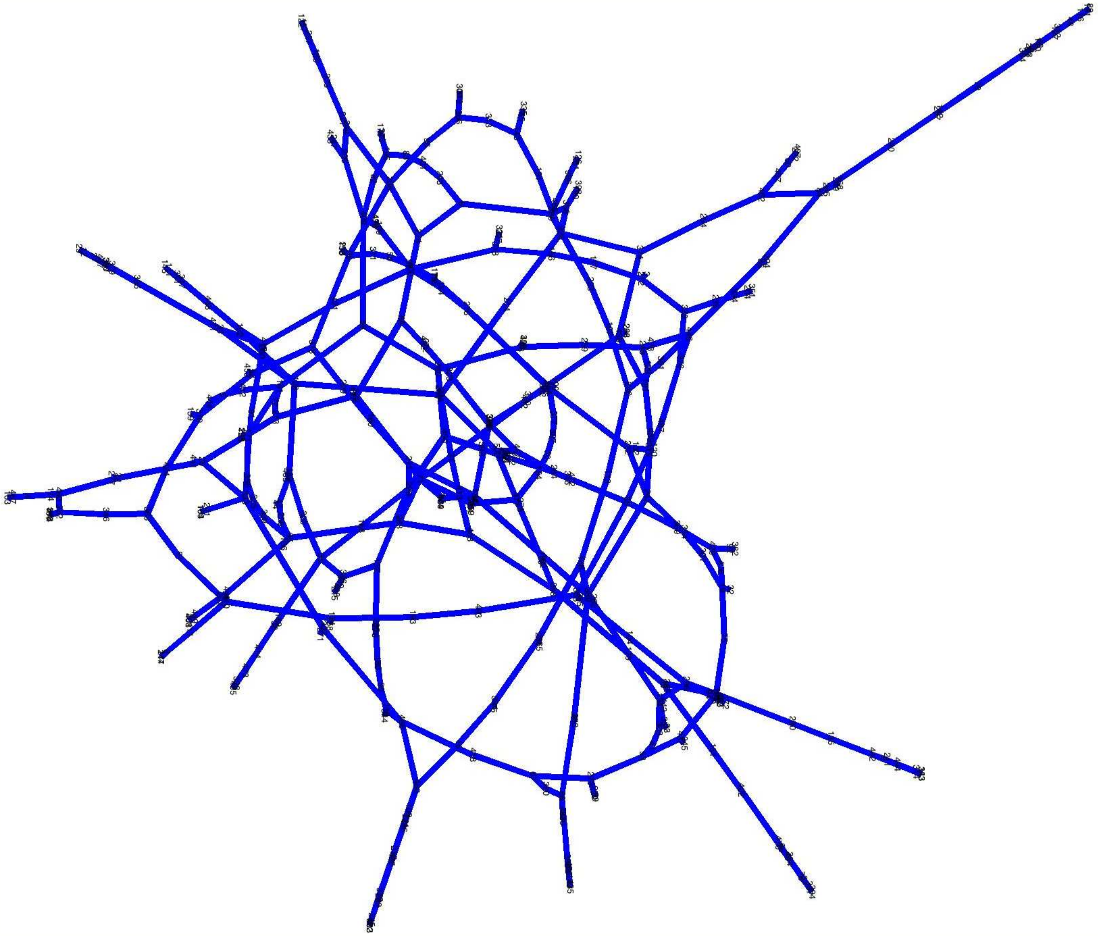
Spectral analysis of large sparse graphs

Justin Salez (LPMA)



Journées d'Informatique Fondamentale de Paris Diderot

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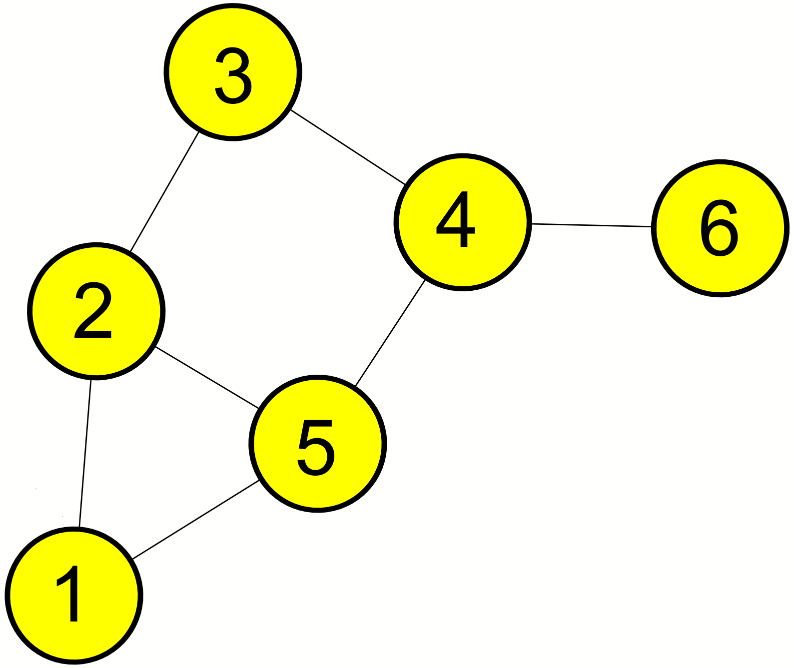
1. Spectra of sparse graphs : a few results and many problems
2. The notion of local weak convergence for graph sequences
3. Back to graph spectra : what have we gained ?

PART I : SPECTRA OF GRAPHS



0	1	0	0	1	0
1	0	1	0	1	0
0	1	0	1	0	0
0	0	1	0	1	1
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Question : How does the spectral distribution μ_G typically look when G is large ?

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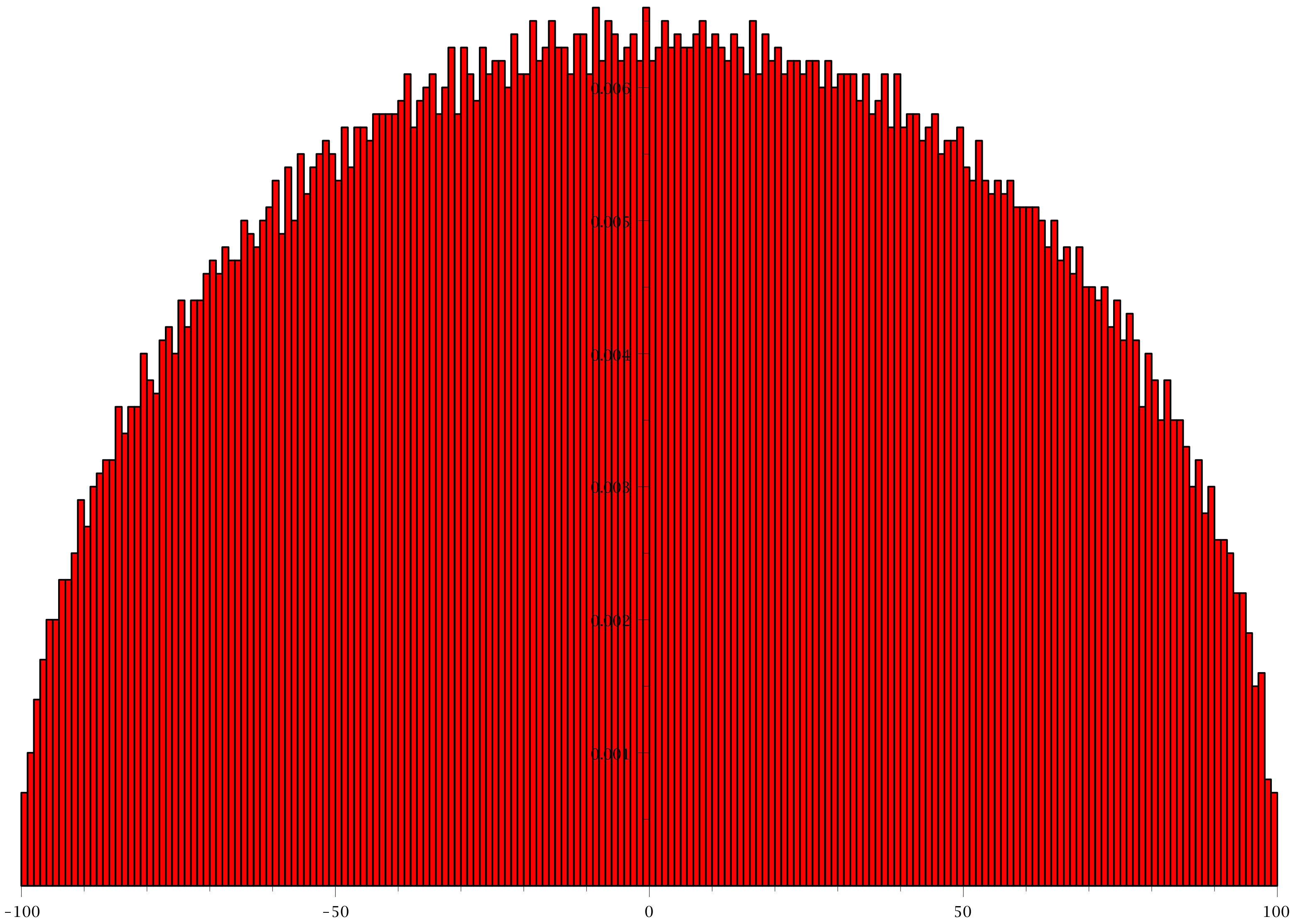

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$$\mu_{G_n}\left(\sqrt{np_n(1-p_n)}d\lambda\right) \xrightarrow[n\to\infty]{} \frac{\sqrt{4-\lambda^2}}{2\pi}\mathbf{1}_{(|\lambda|\le 2)}d\lambda.$$

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▷ What about sparse graphs $|E| \asymp |V|$?

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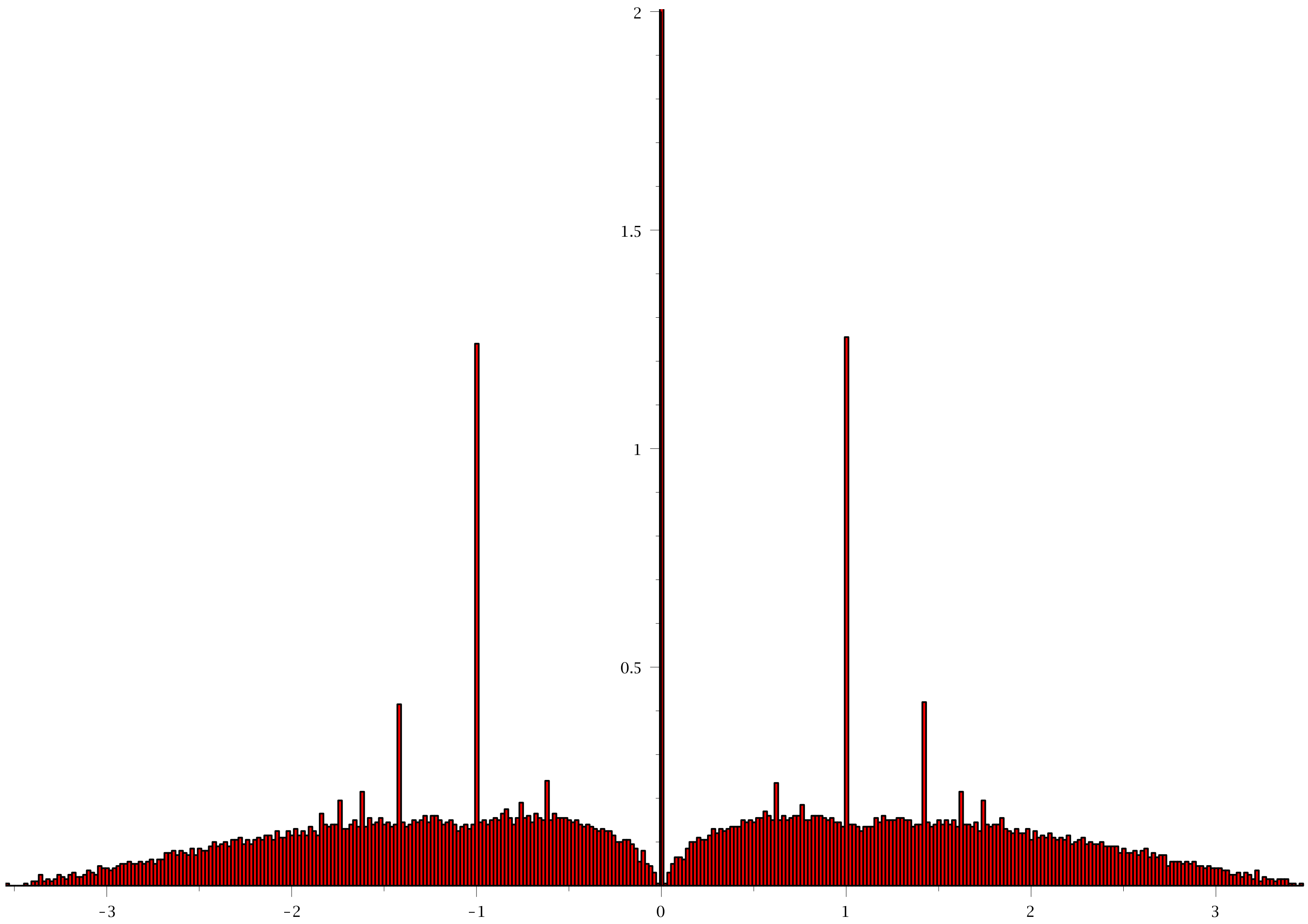

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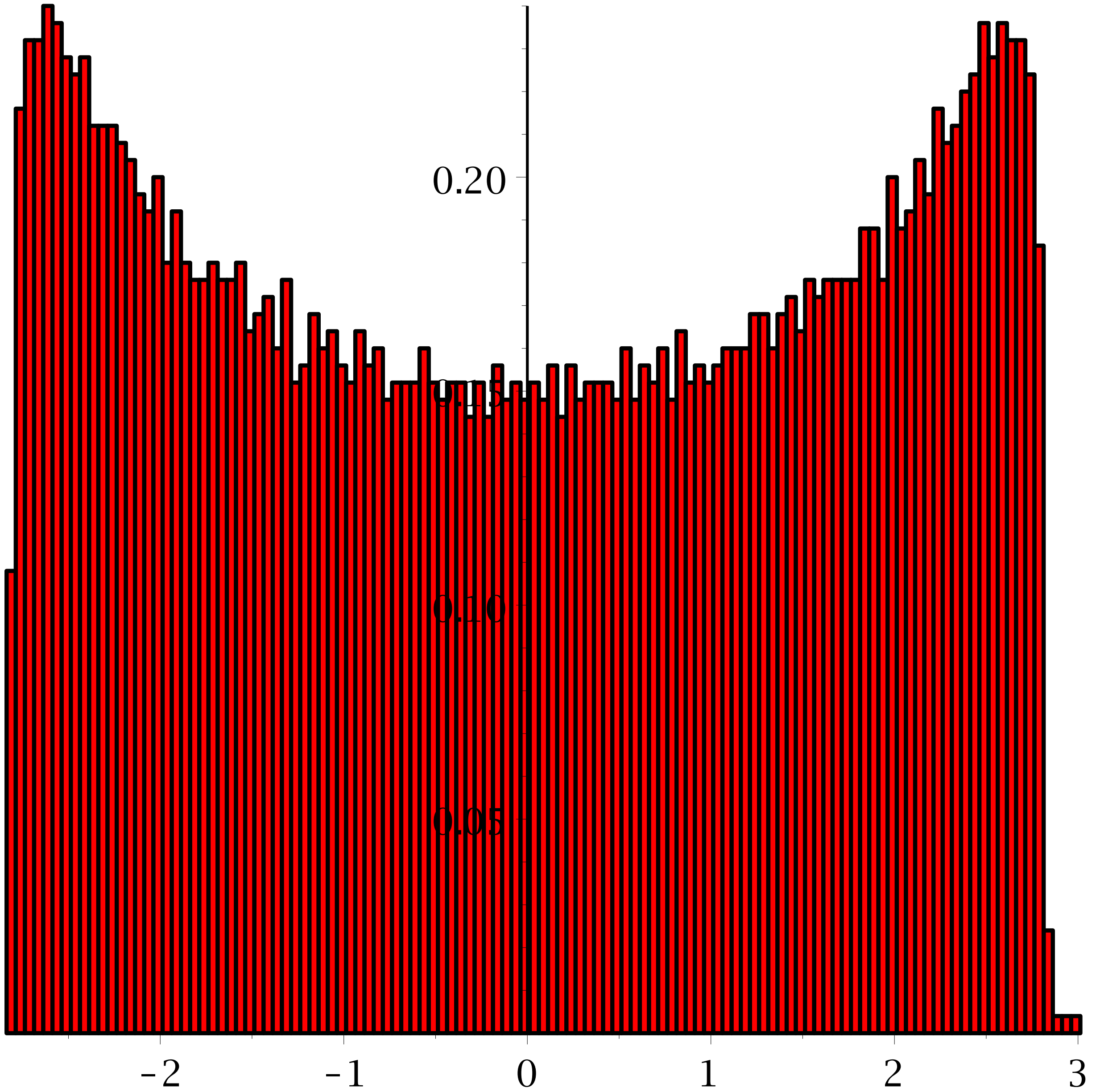

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Issue 3 : Practically nothing is known about $\mu_{\lim}$ (but plenty of fascinating conjectures).

PART II

LOCAL WEAK LIMITS OF FINITE GRAPHS

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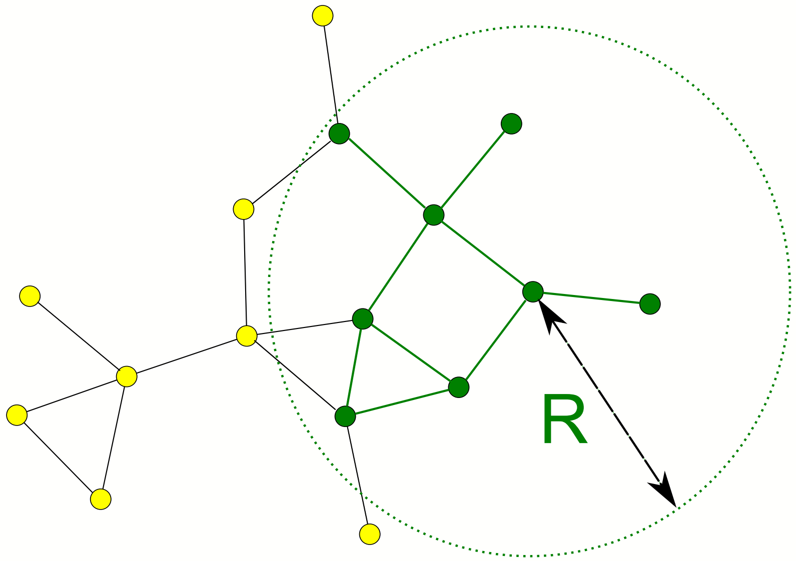

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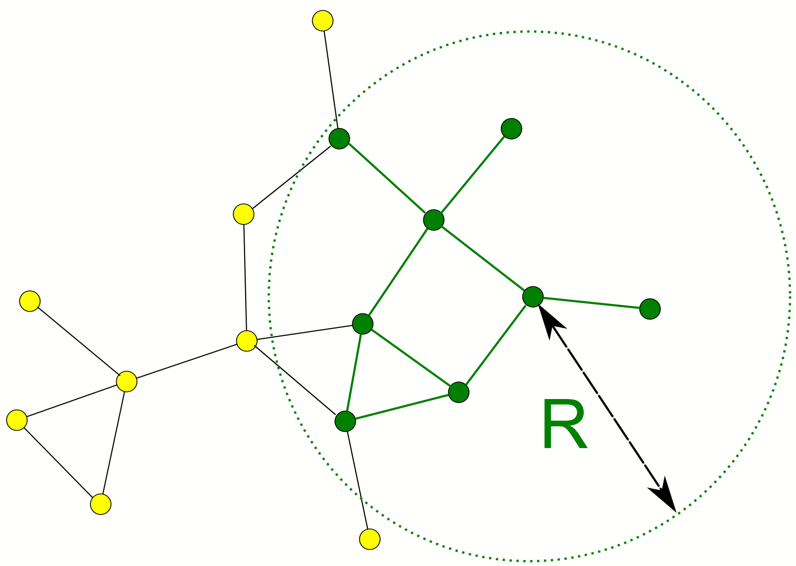

$(G_n, o_n) \to (G, o)$ if for each fixed radius $R \geq 1$, there is some $n_R \geq 1$ such that

$$\forall n \geq n_R,\ \mathrm{Ball}_{G_n}(o_n, R) \equiv \mathrm{Ball}_G(o, R).$$

$\mathrm{Ball}_G(o, R)$: ball of radius R around o in G. $\equiv$ root-preserving isomorphism.

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$\mathcal{L}$ is the asymptotic distribution of G_n when viewed locally from a uniformly chosen node.

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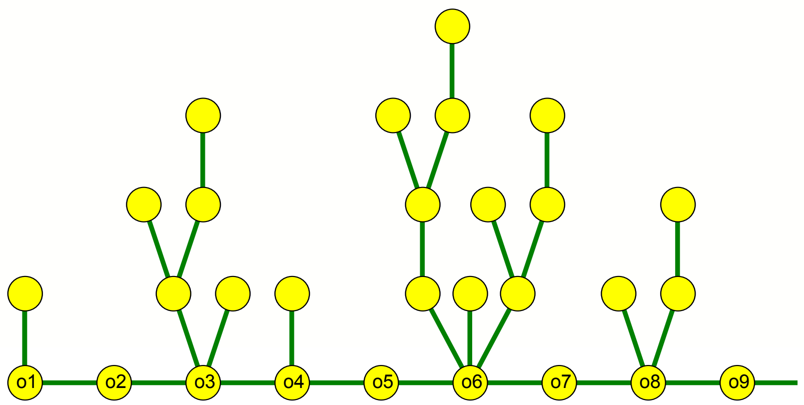

PART III

BACK TO GRAPH SPECTRA

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Implication 2 : opens the way to the study of $\mu_{\mathcal{L}}$, at least when $\mathcal{L}$ is simple.

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▷ If $\mathcal{L}$ is the law of a random locally finite rooted graph (G,o), define $\mu_{\mathcal{L}}$ by

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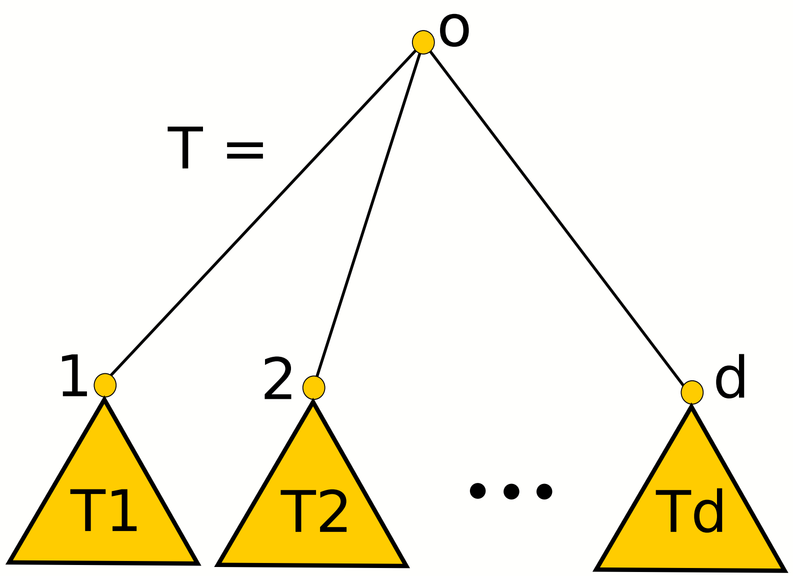

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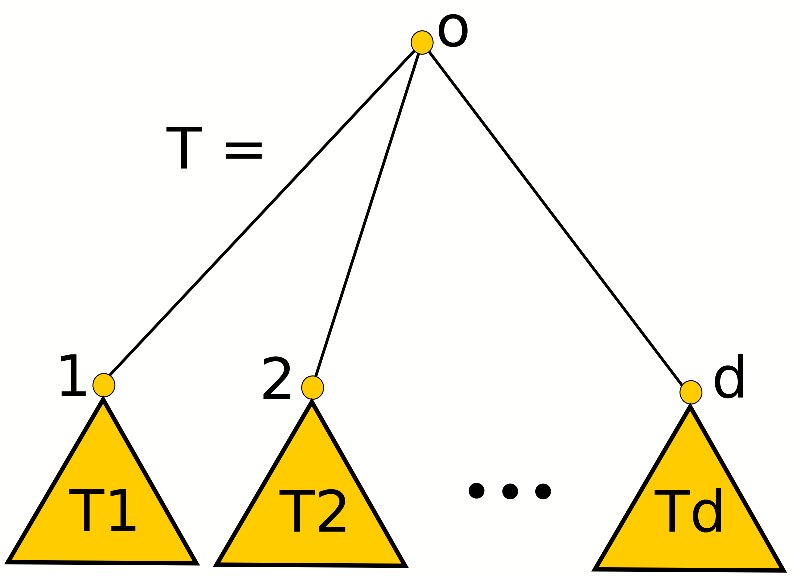

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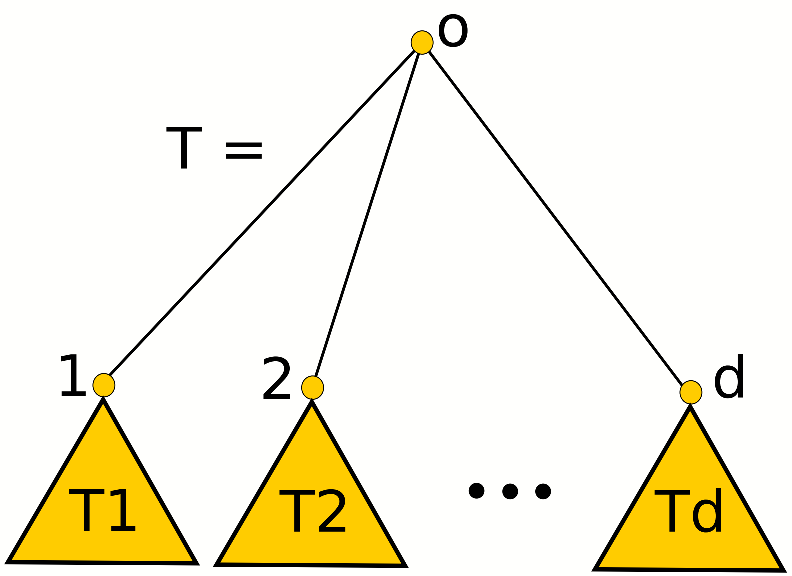

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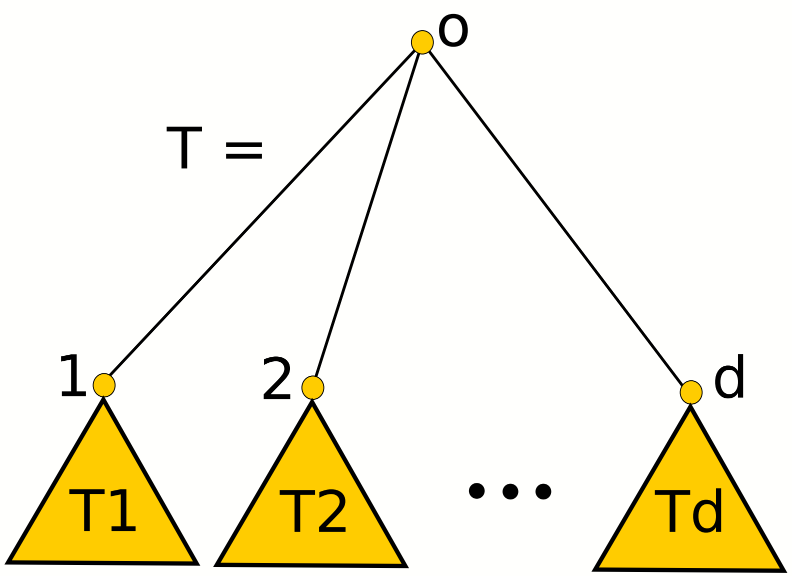

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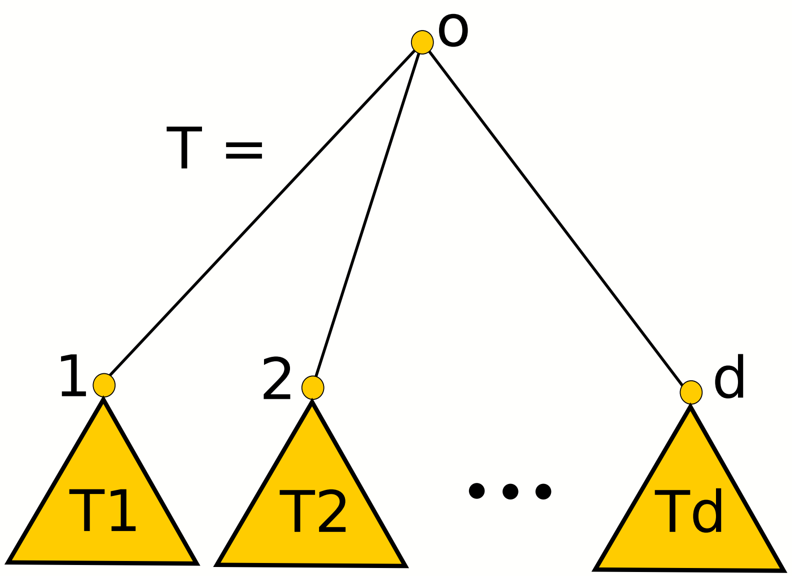

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$\triangleright$ In the case $\pi = \mathrm{Poisson(c)}$, we have $\phi(\lambda) = e^{c(\lambda-1)}$ and the conjecture follows.

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$\triangleright$ **Algorithmic implication :** ϕ is efficiently approximable via local, distributed algorithms, independently of the total size of the network.

THANK YOU FOR YOUR ATTENTION !