List Homomorphisms, Time and Space

Pavol Hell, SFU

Journées d’Informatique Fondamentale de Paris Diderot,
April 22, 2013

Colourings

Colouring the vertices of a graph G

Adjacent vertices have distinct colours

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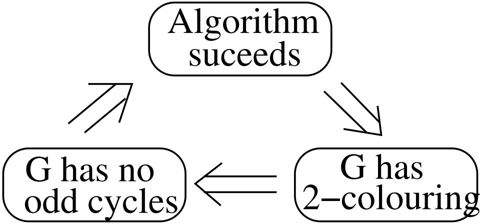

Homomorphisms

Given digraphs G and H

A *homomorphism* $f : G \to H$ is a mapping $f : V(G) \to V(H)$ such that $xy \in E(G) \implies f(x)f(y) \in E(H)$

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Undirected graphs are viewed as symmetric digraphs

Homomorphism Problems

Given a fixed digraph H

Does an input digraph G admit a homomorphism to H?

The "H-colouring problem"

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Example: $H = K_t$

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H-colouring problems

Is there dichotomy?

Constraint Satisfaction Problems

CSP with fixed template H

H with $V(H)$ and relations $R_1(H), \ldots, R_k(H)$

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Does an input G admit a homomorphism to H?

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Homomorphisms preserve all relations

Dichotomy Conjecture

Feder - Vardi, 1993, conjectured for any template H

The CSP problem for H is polynomial or is NP-complete

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If *H* is a digraph

If dichotomy holds for all digraphs *H* then the dichotomy conjecture holds for all CSP

Feder+Vardi 1993

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Each vertex x of the input digraph G has a *list* $L(x) \subseteq V(H)$

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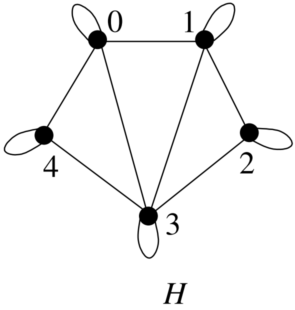
Given a fixed digraph H

Each vertex x of the input digraph G has a *list* $L(x) \subseteq V(H)$
Is there a homomorphism $f : G \to H$ for which all $f(x) \in L(x)$?

Fixed graph *H*

Processors and connections



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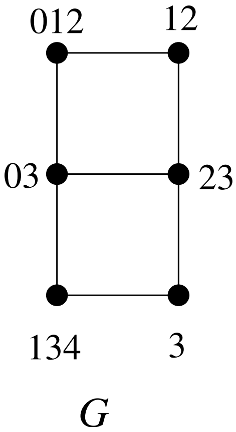
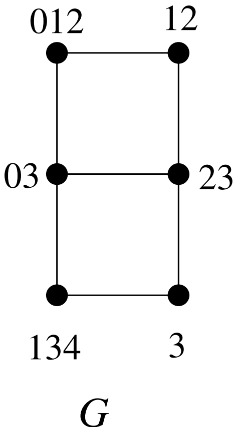
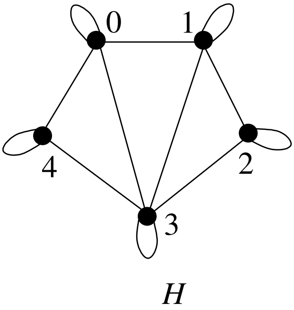


Input graph G

Tasks and communications

List Homomorphism Problems





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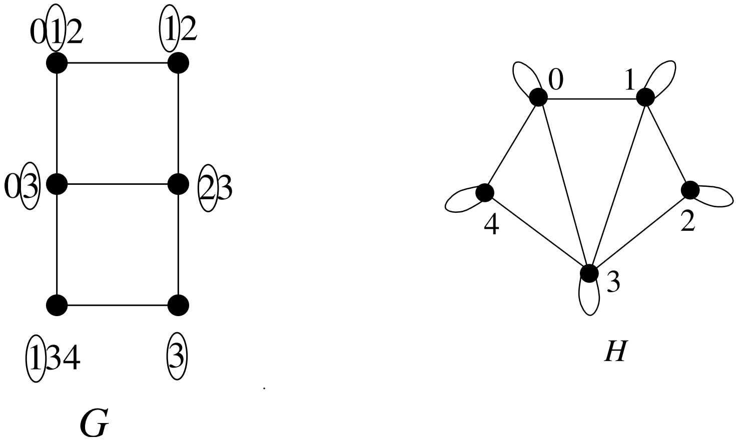
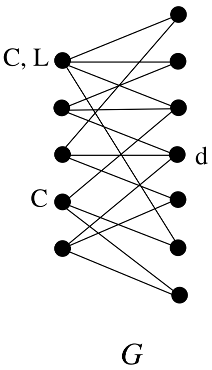
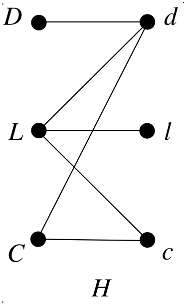


Another Example Application





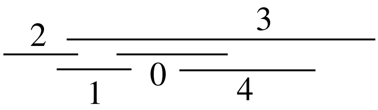
Lists

Lists of allowed decisions

Reflexive Undirected Graphs

Interval graph

Vertices correspond to intervals and adjacency to intersection



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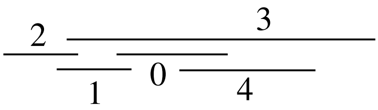

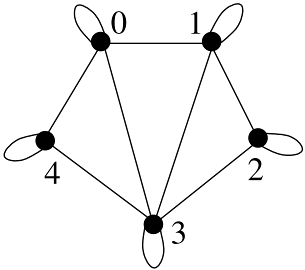

H

Interval Graphs

Lekkerkerker, Boland

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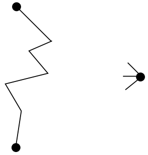

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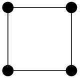

List Homomorphism Problem for Graphs

For a reflexive graph H

If H is NOT an interval graph, then the problem for H is NP-complete

Feder+H 1998

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For a reflexive graph H

If H is an interval graph, then the problem for H is polynomial

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(Will discuss later)

List Homomorphism Problem for Graphs

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If H is an interval graph, then the problem for H is polynomial
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For an irreflexive graph H

If H is a bipartite graph whose complement is a circular arc graph, then the problem for H is polynomial
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Feder+H+Huang 1999

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Similarly for general graphs

Complements of Circular Arc Graphs

For a bipartite graph H

$\overline{H}$ is a circular arc graph $\Leftrightarrow$ H is an intersection graph of a family of two-directional rays

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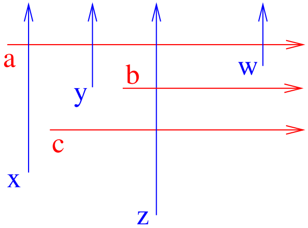

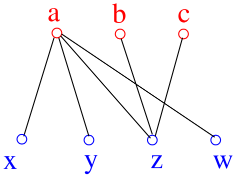

Shresta+Tayu+Ueno 2010

Polymorphisms

A k-ary polymorphism on H

A homomorphism $\phi : H^k \to H$

Polymorphisms

A k-ary polymorphism on H

A homomorphism $\phi : H^k \to H$

A conservative polymorphism f

A polymorphism $\phi : H^k \to H$ with $\phi(u_1, \ldots, u_k) \in \{u_1, \ldots, u_k\}$

Min Ordering

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A min ordering $<$ of $V(H)$

$$uv, u'v' \in E(H) \implies \min(u, u')\min(v, v') \in E(H)$$

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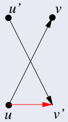

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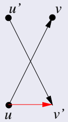

Theorem

Each interval graph has a min ordering

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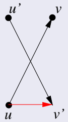

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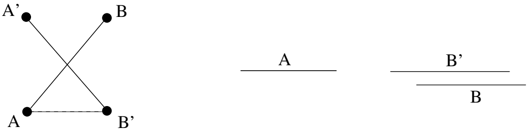

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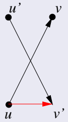
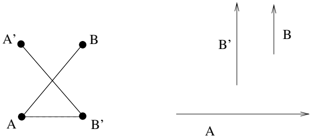


Theorem

Each 2DR graph has a min ordering



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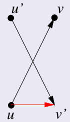

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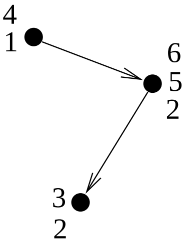
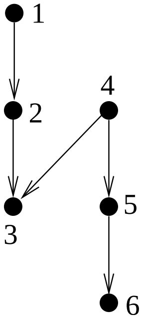
Theorem

If H admits a min ordering, then the list homomorphism problem for H is polynomial

Gutjahr, Welzl, Woeginger 1992

Local Consistency Algorithm





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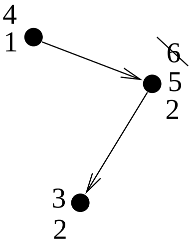

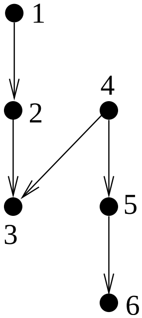

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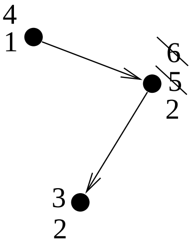

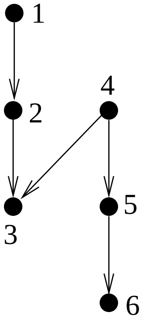

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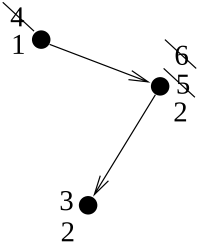

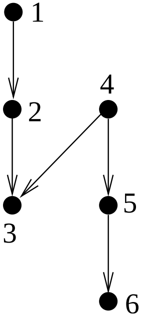

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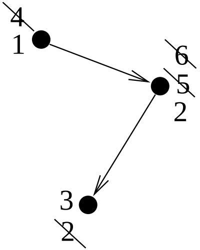

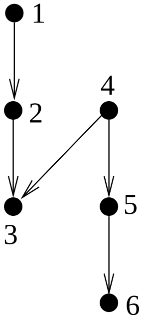

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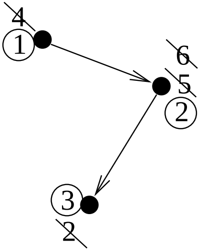

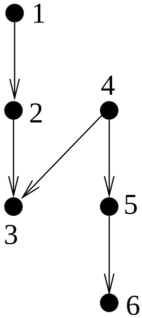

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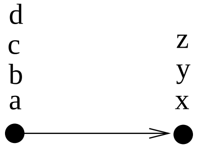

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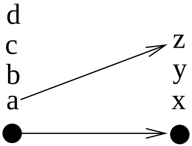

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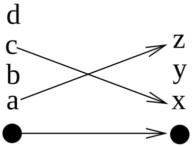

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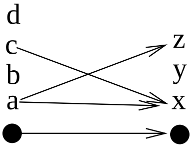

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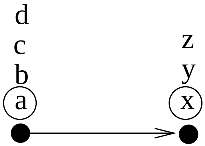

Conservative Ternary Polymorphisms

A conservative majority on H

A conservative ternary polymorphism $g : H^3 \to H$ such that

- $g(u, u, v) = g(u, v, u) = g(v, u, u) = u$

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Theorem

If H admits a conservative majority, then the list homomorphism problem for H is polynomial

Feder+Vardi 1993; Jeavons 1998

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Jeavons, Cohen, Gyssens 1997

List Homomorphism Problems for CSPs

List homomorphism problem for template H

If each pair $u, v \in V(H)$ admits a conservative polymorphism f of H such that $f|\{u, v\}$ is min-ordering, majority, or Maltsev, then the problem is polynomial.

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Otherwise the problem is NP-complete

Bulatov 2011

Barto 2012

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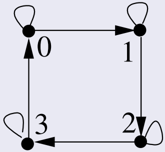
Is NP-completeness again caused by obstructions?

Reflexive 4-cycle does not have a min ordering



Existence of Polymorphisms of Digraphs

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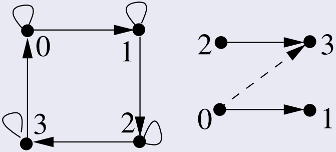

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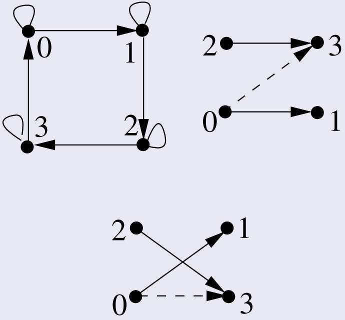

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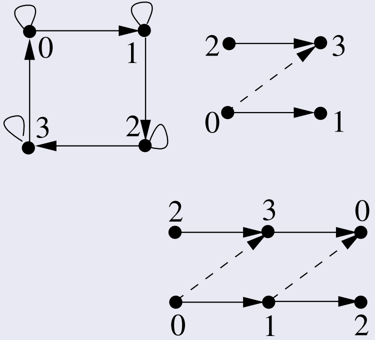

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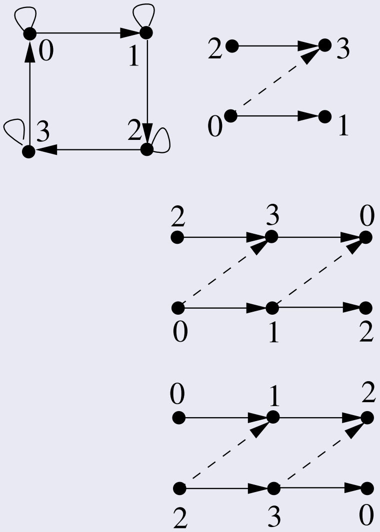
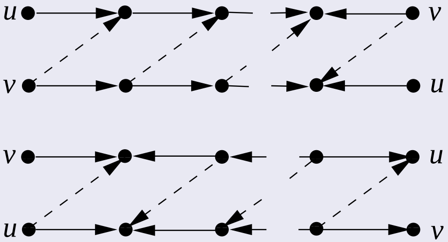


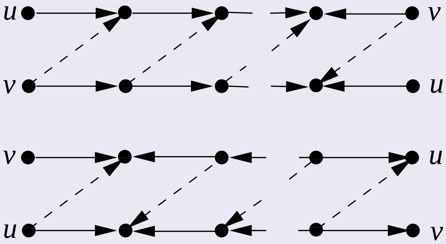
Existence of Polymorphisms of Digraphs

Invertible pair u, v



Existence of Polymorphisms of Digraphs

Invertible pair u, v



Min ordering

A digraph H admits a min ordering if and only if it has no invertible pair

Feder+H+Huang+Rafiey 2009

Existence of Polymorphisms of Digraphs

Majority

A digraph H admits a conservative majority if and only if it has no *permutable triple*

H+Rafiey 2010

Majority

A digraph *H* admits a conservative majority if and only if it has no *permutable triple*

H+Rafiey 2010

Recall $g(u,u,v) = g(u,v,u) = g(v,u,u) = u$

Existence of Polymorphisms of Digraphs

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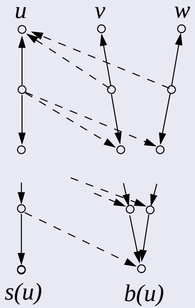
H+Rafiey 2010

Recall $g(u,u,v) = g(u,v,u) = g(v,u,u) = u$

$g(u,v,w) \neq u$



Existence of Polymorphisms of Digraphs

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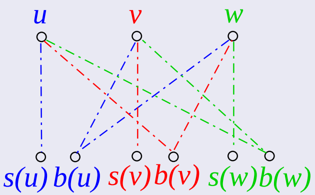
H+Rafiey 2010

Permutable triple



Digraph Asteroidal Triples

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An obstruction for both min ordering and conservative majority:

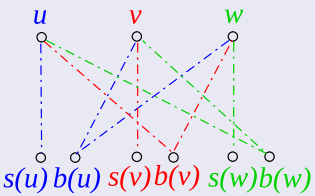
Digraph Asteroidal Triples

An obstruction for both min ordering and conservative majority:

DAT

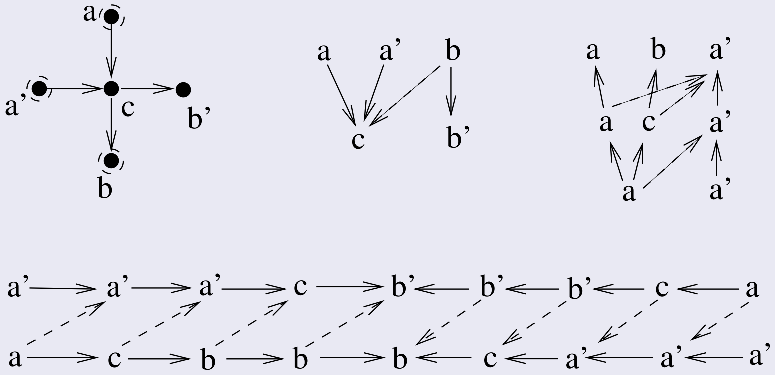
A permutable triple u, v, w with each pair $(s(u), b(u)), (s(v), b(v)), (s(w), b(w))$ being invertible

Recall permutable triple



Digraph Asteroidal Triples

A DAT



Polynomial Dichotomy Classification for Digraphs

For a digraph H

If H is DAT-free, the list homomorphism problem for H is polynomial

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H+Rafiey 2010

Testing for the existence of a DAT is polynomial

How about space?

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Trichotomy for list homomorphisms to a reflexive graph H

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- If H is not an interval graph, then the problem is NP-complete

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- If H is an interval graph but not a cograph, then the problem is polynomial time solvable, but NL-complete

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Trichotomy for list homomorphisms to a reflexive graph H

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Cograph = no induced P_4

Interval + Cograph = "Trivially Perfect")

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Interval + Cograph = "Trivially Perfect")

Similarly for irreflexive graphs, and general graphs.

Reformulation of Egri+Krokhin+Larose+Tesson 2012

Reflexive Graphs H with Logspace Algorithms

Dichotomy for reflexive graphs H

If H is trivially perfect then the list homomorphism problem to H is in logspace. Otherwise it is NL-complete.

Reflexive Graphs H with Logspace Algorithms

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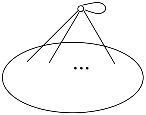
Trivially perfect graphs H

H be obtained from K_1^* (the one-vertex graph with a loop) by taking disjoint unions and joins with K_1^*.



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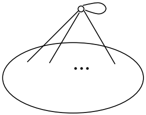

By induction, the list homomorphism problems to trivially perfect graphs H are in logspace.

Reflexive Graphs H with Logspace Algorithms

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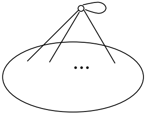


By induction, the list homomorphism problems to trivially perfect graphs H are in logspace. (Uses Reingold)

Reflexive Graphs H with Logspace Algorithms

A graph H

H is trivially perfect if and only if it has no induced C_4, P_4

Reflexive Graphs H with Logspace Algorithms

A graph H

H is trivially perfect if and only if it has no induced C_4, P_4

Reflexive graphs with C_4 or P_4 have NL-complete list H-colouring problems.

Irreflexive Graphs

Dichotomy for irreflexive graphs H

If H is a co-bigraph then the list homomorphism problem to H is in logspace. Otherwise it is NL-complete.

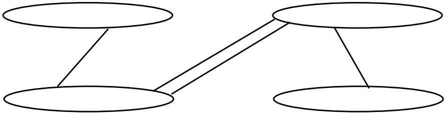
Irreflexive Graphs

Dichotomy for irreflexive graphs H

If H is a co-bigraph then the list homomorphism problem to H is in logspace. Otherwise it is NL-complete.

Co-bigraph H

H be obtained from K_2 (the single edge) by taking disjoint unions and bi-joins.



By induction, the list homomorphism problems to co-bigraphs H are in logspace.

Irreflexive Graphs

A bigraph H

H is a co-bigraph if and only if it does not have an induced C_6, P_6

Irreflexive Graphs

A bigraph H

H is a co-bigraph if and only if it does not have an induced C_6, P_6

Irreflexive graphs with C_6 or P_6 have NL-complete list H-colouring problems.

A General Classification for Digraphs

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Trichotomy for list homomorphisms to a digraph H

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A General Classification for Digraphs

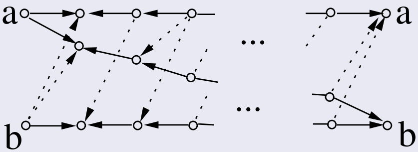
Trichotomy for list homomorphisms to a digraph H

- If H contains a DAT, then the problem is NP-complete
- If H is DAT-free but contains a circular N, then the problem is polynomial time solvable, but NL-complete
- If H contains no circular N, then the problem is solvable in logspace

Egri+H+Larose+Rafiey 2013

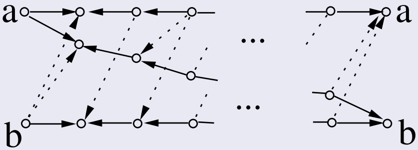
A General Classification for Digraphs

A circular *N*



A General Classification for Digraphs

A circular *N*



Testing for the existence of a circular *N* is polynomial