(u+\frac{1}{2}\right)}{i\varepsilon};\frac{2\pi i}{\varepsilon}\right)$$

- Laurent expansion of Li_2

$$\mathrm{Li}_2\left(e^{-B\varepsilon}x;e^{-\varepsilon}\right) = \frac{1}{\varepsilon}\mathrm{Li}_2(x) + \left(B-\frac{1}{2}\right)\log(1-x) + O(\varepsilon).$$

Proof continued . . .

- Also use quantum dilogarithm

$$\mathrm{Li}_2(x;q) := -\log(x)_\infty$$

- Modular inversion

$$\vartheta\left(u+\frac{1}{2};\frac{i\varepsilon}{2\pi}\right) = i\sqrt{\frac{2\pi}{\varepsilon}}e^{-\frac{2\pi^2}{\varepsilon}\left(u+\frac{1}{2}\right)^2}\vartheta\left(\frac{2\pi\left(u+\frac{1}{2}\right)}{i\varepsilon};\frac{2\pi i}{\varepsilon}\right)$$

- Laurent expansion of Li_2

$$\mathrm{Li}_2\left(e^{-B\varepsilon}x;e^{-\varepsilon}\right) = \frac{1}{\varepsilon}\mathrm{Li}_2(x) + \left(B-\frac{1}{2}\right)\log(1-x) + O(\varepsilon).$$

- Saddle point method

Proof continued ...

- Also use quantum dilogarithm

$$\mathrm{Li}_2(x;q) := -\log(x)_\infty$$

- Modular inversion

$$\vartheta\left(u+\frac{1}{2};\frac{i\varepsilon}{2\pi}\right) = i\sqrt{\frac{2\pi}{\varepsilon}}e^{-\frac{2\pi^2}{\varepsilon}\left(u+\frac{1}{2}\right)^2}\vartheta\left(\frac{2\pi\left(u+\frac{1}{2}\right)}{i\varepsilon};\frac{2\pi i}{\varepsilon}\right)$$

- Laurent expansion of Li_2

$$\mathrm{Li}_2\left(e^{-B\varepsilon}x;e^{-\varepsilon}\right) = \frac{1}{\varepsilon}\mathrm{Li}_2(x) + \left(B-\frac{1}{2}\right)\log(1-x) + O(\varepsilon).$$

- Saddle point method
- Tauberian Theorem

Positive crank and rank moments

Definition: For a partition λ define

$$\text{rank}(\lambda) := \text{largest part of } \lambda - \text{number of parts of } \lambda,$$

Positive crank and rank moments

Definition: For a partition λ define

$$\text{rank}(\lambda) := \text{largest part of } \lambda - \text{number of parts of } \lambda,$$

$$\text{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0, \end{cases}$$

Positive crank and rank moments

Definition: For a partition λ define

$$\text{rank}(\lambda) := \text{largest part of } \lambda - \text{number of parts of } \lambda,$$

$$\text{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0, \end{cases}$$

where

$$o(\lambda) := \#\text{ of 1s in } \lambda,$$

$$\mu(\lambda) := \#\text{parts } > o(\lambda).$$

Rank and crank

Generating functions (basically)

$$M(m,n) := \# \text{ of partitions of } n, \text{ crank } m$$
$$N(m,n) := \# \text{ of partitions of } n, \text{ rank } m$$

Rank and crank

Generating functions (basically)

$$M(m,n) := \# \text{ of partitions of } n, \text{ crank } m$$
$$N(m,n) := \# \text{ of partitions of } n, \text{ rank } m$$

Andrews–Garvan

$$C(w;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} M(m,n) w^m q^n = \frac{1-w}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1-wq^n},$$

Rank and crank

Generating functions (basically)

$$M(m,n) := \#\text{ of partitions of } n, \text{ crank } m$$
$$N(m,n) := \#\text{ of partitions of } n, \text{ rank } m$$

Andrews–Garvan

$$C(w;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} M(m,n)w^m q^n = \frac{1-w}{(q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1-wq^n},$$

Atkin–Swinnerton-Dyer

$$R(w;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} N(m,n)w^m q^n = \frac{1-w}{(q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1-wq^n}.$$

Moments

Crank and Rank moments ($r \in \mathbb{N}$) (Atkin–Garvan)

$$M_r(n) := \sum_{m \in \mathbb{Z}} m^r M(m,n)$$

$$N_r(n) := \sum_{m \in \mathbb{Z}} m^r N(m,n)$$

Moments

Crank and Rank moments ($r \in \mathbb{N}$) (Atkin–Garvan)

$$M_r(n) := \sum_{m \in \mathbb{Z}} m^r M(m,n)$$
$$N_r(n) := \sum_{m \in \mathbb{Z}} m^r N(m,n)$$

Note: $N_{2r+1}(n) = M_{2r+1}(n) = 0$.

Moments

Crank and Rank moments ($r \in \mathbb{N}$) (Atkin–Garvan)

$$M_r(n) := \sum_{m \in \mathbb{Z}} m^r M(m,n)$$

$$N_r(n) := \sum_{m \in \mathbb{Z}} m^r N(m,n)$$

Note: $N_{2r+1}(n) = M_{2r+1}(n) = 0$.

Theorem (B-Mahlburg, B-Mahlburg-Rhoades, Garvan)

(i) *As* $n \to \infty$

$$M_{2k}(n) \sim N_{2k}(n)$$

Moments

Crank and Rank moments ($r \in \mathbb{N}$) (Atkin–Garvan)

$$M_r(n) := \sum_{m \in \mathbb{Z}} m^r M(m, n)$$

$$N_r(n) := \sum_{m \in \mathbb{Z}} m^r N(m, n)$$

Note: $N_{2r+1}(n) = M_{2r+1}(n) = 0.$

Theorem (B-Mahlburg, B-Mahlburg-Rhoades, Garvan)

(i) *As* $n \to \infty$

$$M_{2k}(n) \sim N_{2k}(n)$$

(ii) *For all* $n > 0$

$$M_{2k}(n) > N_{2k}(n)$$

Moments

Crank and Rank moments ($r \in \mathbb{N}$) (Atkin–Garvan)

$$M_r(n) := \sum_{m \in \mathbb{Z}} m^r M(m, n)$$
$$N_r(n) := \sum_{m \in \mathbb{Z}} m^r N(m, n)$$

Note: $N_{2r+1}(n) = M_{2r+1}(n) = 0.$

Theorem (B-Mahlburg, B-Mahlburg-Rhoades, Garvan)

(i) *As* $n \to \infty$
$$M_{2k}(n) \sim N_{2k}(n)$$

(ii) *For all* $n > 0$
$$M_{2k}(n) > N_{2k}(n)$$

Important tool: modularity

Positive moments

Positive crank and rank moments ($r \in \mathbb{N}$) (Andrews–Chan–Kim)

$$M_r^+(n) := \sum_{m\in\mathbb{N}} m^r M(m,n)$$

$$N_r^+(n) := \sum_{m\in\mathbb{N}} m^r N(m,n)$$

Positive moments

Positive crank and rank moments ($r \in \mathbb{N}$) (Andrews–Chan–Kim)

$$M_r^+(n) := \sum_{m\in\mathbb{N}} m^r M(m,n)$$

$$N_r^+(n) := \sum_{m\in\mathbb{N}} m^r N(m,n)$$

Note:

$$M_{2k}^+(n) = \tfrac{1}{2} M_{2k}(n)$$

$$N_{2k}^+(n) = \tfrac{1}{2} N_{2k}(n)$$

Positive moments

Positive crank and rank moments ($r \in \mathbb{N}$) (Andrews–Chan–Kim)

$$M_r^+(n) := \sum_{m \in \mathbb{N}} m^r M(m,n)$$

$$N_r^+(n) := \sum_{m \in \mathbb{N}} m^r N(m,n)$$

Note:

$$M_{2k}^+(n) = \frac{1}{2} M_{2k}(n)$$

$$N_{2k}^+(n) = \frac{1}{2} N_{2k}(n)$$

Theorem (B-Mahlburg, Andrews-Chan-Kim)

(i) *As* $n \to \infty$

$$M_r^+(n) \sim N_r^+(n)$$

Positive moments

Positive crank and rank moments ($r \in \mathbb{N}$) (Andrews–Chan–Kim)

$$M_r^+(n) := \sum_{m\in\mathbb{N}} m^r M(m,n)$$

$$N_r^+(n) := \sum_{m\in\mathbb{N}} m^r N(m,n)$$

Note:

$$M_{2k}^+(n) = \frac{1}{2} M_{2k}(n)$$

$$N_{2k}^+(n) = \frac{1}{2} N_{2k}(n)$$

Theorem (B-Mahlburg, Andrews-Chan-Kim)

(i) *As* $n \to \infty$

$$M_r^+(n) \sim N_r^+(n)$$

(ii) *For all* $n > 0$

$$M_r^+(n) > N_r^+(n)$$

Idea of proof

Difficulty: non-modularity

Idea of proof

Difficulty: non-modularity

- Relate generating functions to "false Lerch sums" ($\ell \in \{1, 3\}$)

$$\frac{1}{(q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{\frac{\ell n^2}{2} + \frac{rn}{2}}}{(1-q^n)^r}$$

Idea of proof

Difficulty: non-modularity

- Relate generating functions to "false Lerch sums" ($\ell \in \{1, 3\}$)

$$\frac{1}{(q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{\frac{\ell n^2}{2} + \frac{rn}{2}}}{(1 - q^n)^r}$$

- Understand the asymptotic behavior near $q = 1$

Idea of proof

Difficulty: non-modularity

- Relate generating functions to "false Lerch sums" ($\ell \in \{1,3\}$)

$$\frac{1}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^{n+1}q^{\frac{\ell n^2}{2}+\frac{rn}{2}}}{(1-q^n)^r}$$

- Understand the asymptotic behavior near $q=1$

- Use Circle Method