A tutorial on order- and arb-invariant logics

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In this talk

- consider finite relational structures $\mathcal{A} = (A, R_1^{\mathcal{A}}, \ldots, R_\ell^{\mathcal{A}})$ over a finite relational signature $\tau = \{R_1, \ldots, R_\ell\}$

- p is a τ-property, if the following is true for all finite τ-structures $\mathcal{A}$ and $\mathcal{B}$:

  if $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A}$ has property $p \iff \mathcal{B}$ has property p

- q is a k-ary τ-query, if the following is true:

  if $\pi : \mathcal{A} \cong \mathcal{B}$, then for all $a_1, \ldots, a_k \in A$,

  $$(a_1, \ldots, a_k) \in q(\mathcal{A}) \iff (\pi(a_1), \ldots, \pi(a_k)) \in q(\mathcal{B})$$

- I.e., τ-properties and queries are closed under isomorphisms.

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Logics expressing τ-properties and queries

Classical logics like, e.g.

- FO (first-order logic: Boolean combinations + quantification over nodes)
- LFP (least fixed point logic: FO + inductive definitions of relations)

express τ-properties and queries in a straightforward way.

Example: Consider graphs $\mathcal{A} = (A, E^{\mathcal{A}})$. The query

$$q(\mathcal{A}) \;=\; \{\, x \in A \;:\; x \text{ lies on a triangle} \,\}$$

is expressed in FO via

$$\varphi(x) \;:=\; \exists y\, \exists z\, \big(\, E(x,y) \,\wedge\, E(y,z) \,\wedge\, E(z,x) \,\big)$$

Drawback:

FO and LFP are too weak to express (some) computationally easy properties, e.g., properties concerning the size of A or $E^{\mathcal{A}}$.
Stronger logics like, e.g., SO or ESO can express computationally hard properties and queries.

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Overview

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Invariant logics

Idea:

- Extend the expressive power of a logic by allowing formulas to also use arithmetic predicates like $<, +, \times, \ldots, \mathit{Halt}, \ldots$ on A.
- For this, identify A with the set $[n] := \{0, 1, \ldots, n{-}1\}$ for $n = |A|$ and interpret $<, +, \times, \ldots, \mathit{Halt}, \ldots$ in the natural way.
- To ensure closure under isomorphisms, restrict attention to formulas independent of the particular way of identifying A with $[n]$. These formulas are called Arb-invariant.

Definition: Let $\mathcal{L}$ be a logic (e.g., FO, MSO, LFP).
A -formula $\varphi(\vec{x})$ is -invariant on $\mathcal{A} = (A, R_1^A, \ldots, R_\ell^A)$ $\iff$
for all tuples of elements $\vec{a}$ in A, for all linear orders $\prec_1$ and $\prec_2$ on A,

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Definition: Let $\mathcal{L}$ be a logic (e.g., FO, MSO, LFP).

A $\mathcal{L}(\tau, \prec)$-formula $\varphi(\vec{x})$ is order-invariant on $\mathcal{A} = (A, R_1^{\mathcal{A}}, \ldots, R_\ell^{\mathcal{A}}) \iff$ for all tuples of elements $\vec{a}$ in A, for all linear orders $\prec_1$ and $\prec_2$ on A,

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A $\mathcal{L}(\tau, \prec, +)$-formula $\varphi(\vec{x})$ is addition-invariant on $\mathcal{A} = (A, R_1^{\mathcal{A}}, \ldots, R_\ell^{\mathcal{A}})$ $\iff$ for all tuples of elements $\vec{a}$ in A, for all linear orders $\prec_1$ and $\prec_2$ on A, and the matching addition relations $+_1, +_2$,

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A $\mathcal{L}(\tau, \prec, +, \times)$-formula $\varphi(\vec{x})$ is $(+, \times)$-invariant on $\mathcal{A} = (A, R_1^{\mathcal{A}}, \ldots, R_\ell^{\mathcal{A}})$ $\iff$ for all tuples of elements $\vec{a}$ in A, for all linear orders $\prec_1$ and $\prec_2$ on A, and the matching addition relations $+_1, +_2$, and the according multiplications $\times_1, \times_2$,

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For Arb-invariant sentences, shortly write $\mathcal{A} \models \varphi$ for $(\mathcal{A}, \prec_1, +_1, \times_1 \ldots) \models \varphi$.

Example

- Let $\tau = \emptyset$. An addition-invariant FO$(\tau, \prec, +)$-sentence φ such that

$$\mathcal{A} \models \varphi \iff |A| \text{ is odd.}$$

$$\varphi \ := \ \exists x\, \exists z \Big(x + x = z \ \wedge \ \forall y \, (y \prec z \ \vee \ y = z) \Big)$$

- Similarly, there is an $(+, \times)$-invariant FO$(\tau, \prec, +, \times)$-sentence ψ such that

$$\mathcal{A} \models \psi \iff |A| \text{ is a prime number.}$$

- And there is an Arb-invariant FO$(\tau, \prec, \mathit{Halt})$-sentence χ such that

$$\mathcal{A} \models \chi \iff |A| - 1 \text{ is the index of a Turing machine halting on empty input :}$$

Simply choose $\chi \ := \ \exists x \, \big(\mathit{Halt}(x) \ \wedge \ \forall y \, (y \prec x \ \vee \ y = x) \big)$.

Thus:

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Expressive power of invariant logics

Known results:

- Order-invariant LFP precisely captures the polynomial time computable τ-properties and queries. (Immerman, Vardi, 1982)
- Arb-invariant LFP precisely captures the τ-properties and queries that belong to the comlexity class $P_{/\text{poly}}$. (Makowsky, 1998)

  $P_{/\text{poly}}$ consists of all problems solvable by circuit families of polynomial size
- Arb-invariant FO precisely captures the τ-properties and queries that belong to the circuit complexity class AC^0.

  AC^0 consists of all problems solvable by circuit families of polynomial size and constant depth
- $(+, \times)$-invariant FO precisely captures the τ-properties and queries that belong to uniform AC^0.

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Order-invariance is undecidable

Invariant logics are not logics in the strict formal sense:
They have an undecidable syntax. Precisely:

The following problem is undecidable (for binary symbol E and unary symbol C):

ORDER-INVARIANCE ON FINITE $\{E, C\}$-STRUCTURES:

Input: a FO($E, C, \prec$)-sentence φ

Question: Is φ order-invariant on all finite $\{E, C\}$-structures?

Proof: By a reduction using Trakhtenbrot's theorem.

- Assume, for contradiction, that order-invariance is decidable.
- Then, also the problem "Is a given FO(E)-sentence ψ true for all finite graphs?" is decidable as follows:
  (1) If there is a one-vertex-graph, in which ψ is not true, then stop with output "no". Otherwise, proceed with (2).
  (2) Let χ be a formula that is not order-invariant on structures of size $\geqslant 2$.
  E.g., $\chi := \exists x\big(C(x) \wedge \forall y(x \preceq y)\big)$.
  Stop with output "yes" iff the formula $(\neg\psi \rightarrow \chi)$ is order-invariant.

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Order- and addition-invariance for unary signatures

Let $\tau = \{C_1, \ldots, C_\ell\}$ consist of unary relation symbols.

Theorem: Order-invariance of a given FO(τ)-sentence φ
(on the class of all finite τ-structures) is decidable.

Decision procedure:

- φ defines a language L of finite strings.
- φ is order-invariant $\iff$ L is commutative.
- Commutativity of regular string-languages is decidable.

Theorem: Addition-invariance of a given FO($C, \prec, +$)-sentence φ
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Overview

FO $<$ order-invariant FO

Theorem (Gurevich):

Let $\tau := \{\subseteq\}$ be a signature consisting of a single binary relation symbol $\subseteq$.
For a finite set X let $\mathcal{B}_X := (2^X, \subseteq)$ be the Boolean algebra over X.

There is an order-invariant FO($\tau, \prec$)-sentence φ_{even} such that for every finite set X:

$$\mathcal{B}_X \models \varphi_{even} \iff |X| \text{ is even.}$$

But there is no FO(τ)-sentence ψ_{even} such that for every finite set X:

$$\mathcal{B}_X \models \psi_{even} \iff |X| \text{ is even.}$$

Proof:

Part 1: φ_{even} expresses that there is a set z that contains the first (w.r.t. $\prec$) atom of X, every other (w.r.t. $\prec$) atom of X, but not the last (w.r.t. $\prec$) atom of X.

Part 2: Use an Ehrenfeucht-Fraïssé game argument to show that $\mathcal{B}_X \equiv_r \mathcal{B}_Y$ for all finite X, Y of cardinality $> 2^r$. □

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Successor-invariant FO

By a much more elaborate construction, one can also show:

Theorem (Rossman, LICS'03)

On the class of all finite structures,
successor-invariant FO is strictly more expressive than FO.

FO+MOD$_2$ $<$ order-invariant FO+MOD$_2$

FO+MOD$_2$: the extension of FO by modulo 2 counting quantifiers

$\exists^{r \bmod 2} x\, \psi(x)$: the number of nodes x satisfying $\psi(x)$ is congruent r modulo 2.

Theorem (Niemistö):

Let $\tau := \{E\}$ be a signature consisting of a single binary relation symbol E.

There is an order-invariant FO+MOD$_2$(E)-sentence $\varphi_{\textit{even cycles}}$ that is satisfied by a finite directed graph $G = (V, E)$ iff

(1) G is a disjoint union of directed cycles, and

(2) the number of even-length cycles is even.

Proof:

- (1) can be expressed in FO: "every node has in- and out-degree 1"
- Every G satisfying (1) is the cycle decomposition of a permutation π.
- G has an even number of even-length cycles $\iff$
  π is an even permuatation, i.e., sgn(π) $= 1$ $\iff$
  π has an even number of inversions (i, j) such that $i < j$ and $\pi(i) > \pi(j)$.

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Order-invariant MSO

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$\exists^{r \bmod m} x\ \psi(x)$: the number of nodes x satisfying $\psi(x)$ is congruent r modulo m.

Known:

- On trees:
  Order-invariant MSO $=$ CMSO (Courcelle 1996, Lapoire 1998)

- On the class of all finite structures:
  Order-invariant MSO $>$ CMSO (Ganzow, Rubin 2008)

  The separating example:

  - Consider 2-dimensional grids, represented as structures of the form $(A, Same\_Row, Same\_Column)$.
  - Order-invariant MSO can express that the number of columns is a multiple of the number of rows.
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Overview

Neighborhoods

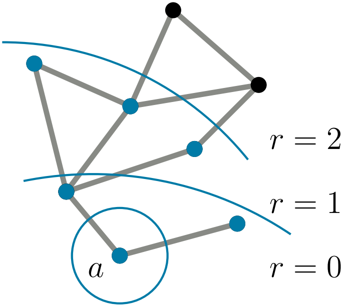
Graph $G = (V, E)$

Distance $dist(u, v)$: length of a shortest path between u, v in G.

Shell $S_r(a)$ of nodes at distance exactly r from a.

Ball $N_r(a)$ of radius r at a in G.

Neighborhood $\mathcal{N}_r(a)$ of radius r at a in G.



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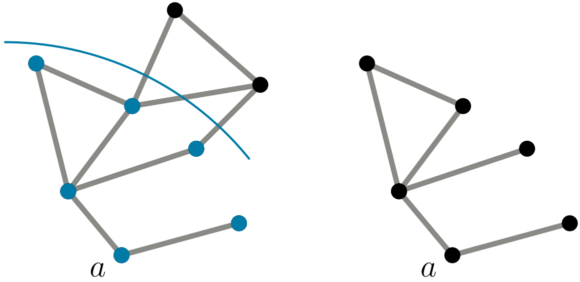

Gaifman-local queries

- For a list $a = a_1, \ldots, a_k$ of nodes, $N_r^G(a) = N_r^G(a_1) \cup \cdots \cup N_r^G(a_k)$.
- The r-neighborhood $\mathcal{N}_r^G(a)$ is the structure $(G_{|N_r^G(a)}, a)$ consisting of the induced subgraph of G on $N_r^G(a)$, together with the distinguished nodes a.

Definition: Let q be a k-ary graph query. Let $f : \mathbb{N} \to \mathbb{N}$.
q is called $f(n)$-local if there is an n_0 such that for every $n \geqslant n_0$ and every graph G with $|V^G| = n$, the following is true for all k-tuples a and b of nodes:

$$\text{if} \quad \mathcal{N}_{f(n)}^G(a) \cong \mathcal{N}_{f(n)}^G(b) \quad \text{then} \quad a \in q(G) \iff b \in q(G).$$

Gaifman-locality of FO

Theorem:

- For every graph query q that is FO-definable,
  there is a constant c such that q is c-local.
  (Hella, Libkin, Nurmonen 1990s; Gaifman '82)

- For every graph query q that is FO-definable on ordered graphs
  (for short: q is definable in order-invariant FO),
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  (Grohe, Schwentick '98)

- For every graph query q that is FO-definable on graphs with arbitrary numerical predicates (for short: q is definable in Arb-invariant FO),
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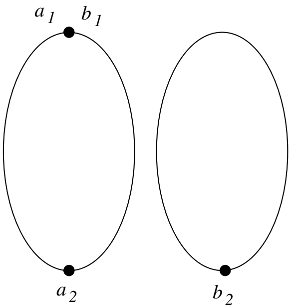
Use locality for proving non-expressibility

Example: The reachability query

$$\text{REACH}(G) := \{ (a_1, a_2) : \text{there is a directed path from } a_1 \text{ to } a_2 \text{ in } G \}$$

is not $\frac{n}{5}$-local an thus cannot be expressed in Arb-invariant FO.

Proof: Consider the graph G:



Use locality for proving non-expressibility

Similarly, one obtains that the following queries are not definable in Arb-invariant FO:

- Does node x lie on a cycle?
- Does node x belong to a connected component that is acyclic?
- Is node x reachable from a node that belongs to a triangle?
- Do nodes x and y have the same distance to node z?

Proof of Gaifman-locality theorem (1/5)

For every query q expressible by Arb-invariant FO, there is a $c \in \mathbb{N}$ such that q is $(\log n)^c$-local.

Idea: Use known lower bounds in circuit complexity!

- Let q be expressible by an Arb-invariant FO formula.
- Then, q can be computed by an AC^0 circuit family $\mathcal{C}$ (Immerman '87).
- Assume that q is *not* $(\log n)^c$-local (for any $c \in \mathbb{N}$), and modify $\mathcal{C}$ to obtain an AC^0 circuit family computing

$$\text{PARITY} := \{w \in \{0,1\}^* : |w|_1 \text{ is even}\}.$$

- This contradicts known lower bounds in circuit complexity theory (Håstad'86).

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Proof of Gaifman-locality theorem (2/5)

How to compute a graph query $q(x)$ by an AC^0 circuit family $\mathcal{C}$?

- Represent graph $G = (V, E)$ by a bitstring $\beta(G)$ corresponding to an adjacency matrix for G.
- Represent a node $a \in V$ by the bitstring $\beta(a)$ of the form 0^*10^*, carrying the 1 at position i iff node a corresponds to the i-th row/column of the adjacency matrix.
- Let $Rep(G, a)$ be the set of all bitstrings $\beta(G)\beta(a)$, corresponding to all adjacency matrices of G (i.e., all ways of embedding V in $\{1, \dots, |V|\}$). Thus, $Rep(G, a)$ is the set of all bitstrings representing (G, a).
- A unary graph query $q(x)$ is computed by a circuit family $\mathcal{C} = (C_n)_{n \in \mathbb{N}}$ iff the following is true:
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- *Known:* A unary graph query $q(x)$ is definable in Arb-invariant FO $\iff$ it is computed by a circuit family of constant depth and polynomial size.
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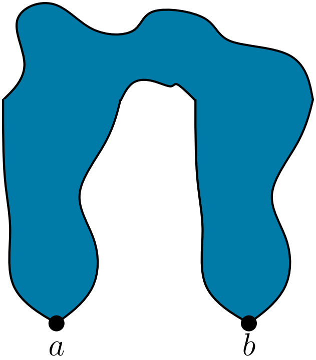

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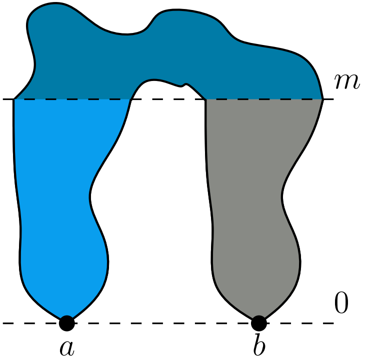

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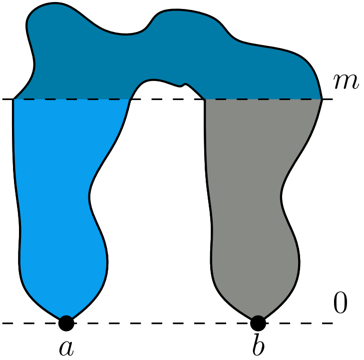

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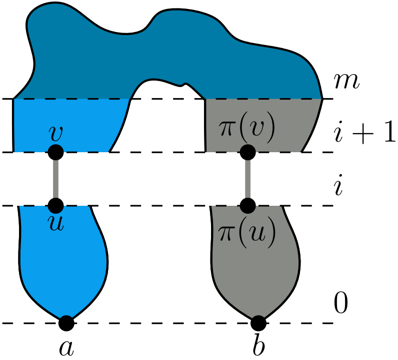

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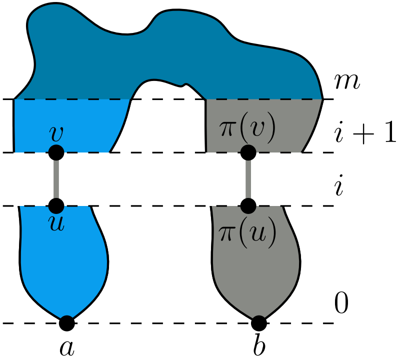

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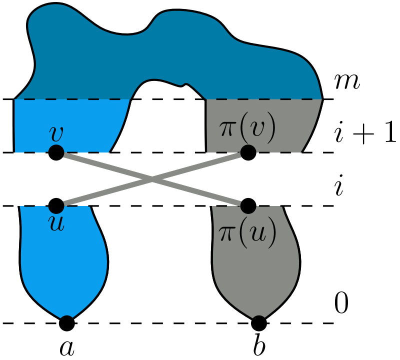

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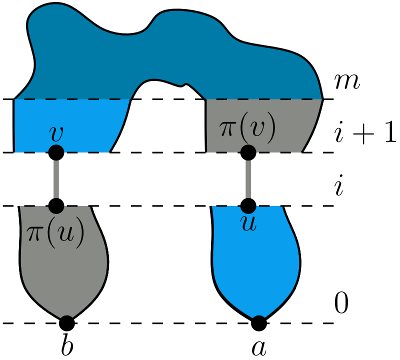

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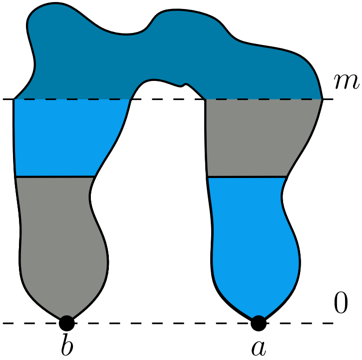

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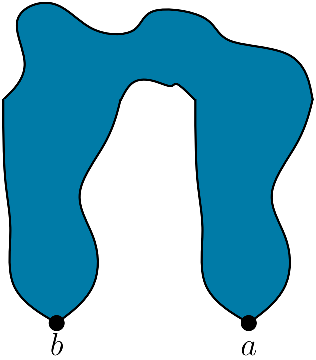

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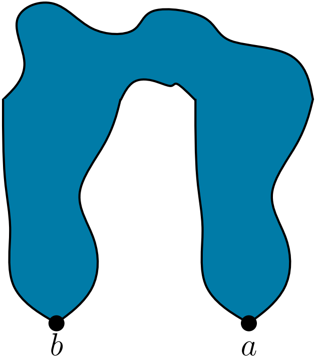

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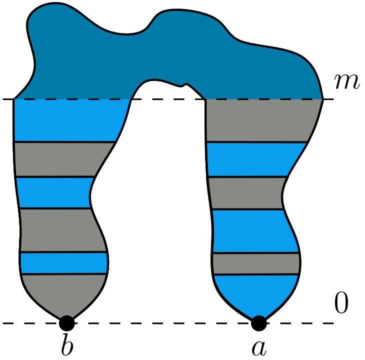

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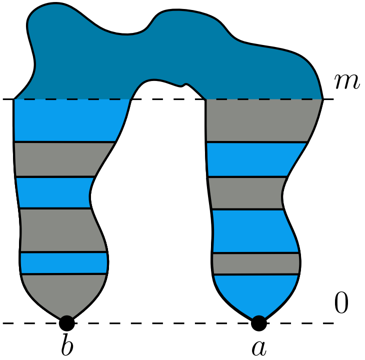

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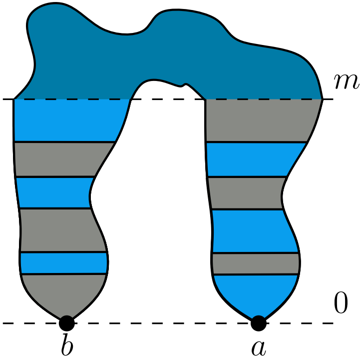

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- Consider C for a fixed input string $\gamma \in Rep(G, a)$.
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$$e \text{ by } (e \wedge \neg w_i) \qquad e' \text{ by } (e' \wedge \neg w_i)$$
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Summary: Gaifman-locality of Arb-invariant FO (1/2)

Theorem: *(Anderson, Melkebeek, S., Segoufin '11)*

(a) *For every query q expressible by Arb-invariant FO there is a $c \in \mathbb{N}$ such that q is $(\log n)^c$-local.*

(b) *For every $d \in \mathbb{N}$ there is a $(+,\times)$-invariant FO query that is not $(\log n)^d$-local.*

The query $q_d(x)$ states:

(1) The graph has at most $(\log n)^{d+1}$ non-isolated vertices.

(Use the polylog-counting capability of FO$(+,\times)$)

(2) Node x is reachable from a node that belongs to a triangle.

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Goal: Show that in graphs with $\leqslant (\log n)^c$ non-isolated vertices, reachability by paths of length $(\log n)^c$ can be expressed in $(+, \times)$-invariant FO.

Lemma: *(Durand, Lautemann, More '07)*

For every $c \in \mathbb{N}$ there is a $\mathrm{FO}(<, +, \times, S)$*-formula $bij_c(x, y)$ such that for all $n \in \mathbb{N}$, all $S \subseteq [n] := \{0, \ldots, n-1\}$, all $a, i < n$ we have*

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Locality of Arb-invariant FO+MOD$_p$

In a similar way, we can also prove:

Theorem: *(Harwath, S., 2013)*

Let p be a prime power and let $k \in \mathbb{N}$ be coprime with p.

For every k-ary query q expressible in Arb-invariant FO+MOD$_p$, there is a $c \in \mathbb{N}$ such that q is $(\log n)^c$-shift-local w.r.t. k.

Definition: Let q be a k-ary graph query. Let $f : \mathbb{N} \to \mathbb{N}$.

q is called $f(n)$-shift-local w.r.t. k if there is an n_0 such that for every $n \geqslant n_0$ and every graph G with $|V^G| = n$, the following is true for all k-tuples $(a_0, \ldots, a_{k-1})$ of nodes:

if the $f(n)$-neighborhoods of the a_i are disjoint and isomorphic,

then $(a_0, a_1, \ldots, a_{k-1}) \in q(G) \iff (a_1, \ldots, a_{k-1}, a_0) \in q(G)$.

Proof: Use Smolensky's result for AC$^0[p]$-circuits.

Corollary: Reachability is not definable in Arb-invariant FO+MOD$_p$ (for prime power p).

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then $(a_0, a_1, \ldots, a_{k-1}) \in q(G) \iff (a_1, \ldots, a_{k-1}, a_0) \in q(G)$.

Proof: Use Smolensky's result for AC$^0[p]$-circuits.

Corollary: Reachability is not definable in Arb-invariant FO+MOD$_p$ (for prime power p).

Locality of Arb-invariant FO+MOD$_p$

In a similar way, we can also prove:

Theorem: *(Harwath, S., 2013)*

Let p be a prime power and let $k \in \mathbb{N}$ be coprime with p.

For every k-ary query q expressible in Arb-invariant FO+MOD$_p$, there is a $c \in \mathbb{N}$ such that q is $(\log n)^c$-shift-local w.r.t. k.

Definition: Let q be a k-ary graph query. Let $f : \mathbb{N} \to \mathbb{N}$.

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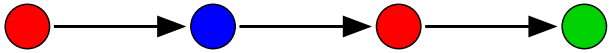
Overview

Represent words as labeled graphs

(labeled) chain-graphs . . . . this chain-graph represents the string *rbrg*.



Edges correspond to the successor relation "*succ*" on the positions of the string.

Write $\prec$ -inv-FO(*succ*) for order-invariant FO on these graphs.

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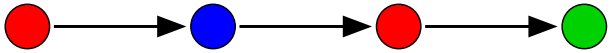

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Known results

$\mathrm{FO}(\mathit{succ}) = $ locally threshold testable languages (Thomas)

$\mathrm{FO}(<) = $ star-free regular languages (McNaughton, Papert)

$\mathrm{MSO}(<) = $ regular languages (Büchi, Elgot, Trakhtenbrot)

$\mathrm{MSO}(<) = \mathrm{MSO}(\mathit{succ}) = \mathrm{MLFP}(<) = \mathrm{MLFP}(\mathit{succ})$

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$+$-inv-$\mathrm{MLFP}(\mathit{succ}) \supseteq$ DLIN (S., 2004)

$+$-inv-$\mathrm{MSO}(\mathit{succ}) = $ the linear time hierarchy (More & Olive 1997, S. 2004)

$\mathrm{FO}(<) = \prec\text{-inv-}\mathrm{FO}(<) \subsetneq +\text{-inv-}\mathrm{FO}(<) \subsetneq (+,\times)\text{-inv-}\mathrm{FO}(<) \subseteq$ uniform AC^0.

The following problem is undecidable: (Benedikt, Segoufin, 2005)

ORDER-INVARIANCE ON FINITE LABELED CHAIN-GRAPHS:

Input: a $\mathrm{FO}(\prec, E)$-sentence φ

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The “Algebraic” Approach

Let L_1 and L_2 be logics, and let C be a class of structures.

Goal: Show that L_1 can define exactly the same properties of C-structures as L_2.

Approach:

(0) Identify a suitable set of operations $\mathcal{O}$ on structures in C.

(1) Show that a property p of C-structures is definable in L_1 iff it is closed under every operation $op \in \mathcal{O}$. I.e., for every $\mathcal{A} \in C$:

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Theorem (Benedikt, Segoufin, '09):
A string-language is definable in $\prec$-inv-FO(*succ*) iff it is definable in FO(*succ*).

Main ingredients of the proof:

- Use a result by Beauquier and Pin (1989) stating that a string-language is definable in FO(*succ*) iff it is aperiodic and closed under swaps.
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Some further results proved using this method

Theorem:

- A tree-language is definable in $<$-invariant FO(*Succ*) iff it is definable in FO(*Succ*). (Benedikt, Segoufin '09)
  (They use aperiodicity and closure under guarded swaps.)

- A colored finite set is definable in $+$-invariant FO iff it is definable in $\mathrm{FO}_{\mathrm{card}}$ (i.e., FO with predicates testing the cardinality of the universe modulo fixed numbers). (S., Segoufin '10)

- A regular string- or tree-language is definable in $+$-invariant FO(*Succ*) iff it is definable in $\mathrm{FO}_{\mathrm{card}}(succ)$. (S., Segoufin '10 and Harwath, S. '12)
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An open question

Open Question:

Are all languages definable in addition-invariant FO regular?

Known: (S., Segoufin, 2010)

- Arb-invariant FO can define non-regular languages, e.g.,
  $L = \{w \in \{1\}^* : |w| \text{ is a prime number }\}$.
- Every deterministic context-free language definable in addition-invariant FO is regular.
- Every commutative language definable in addition-invariant FO is regular.
- Every bounded language definable in addition-invariant FO is regular.

Bounded languages

Definition: (Ginsburg & Spanier, 1964)

$L \subseteq \Sigma^*$ is bounded $\iff$

$\exists\, k \in \mathbb{N}$ and k strings $w_1, \ldots, w_k \in \Sigma^*$ such that $L \subseteq w_1^* w_2^* \cdots w_k^*$.

Theorem: *(S., Segoufin, 2010)*

Every bounded language definable in $+$-inv-FO($<$) is regular.

Proof method:

- Identify $w_1^* w_2^* \cdots w_k^*$ with $\mathbb{N}^k$ via $(x_1, \ldots, x_k) \in \mathbb{N}^k \;\widehat{=}\; w_1^{x_1} w_2^{x_2} \cdots w_k^{x_k}$.
  Thus: $L \subseteq w_1^* w_2^* \cdots w_k^* \;\widehat{=}\; S(L) \subseteq \mathbb{N}^k$.
- Note that $S(L)$ is semi-linear, since L is definable in $+$-inv-FO($<$).
- Reason about semi-linear sets …

Corollary:

Every commutative language definable in $+$-inv-FO($<$) is regular.

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Characterization of colored sets definable in $+$-inv-FO

Definition: A colored finite set is a finite relational structure over a finite signature that contains only unary relation symbols.

> *Theorem:* *(S., Segoufin, 2010)*
>
> *Over the class of colored finite sets, $+$-inv-FO$(=)$ and $\mathrm{FO}_{\mathrm{Card}}(=)$ have the same expressive power.*

Proof:

- Every $+$-inv-FO$(=)$ sentence over colored sets defines a commutative language.
- Every commutative language definable in $+$-inv-FO$(<)$ is regular.
- Every regular language definable in $+$-inv-FO$(=)$ is definable in $\mathrm{FO}_{\mathrm{Card}}(=)$.

□

Characterization of colored sets definable in $+$-inv-FO

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Theorem: *(S., Segoufin, 2010)*

Over the class of colored finite sets, $+$-inv-FO$(=)$ and $\mathrm{FO}_{\mathrm{Card}}(=)$ have the same expressive power.

Note: $\mathrm{FO}_{\mathrm{Card}}(=)$ is a logic (with a decidable syntax); $+$-inv-FO$(=)$ is not.

More precisely: The following problem is undecidable:

Input: a FO$(\prec, +, C)$-sentence φ (C a unary relation symbol)

Question: Is φ addition-invariant on all finite $\{C\}$-structures ?

Overview

Gaifman-locality

If $\mathcal{N}_r^G(a) \cong \mathcal{N}_r^G(b)$ then $\big(a \in q(G) \iff b \in q(G)\big)$.

Known:

- Queries definable in order-invariant FO are Gaifman-local with respect to a constant locality radius. (Grohe, Schwentick '98)
- Queries definable in Arb-invariant FO are Gaifman-local with respect to a poly-logarithmic locality radius. (Anderson, Melkebeek, S., Segoufin '11)

Open Question:

- How about addition-invariant FO — is it Gaifman-local with respect to a constant locality radius?

Hanf-locality

A graph property p is Hanf-local w.r.t. locality radius r, if
any two graphs having the same r-neighbourhood types with the same multiplicities,
are not distinguished by p.

Known:

- Properties of graphs definable in FO are Hanf-local w.r.t. a constant locality radius. (Fagin, Stockmeyer, Vardi '95)
- Properties of strings or trees definable by order-invariant FO are Hanf-local w.r.t. a constant locality radius. (Benedikt, Segoufin '09)
- Properties of strings definable by Arb-invariant FO are Hanf-local w.r.t. a poly-logarithmic locality radius. (Anderson, van Melkebeek, S., Segoufin '11)
- Properties of strings definable by Arb-invariant FO+MOD$_p$, for odd prime powers p, are Hanf-local w.r.t. a poly-logarithmic locality radius. (Harwath, S. '13)

Open Question:

- Which of these results generalise from strings to arbitrary finite graphs?

Decidable Characterisations

Open Question:

Are there decidable characterisations of

- order-invariant FO?
- addition-invariant FO?
- $(+, \times)$-invariant FO?

Known:

- On finite strings and trees: order-invariant FO $\equiv$ FO. (Benedikt, Segoufin '10)
- On finite coloured sets: addition-invariant FO $\equiv$ FO enriched by "cardinality modulo" quantifiers. (S., Segoufin '10)

Thank You!