REGULARITIES OF THE DISTRIBUTION OF β-ADIC VAN DER CORPUT SEQUENCES

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Abstract. For Pisot numbers β with irreducible β-polynomial, we prove that the discrepancy function $D(N,[0,y))$ of the β-adic van der Corput sequence is bounded if and only if the β-expansion of y is finite or its tail is the same as that of the expansion of 1. If β is a Parry number, then we can show that the discrepancy function is unbounded for all intervals of length $y \notin \mathbb{Q}(\beta)$. We give explicit formulae for the discrepancy function in terms of lengths of iterates of a reverse β-substitution.

1. Introduction

Let $(x_n)_{n\ge 0}$ be a sequence with $x_n \in [0,1)$ and

$$D(N,I) = \#\{0 \le n < N:\ x_n \in I\} - N\lambda(I)$$

its *discrepancy function* on the interval I, where $\lambda(I)$ denotes the length of the interval. Then $(x_n)_{n\ge 0}$ is uniformly distributed if and only if $D(N,I) = o(N)$ for all intervals $I \subseteq [0,1)$. Van Aardenne-Ehrenfest [25] proved that the discrepancy function cannot be bounded (in N) for all intervals $I \subseteq [0,1)$. W.M. Schmidt showed in [23] that the set of lengths of intervals with bounded discrepancy function, $\{\lambda(I):\ \sup_{N\ge 0} D(N,I) < \infty\}$, is at most countable and in [22] that $\sup_{I\subseteq[0,1)} D(N,I) \ge C \log N$ for some constant $C > 0$. For more details on the discrepancy, see Drmota and Tichy [4].

For some special sequences, the intervals with bounded discrepancy function were determined. If $x_n = \{n\alpha\}$, then $D(N,I)$ is bounded if and only if $\lambda(I) = \{m\alpha\}$ for some $m \ge 0$ (Hecke [10] and Kesten [13]). More generally, Rauzy [18] and Ferenczi [8] characterized bounded remainder sets for irrational rotations on the torus $\mathbb{T}^s$. Liardet [14] extended Hecke's and Kesten's result on these rotations and studied bounded remainder sets for $x_n = \{p(n)\}$, where $p(n)$ is a real polynomial with irrational leading coefficient, as well as for q-multiplicative sequences.

If $(x_n)_{n\ge 0}$ is the van der Corput sequence in base q, then $D(N,I)$ is bounded if and only if $\lambda(I)$ has finite q-ary expansion (W.M. Schmidt [23] and Shapiro [24] for $q = 2$, Hellekalek [11] for integers $q \ge 2$). Faure extended this result in [6] on generalized van

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der Corput sequences and recently in [7] on digital $(0,1)$-sequences over $\mathbb{Z}_q$ generated by a nonsingular upper triangular matrix where q is a prime number (see also Drmota, Larcher and Pillichshammer [3]). Hellekalek [12] considered generalizations of the Halton sequences in higher dimensions.

The aim of this article is to determine the intervals with bounded discrepancy function for the β-adic van der Corput sequences, which were introduced by Ninomiya [15] who proved that these sequences are low discrepancy sequences, i.e. $\sup_{I\subseteq[0,1)} D(N,I) = \mathcal{O}(\log N)$, if β is a Pisot number with irreducible β-polynomial.

For a given real number $\beta > 1$, the *expansion of 1* with respect to β is the sequence of nonnegative integers $(a_j)_{j\geq 1}$ satisfying

$$1 = .a_1a_2\ldots = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots \text{ with } a_ja_{j+1}\ldots < a_1a_2\ldots \text{ for all } j \geq 2$$

(Throughout this article, let $<$ denote the lexicographical order for words.) For $x \in [0,1)$, the *β-expansion* of x, introduced by Rényi [19] and characterized by Parry [16], is given by

$$x = .\epsilon_1\epsilon_2\ldots = \frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \cdots \text{ with } \epsilon_j\epsilon_{j+1}\ldots < a_1a_2\ldots \text{ for all } j \geq 1.$$

The elements of the *β-adic van der Corput sequence* $(x_n)_{n\geq 0}$ are the real numbers $x \in [0,1)$ with finite β-expansion,

$$\{x_n : n \geq 0\} = \{.\epsilon_1\epsilon_2\ldots : \epsilon_j\epsilon_{j+1}\ldots < a_1a_2\ldots \text{ for all } j \geq 1,\ \epsilon_\ell\epsilon_{\ell+1}\ldots = 0^\infty \text{ for some } \ell \geq 1\},$$

ordered lexicographically with respect to the (inversed) word $\ldots\epsilon_2\epsilon_1$, i.e. for $x_n = .\epsilon_1\epsilon_2\ldots$ and $x_{n'} = .\epsilon'_1\epsilon'_2\ldots$, we have $n < n'$ if we have some $k \geq 1$ such that $\epsilon_k < \epsilon'_k$ and $\epsilon_j = \epsilon'_j$ for all $j > k$.

If the expansion of 1 is finite, $a_1a_2\ldots = a_1\ldots a_d0^\infty$, or eventually periodic, $a_1a_2\ldots = a_1\ldots a_{d-p}(a_{d-p+1}\ldots a_d)^\infty$, then β is a *Parry number* and it is the dominant root of the *β-polynomial* $x^d - a_1x^{d-1} - \cdots - a_d$ (with $a_d > 0$) and $(x^d - a_1x^{d-1} - \cdots - a_d) - (x^{d-p} - a_1x^{d-p-1} - \cdots - a_{d-p})$ (where p is assumed to be minimal) respectively. In this case, we obtain results for the discrepancy function.

Theorem 1. *If β is a Parry number and $D(N,I)$ is bounded (in N), then $\lambda(I) \in \mathbb{Q}(\beta)$.*

Bertrand [1] and K. Schmidt [21] proved that all *Pisot number*s (algebraic integers for which all algebraic conjugates have modulus < 1) are Parry numbers. If furthermore the β-polynomial is the minimal polynomial of β, then we can completely characterize the intervals $[0,y)$ with bounded discrepancy function.

Theorem 2. *If β is a Pisot number with irreducible β-polynomial, then $D(N,[0,y))$ is bounded (in N) for $y \in [0,1)$ if and only if the β-expansion of y is finite or its tail is the same as that of the expansion of 1 with respect to β, i.e. if $y = .y_1y_2\ldots$ with $y_ky_{k+1}\ldots = 0^\infty$ or $y_ky_{k+1}\ldots = (a_{d-p+1}\ldots a_d)^\infty$ for some $k \geq 1$.*

Remark. Another way to formulate the condition on y is: the infinite β-expansion of y has the same tail as the infinite expansion of 1 (which is $1 = .(a_1\ldots a_{d-1}(a_d-1))^\infty$ if $1 = .a_1\ldots a_d$).

The classification for general intervals I seems to be more difficult. Of course, $D(N,[y,y'))$ is bounded if $D(N,[0,y))$ and $D(N,[0,y'))$ are bounded because of $D(N,[y,y')) = D(N,[0,y')) - D(N,[0,y))$. From the proof of Theorem 2 we see that $D(N,[y,y'))$ is bounded if $y = .y_1y_2\ldots$ and $y' = .y'_1y'_2\ldots$ with $y_ky_{k+1}\ldots = y'_ky'_{k+1}\ldots$ for some $k \geq 1$.

The boundedness of $D(N,I)$ is not necessarily invariant under translation of the interval. E.g. for $1 = .31^\infty$, $D(N,[0,.1^\infty))$ is bounded, but $D(N,[.1^\infty,.2^\infty))$ is unbounded. It is also possible that $D(N,[y,y'))$ is bounded and $D(N,[0,y'-y))$ is unbounded: $D(N,[.02,1))$ is bounded and $D(N,[0,1-.02)) = D(N,[0,.2^\infty))$ is unbounded.

This article is organised as follows. In Section 2 we recapitulate some facts about number systems defined by substitutions (due to Dumont and Thomas [5]) and define a reverse β-substitution which determines x_n. Theorem 1 is proved in Section 3 similarly to Shapiro [24]. The remaining parts of Theorem 2 are proved in Section 4, where explicit formulae for the discrepancy function in terms of lengths of iterates of the reverse β-substitution are given.

2. Number systems defined by substitutions

2.1. **Generalities.** Let σ be a substitution on the alphabet $\mathcal{A} = \{1,\ldots,d\}$, i.e. a mapping from $\mathcal{A}$ into the set of nonempty finite words on $\mathcal{A}$, which is extended to a mapping on words by concatenation, $\sigma(ww') = \sigma(w)\sigma(w')$. A sequence of words $m_k,\ldots,m_1$ is called *σ-b-admissible* if we have a companion sequence of letters b_j with $b_{k+1} = b$ such that $m_jb_j \leq_p \sigma(b_{j+1})$ for all $j \leq k$ (where $w \leq_p w'$ means that w is a prefix of w'). For a given sequence $m_k,\ldots,m_1$, clearly the sequence $b_k,\ldots,b_1$ is unique.

If $\sigma(1) = 1w$ for some word w, then the limit $\sigma^\infty(1) = \lim_{k\to\infty}\sigma^k(1)$ exists because of $\sigma^{k+1}(1) = \sigma^k(1w) = \sigma^k(1)\sigma^k(w)$ and we have

$$\sigma^{k-1}(m_k)\ldots\sigma^0(m_1) \leq_p \sigma^k(1) \leq_p \sigma^\infty(1) \tag{1}$$

for all σ-1-admissible sequences $m_k,\ldots,m_1$. Furthermore, every prefix $u_1\ldots u_n \leq_p \sigma^\infty(1)$, $n \geq 1$, can be written as the left hand side of (1) with a unique σ-1-admissible sequence $m_k,\ldots,m_1$ with $|m_k| > 0$ (where $|m|$ denotes the length of m). Denote these m_j by $m_{j,\sigma}(n)$ and set $m_{j,\sigma}(n) = \varepsilon$ (the empty word) for all $j > k$. For $n = 0$, set $m_{j,\sigma}(0) = \varepsilon$ for all $j \geq 1$. Then

$$n = \sum_{j=1}^\infty |\sigma^{j-1}(m_{j,\sigma}(n))| = \sum_{j=1}^\infty\sum_{b=1}^d |m_{j,\sigma}(n)|_b\,|\sigma^{j-1}(b)|,$$

where $|m|_b$ denotes the number of b's in m. If $m_{j,\sigma}(n') = m_{j,\sigma}(n)$ for all $j > k$ and $|m_{k,\sigma}(n')| > |m_{k,\sigma}(n)|$, i.e. $m_{k,\sigma}(n') = m_{k,\sigma}(n)b_jw$ for some word w, then $\sigma^{k-2}(m_{k-1,\sigma}(n))\ldots\sigma^0(m_{1,\sigma}(n))$ is a strict prefix of $\sigma^{k-1}(b_k)$, hence $\sum_{j=1}^{k-1}|\sigma^{j-1}m_{j,\sigma}(n)| < \sigma^{k-1}(b_j)$ and we have

$$n' \geq \sum_{j=k}^\infty |\sigma^{j-1}(m_{j,\sigma}(n'))| \geq \sum_{j=k}^\infty |\sigma^{j-1}(m_{j,\sigma}(n))| + |\sigma^{k-1}(b_k)| > n,$$

thus

$$n < n' \text{ if } \ldots |m_{2,\sigma}(n)|\,|m_{1,\sigma}(n)| < \ldots |m_{2,\sigma}(n')|\,|m_{1,\sigma}(n')| \tag{2}$$

2.2. **β-substitution.** If β is a Parry number, then the *β-substitution* σ is defined by

$$\sigma(b) = \begin{cases} 1^{a_b}(b+1) & \text{if } 1 \le b < d \\ 1^{a_d} & \text{if } b = d, 1 = .a_1 \ldots a_d \\ 1^{a_d}(d-p+1) & \text{if } b = d, 1 = .a_1 \ldots a_{d-p}(a_{d-p+1} \ldots a_d)^\infty \end{cases}$$

(where 1^{a_j} denotes the concatenation of a_j letters 1).

If we set $G_k = |\sigma^k(1)|$ for all $k \ge 0$, then

$$G_k = \sum_{j=1}^{k} a_j G_{k-j} + \begin{cases} 1 & \text{if } a_j = 0 \text{ for all } j > k \\ 0 & \text{else} \end{cases}$$

(in particular $G_k = \sum_{j=1}^{d} a_j G_{k-j}$ if $1 = .a_1 \ldots a_d$ and $k > d$) and

$$n = \sum_{j=1}^{\infty} |m_{j,\sigma}(n)|\,|\sigma^{j-1}(1)| = \sum_{j=1}^{\infty} |m_{j,\sigma}(n)| G_{j-1}$$

since the words $m_{j,\sigma}(n)$ consist only of ones. Thus the $|m_{j,\sigma}(n)|$ are the digits in the G-ary expansion of n with $G = (G_j)_{j \ge 0}$ and the σ-1-admissible sequences $m_k, \ldots, m_1$ are exactly those sequences consisting only of ones with $|m_j| \ldots |m_1| 0^\infty < a_1 a_2 \ldots$ for all $j \le k$.

Example. If $1 = .402$, then

$$\sigma(1) = 11112, \quad \sigma(2) = 3, \quad \sigma(3) = 11.$$

An example of a σ-1-admissible sequence with $k = 5$ is

$$(m_5, b_5), \ldots, (m_1, b_1) = (11, 1), (1111, 2), (\varepsilon, 3), (\varepsilon, 1), (1, 1)$$

which corresponds to

$$n = |\sigma^4(11)\sigma^3(1111)\sigma^2(\varepsilon)\sigma(\varepsilon)1| = 2G_4 + 4G_3 + 1 = 1053.$$

2.3. **Reverse β-substitution.** For a Parry number β, set $t_1 = 0^\infty$ and let $\{t_2, \ldots, t_{d+1}\}$ be the set of words $\{a_j a_{j+1} \ldots : \; j \ge 2\}$ with

$$0^\infty = t_1 < t_2 < \cdots < t_d < t_{d+1} = a_1 a_2 \ldots$$

For $1 \le b \le d$ set

$$\tau(b) = \begin{cases} u_0(b) \ldots u_{a_1}(b) & \text{if } a_1 t_b < a_1 a_2 \ldots \\ u_0(b) \ldots u_{a_1 - 1}(b) & \text{else} \end{cases}$$

with

$$u_j(b) = b' \text{ if } \; t_{b'} \le j t_b < t_{b'+1}.$$

We clearly have $u_0(1) = 1$, thus $\tau^\infty(1)$ exists and every $n \ge 1$ corresponds to a unique τ-1-admissible sequence $m_k, \ldots, m_1$ with $|m_k| > 0$.

The following example and proposition show (for $b = 1$) that the possible sequences of "digits" $|m_{j,\tau}(n)|$ are the same as for $|m_{j,\sigma}(n)|$, but in reversed order. Therefore we call τ *reverse β-substitution.*

Example. For $1 = .402$, we have $t_1 = 0^\infty$, $t_2 = 020^\infty$, $t_3 = 20^\infty$, $t_4 = 4020^\infty$, thus

$$\tau(1) = 12333, \quad \tau(2) = 1233, \quad \tau(3) = 2233.$$

We have a τ-1-admissible sequence with $|m_5| \dots |m_1| = 10042$,

$$(m_5, b_5), \dots, (m_1, b_1) = (1, 2), (\varepsilon, 1), (\varepsilon, 1), (1233, 3), (22, 3)$$

which corresponds to

$$n = |\tau^4(1)\tau^3(\varepsilon)\tau^2(\varepsilon)\tau(1233)22| = G_4 + 19 = 373.$$

Proposition 1. *Each τ-b-admissible sequence $m_k, \dots, m_1$ satisfies*

$$|m_j| \dots |m_k| t_b < a_1 a_2 \dots \text{ for all } j \le k. \tag{3}$$

Conversely, for each sequence $\epsilon_1 \dots \epsilon_k$ with $\epsilon_j \dots \epsilon_k t_b < a_1 a_2 \dots$ for all $j \ge 1$, we have a (unique) τ-b-admissible sequence $m_k, \dots, m_1$ with $|m_1| \dots |m_k| = \epsilon_1 \dots \epsilon_k$.

Proof. Assume first that $m_k, \dots, m_1$ is τ-b-admissible and let $b_k, \dots, b_1$ be its companion sequence ($m_j b_j \le_p \tau(b_{j+1})$, $b_{k+1} = b$). Assume further

$$|m_j| \dots |m_{\ell-1}| = a_1 \dots a_{\ell-j} \text{ and } t_{b_\ell} < a_{\ell-j+1} a_{\ell-j+2} \dots$$

(which is trivially true for $j = \ell$). We have $b_\ell = u_{|m_\ell|}(b_{\ell+1})$, hence

$$|m_\ell| t_{b_{\ell+1}} < t_{b_\ell + 1} \le a_{\ell-j+1} a_{\ell-j+2} \dots$$

This implies $|m_j| \dots |m_\ell| < a_1 \dots a_{\ell-j+1}$ or

$$|m_j| \dots |m_\ell| = a_1 \dots a_{\ell-j+1} \text{ and } t_{b_{\ell+1}} < a_{\ell-j+2} a_{\ell-j+3} \dots$$

In the latter case, we proceed inductively and obtain

$$|m_j| \dots |m_k| t_{b_{k+1}} = |m_j| \dots |m_k| t_b < a_1 a_2 \dots$$

Hence, (3) is proved.

For the converse, assume $\epsilon_j \dots \epsilon_k t_b < a_1 a_2 \dots$ for all $j \ge 1$ and

$$t_{b_{\ell+1}} \le \epsilon_{\ell+1} t_{b_{\ell+2}} \text{ for all } \ell \in \{j+1, \dots, k\}$$

(which is trivially true for $j = k$). Then we have

$$\epsilon_j t_{b_{j+1}} \le \epsilon_j \epsilon_{j+1} t_{b_{j+2}} \le \dots \le \epsilon_j \dots \epsilon_k t_{b_{k+1}} = \epsilon_j \dots \epsilon_k t_b < a_1 a_2 \dots,$$

thus $b_j = u_{\epsilon_j}(b_{j+1})$ exists and $m_j = u_0(b_{j+1}) \dots u_{\epsilon_j - 1}(b_{j+1})$. Furthermore, we have $t_{b_j} \le \epsilon_j t_{b_j + 1}$ and obtain, by induction, a (unique) τ-b-admissible sequence $m_k, \dots, m_1$ with $|m_1| \dots |m_k| = \epsilon_1 \dots \epsilon_k$. □

By Proposition 1 ($b = 1$), every finite β-expansion $\epsilon_1 \dots \epsilon_k 0^\infty$ corresponds to some $n < |\tau^k(1)|$ such that $\epsilon_1 \dots \epsilon_k = |m_{1,\tau}(n)| \dots |m_{k,\tau}(n)|$. By (2), we have $n < n'$ for $n, n' < |\tau^k(1)|$ if

$$\epsilon_k \dots \epsilon_1 = |m_{k,\tau}(n)| \dots |m_{1,\tau}(n)| < |m_{k,\tau}(n')| \dots |m_{1,\tau}(n')| = \epsilon'_k \dots \epsilon'_1.$$

Therefore the β-adic van der Corput sequence is given by

$$x_n = \sum_{j=1}^{\infty} |m_{j,\tau}(n)| \beta^{-j}.$$

Note that we have $|\tau^k(1)| = |\sigma^k(1)| = G_k$ for all $k \geq 0$.

3. Proof of Theorem 1

Let $\mathcal{D}$ be the set of all sequences $(m_j, b_j)_{j\geq 1}$ of words m_j and letters b_j with $m_j b_j \leq_p \tau(b_{j+1})$ for all $j \geq 1$. Set

$$\delta((m_j, b_j)_{j\geq 1}, (m'_j, b'_j)_{j\geq 1}) = 1/k$$

if $(m_j, b_j) = (m'_j, b'_j)$ for all $j < k$ and $(m_j, b_j) \neq (m'_j, b'_j)$. Then $\mathcal{D}$ is a compact metric space with the metric δ.

In order to extend the addition of 1 in the number system defined by τ, $(m_{j,\tau}(n))_{j\geq 1} \mapsto (m_{j,\tau}(n+1))_{j\geq 1}$, define the *successor function* (or *odometer* or *adic transformation*) on $\mathcal{D}$ by

$$S((m_j, b_j)_{j\geq 1}) = (m'_j, b'_j)_{j\geq 1} \text{ with } (m'_j, b'_j) = \begin{cases} (m_j, b_j) & \text{if } j > k \\ (m_k b_k, b'_k) & \text{if } j = k \\ (\varepsilon, u_0(b'_{j+1})) & \text{if } j < k \end{cases}$$

where $k \geq 1$ is the smallest integer such that $\tau(b_{k+1}) = m_k b_k b'_k w$ for some letter b'_k and some word w. If $(m_j, b_j)_{j\geq 1}$ is a maximal sequence, i.e. $m_k b_k = \tau(b_{k+1})$ for all $k \geq 1$, then let its successor be the (unique) minimal sequence $(\varepsilon, 1), (\varepsilon, 1), \dots$

If the maximal sequence is unique, then S is a homeomorphism and $(\mathcal{D}, S)$ is a transformation group, but in many cases the maximal sequence is not unique. In particular if $a_2 a_3 \dots > (a_1 - 1)^\infty$, then every maximal sequence satisfies $|m_j| = a_1$, $|m_{j'}| = a_1 - 1$ for some $j, j' \geq 1$, and we obtain a different maximal sequence by shifting this sequence. Hence $(\mathcal{D}, S)$ is only a transformation semigroup.

Define a continuous function $f : \mathcal{D} \to [0, 1)$ by

$$f((m_j, b_j)_{j\geq 1}) = \sum_{j=1}^{\infty} |m_j| \beta^{-j}.$$

Then we have $x_n = f(S^n((\varepsilon, 1), (\varepsilon, 1), \dots))$. If S is invertible, then $(x_0, x_1, \dots)$ can be extended to a bisequence $(x_n)_{n\in\mathbb{Z}}$ by this definition.

Let X denote the orbit closure of $(x_0, x_1, \dots)$ under the shift T, and define $\varphi : \mathcal{D} \to X$ by

$$(\varphi((m_j, b_j)_{j\geq 1}))_k = f(S^k((m_j, b_j)_{j\geq 1}))$$

Then φ is a homeomorphism and $\varphi \circ S = T \circ \varphi$. Hence the transformation (semi)group (X,T) is isomorphic to $(\mathcal{D}, S)$. If S is invertible, then (X,T) is minimal by Theorem 2.2 of Shapiro [24] and we can apply Theorem 5.1 of this article, which states that $\exp(2\pi i\lambda(I))$ is an eigenvalue of T and thus of S if $D(N,I)$ is bounded. Lemma 1 shows that Shapiro's proof is valid for our transformation semigroup as well.

By Théorème 5.2 of Canterini and Siegel [2], we have a continuous and surjective "desubstitution map" $\Gamma : \Omega \to \mathcal{D}$, where Ω is the set of biinfinite words which have the same language as $\tau^\infty(1)$. Let Δ be the shift on Ω. By Théorème 5.1 of this article and since the minimal sequence in $\mathcal{D}$ is unique, we have $S \circ \Gamma = \Gamma \circ \Delta$. Therefore the eigenvalues of S are a subset of the eigenvalues of Δ and, by Proposition 5 of Ferenczi, Mauduit and Nogueira [9], these eigenvalues are of the form $\exp(2\pi i y)$ with $y \in \mathbb{Q}(\beta)$, This concludes the proof of Theorem 1.

Remarks. Ferenczi, Mauduit and Nogueira [9] gave a more precise description of the set of eigenvalues of Δ in their Proposition 4, which is too complicated to be cited here.

For more details on the spectrum of these dynamical systems, see Chapter 7.3 in Pytheas Fogg [17], but note that the result of [9] is cited uncorrectly: According to Theorem 7.3.28 of [17], the eigenvalues of Δ associated with the trivial coboundary are in $\exp(2\pi i\mathbb{Z}[\beta])$, but $\mathbb{Z}[\beta]$ should be $\mathbb{Q}[\beta]$ and the condition on the coboundary is unnecessary. Nevertheless, the author considered the coboundary and showed that all reverse β-substitutions τ have only the trivial coboundary, but the proof is rather lengthy and technical and therefore not given in this article.

Lemma 1. *If $D(N,I)$ is bounded, then $\exp(2\pi i\lambda(I))$ is an eigenvalue of S.*

Proof. Set

$$g((m_j,b_j)_{j\geq 1}) = \chi_I\left(\sum_{j=1}^{\infty} |m_j|\beta^{-j}\right) - \lambda(I)$$

where χ_I denotes the indicator function of I. Let $\omega = (m_j,b_j)_{j\geq 1}$ be a sequence with $|m_1|\,|m_2|\ldots = y_1y_2\ldots$, hence $\sum_{j=0}^{N-1} g(S^j\omega) = D(N,I)$ is bounded. Set $U(x,\eta) = (Sx, \eta + g(x))$ for $x \in \mathcal{D}$, $\eta \in \mathbb{R}$. Then we have

$$U^k(x,\eta) = \left(S^k x, \eta + \sum_{j=0}^{k-1} g(S^j x)\right).$$

The positive semi-orbit $\{U^k(\omega,0) : k \geq 0\}$ is bounded and has therefore compact closure. Denote by M the set of limit points of this semi-orbit. Then M is nonempty, closed and invariant under U (NCI). It is easy to see that $\{S^k x : k \geq 0\}$ is dense in $\mathcal{D}$ for all $x \in \mathcal{D}$. Since M is NCI, we must therefore have some point $(x,\eta) \in M$ for all $x \in \mathcal{D}$.

Below we show that, for a given x, this η is unique, i.e. $\eta = \eta(x)$. Then the graph $(x,\eta(x))$ is the compact set M, therefore η is continuous. Since $U(x,\eta(x)) = (Sx, \eta(x) +$

$g(x))$, we have

$$\eta(Sx) = \eta(x) + g(x),$$
$$\exp(-2\pi i\lambda(I)) = \exp(2\pi i g(x)) = \exp(2\pi i\eta(Sx))/\exp(2\pi i\eta(x)).$$

Therefore $K(x) = \exp(-2\pi i\eta(x))$ is a continuous function with

$$K(Sx) = \exp(2\pi i\lambda(I))K(x)$$

and $\exp(2\pi i\lambda(I))$ is an eigenvalue of S.

To prove that $\eta(x)$ is unique, we show first $\eta(\omega) = 0$. Suppose $(\omega, \eta) \in M$. Since M consists of limit points of $\{U^k(\omega, 0) : k \geq 0\}$, we have a sequence $k_j \to \infty$ with

$$\lim_{j\to\infty} U^{k_j}(\omega, 0) = (\omega, \eta).$$

This implies

$$\lim_{j\to\infty} S^{k_j}\omega = \omega \quad \text{and} \quad \lim_{j\to\infty} \sum_{i=0}^{k_j-1} g(S^i\omega) = \eta,$$

hence

$$\lim_{j\to\infty} U^{k_j}(\omega, \eta) = \left(\lim_{j\to\infty} S^{k_j}\omega, \eta + \lim_{j\to\infty} \sum_{i=0}^{k_j-1} g(S^i\omega) \right) = (\omega, \eta + \eta).$$

Since M is invariant, we have $U^{k_j}(\omega, \eta) \in M$ for all j and, since M is closed, $(\omega, 2\eta) \in M$. Inductively we obtain $(\omega, k\eta) \in M$ for all M, which implies $\eta = 0$ since M is bounded.

Next suppose $(x, \eta) \in M$ and $(x, \eta') \in M$. Since $\{S^k x : k \geq 0\}$ is dense, we have some $k_j \to \infty$ such that

$$\lim_{j\to\infty} S^{k_j}x = \omega.$$

Since M is compact, we can refine the sequence k_j so that the sequences $U^{k_j}(x, \eta)$ and $U^{k_j}(x, \eta')$ converge (to points in M). Since the first coordinate of the limit points is ω, the second coordinate must be 0 for both points. Therefore

$$\lim_{j\to\infty} \left(\eta + \sum_{\ell=0}^{k_j-1} g(S^\ell x) \right) = \lim_{j\to\infty} \left(\eta' + \sum_{\ell=0}^{k_j-1} g(S^\ell x) \right),$$

hence $\eta = \eta'$ and we have proved that $\eta(x)$ is unique. □

4. Proof of Theorem 2

Because of Theorem 1, we just have to consider $y \in \mathbb{Q}(\beta)$ for Theorem 2, but first we compute formulae for the discrepancy function of arbitrary intervals $[0, y)$. Let $A(N, I) = \#\{x_n \in I : \ 0 \leq n < N\}$. Then we have, for $y = .y_1y_2\ldots$,

$$D(N, [0, y)) = \sum_{k=1}^{\infty} (A(N, [.y_1 \ldots y_{k-1}, .y_1 \ldots y_k)) - N y_k \beta^{-k}).$$

Lemma 2. *We have*

$$A(N,[.y_1\dots y_{k-1},.y_1\dots y_k)) = y_k \sum_{\ell=k+1}^{\infty}\sum_{b=1}^{d} |m_{\ell,\tau}(N)|_b\,|\tau^{\ell-k-1}(b)| + \mu_k(N,y)$$

with

$$\mu_k(N,y) = \begin{cases} y_k & \text{if } |m_{k,\tau}(N)| \geq y_k \\ |m_{k,\tau}(N)|+1 & \text{if } |m_{k,\tau}(N)| < y_k, \\ & |m_{k-1,\tau}(N)|\dots|m_{1,\tau}(N)| > y_{k-1}\dots y_1 \\ |m_{k,\tau}(N)| & \text{else.} \end{cases}$$

Proof. For $G_L \leq N < G_{L+1}$, we have

$$\begin{aligned}&\{(m_{1,\tau}(n),\dots,m_{L,\tau}(n)):\ 0\leq n<N\}\\ &=\bigcup_{\ell=1}^{L}\ \bigcup_{m:\,mb\leq_p m_{\ell,\tau}(N)} \{(m_1,\dots,m_{\ell-1},m,m_{\ell+1,\tau}(N),\dots,m_{L,\tau}(N)) : m_{\ell-1},\dots,m_1 \text{ is } \tau\text{-}b\text{-adm.}\}\end{aligned}$$

and $x_n \in [.y_1\dots,y_{k-1},.y_1\dots y_k)$ if and only if

$$|m_{1,\tau}(n)|\dots|m_{k-1,\tau}(n)| = y_1\dots y_{k-1},\ |m_{k,\tau}(n)| < y_k.$$

Thus, for $\ell > k$, we have to count the τ-b-admissible sequences $m_{\ell-1},\dots,m_1$ with $|m_1|\dots|m_{k-1}| = y_1\dots y_{k-1}$, $|m_k| < y_k$. By Proposition 1, every τ-b-admissible sequence $m_{\ell-1},\dots,m_{k+1}$ can be prolongated to such a sequence for all $|m_k| < y_k$ because of

$$|m_j|\dots|m_{\ell-1}|t_b < y_j\dots y_k \leq a_1a_2\dots \text{ for } j\leq k.$$

Therefore we have $y_k|\tau^{\ell-k-1}(b)|$ such sequences for every letter b in $m_{\ell,\tau}(N)$.

For $\ell = k$, we need $|m| < |m_{k,\tau}(N)|$ and $|m| < y_k$. For each such $|m|$ (and the corresponding b), there is one τ-b-admissible sequence $m_{k-1},\dots,m_1$ with $|m_1|\dots|m_{k-1}| = y_1\dots y_{k-1}$. Thus, the contribution is $\max(|m_{k,\tau}(N)|, y_k)$.

Finally, for $\ell < k$, we need $|m| = y_\ell < |m_{\ell,\tau}(N)|$, $|m_{k,\tau}(N)| < y_k$ and $|m_{\ell+1,\tau}(N)|\dots|m_{k-1,\tau}(N)| = y_{\ell+1}\dots y_{k-1}$. Thus the contribution is 1 if $|m_{k,\tau}(N)| < y_k$, $|m_{k-1,\tau}(N)|\dots|m_{1,\tau}(N)| > y_{k-1}\dots y_1$ and 0 else. □

The characteristic polynomial of the incidence matrix of the β-substitution σ is the β-polynomial. Hence σ is of Pisot type (one eigenvalue is > 1 and all other eigenvalues have modulus < 1) if and only if β is a Pisot number and the β-polynomial is irreducible. Since $|\sigma^k(1)| = |\tau^k(1)|$ for all $k \geq 0$, β is an eigenvalue of τ as well. Furthermore, τ is of Pisot type because the alphabet has the same size as the alphabet of σ. Hence we have some constants $c_{b,j}$ and $\rho < 1$ such that

$$|\tau^k(b)| = c_{b,1}\beta^k + c_{b,2}\beta_2^j + \dots + c_{b,d}\beta_d^k = c_{b,1}\beta^k + \mathcal{O}\left(\rho^k\right),$$

where the β_j, $2 \le j \le d$ are the conjugates of β. Thus

$$\begin{aligned}D(N,[0,y)) &= \sum_{k=1}^{\infty}\Bigg(y_k \sum_{\ell=k+1}^{\infty}\sum_{b=1}^{d} |m_{\ell,\tau}(N)|_b\, |\tau^{\ell-k-1}(b)| + \mu_k(N,y) \\ &\qquad - y_k \sum_{\ell=1}^{\infty}\sum_{b=1}^{d} |m_{\ell,\tau}(N)|_b\, |\tau^{\ell-1}(b)|\beta^{-k}\Bigg) \\ &= \sum_{k=1}^{\infty}\Bigg(y_k \sum_{\ell=k+1}^{\infty}\sum_{b=1}^{d} |m_{\ell,\tau}(N)|_b \sum_{j=2}^{d} c_{b,j}\left(\beta_j^{\ell-k-1} - \beta_j^{\ell-1}\beta^{-k}\right) + \mu_k(N,y) \\ &\qquad - y_k \sum_{\ell=1}^{k}\sum_{b=1}^{d} |m_{\ell,\tau}(N)|_b \left(c_{b,1}\beta^{\ell-1-k} + \sum_{j=2}^{d}\beta_j^{\ell-1}\beta^{-k}\right)\Bigg) = \sum_{k=1}^{\infty} y_k \mathcal{O}(1)\end{aligned}$$

and

$$\begin{aligned}D(N,[0,y)) &= \sum_{\ell=1}^{\infty}\Bigg(\sum_{b=1}^{d} |m_{\ell,\tau}(N)|_b\Bigg(\sum_{k=1}^{\ell-1} y_k \sum_{j=2}^{d} c_{b,j}\left(\beta_j^{\ell-k-1} - \beta_j^{\ell-1}\beta^{-k}\right)\Bigg) \\ &\qquad + \mu_\ell(N,y) - \sum_{b=1}^{d} |m_{\ell,\tau}(N)|_b \sum_{k=\ell}^{\infty} y_k\Bigg(c_{b,1}\beta^{\ell-k-1} + \sum_{j=2}^{d} c_{b,j}\beta_j^{\ell-1}\beta^{-k}\Bigg)\Bigg) \\ &= \sum_{\ell=1}^{\infty}\Bigg(\mu_\ell(N,y) - \sum_{b=1}^{d} |m_{\ell,\tau}(N)|_b\Bigg(c_{b,1}\sum_{k=\ell}^{\infty} y_k\beta^{\ell-k-1} - \sum_{j=2}^{d} c_{b,j}\sum_{k=1}^{\ell-1} y_k\beta_j^{\ell-k-1}\Bigg)\Bigg) + \mathcal{O}(1)\end{aligned}$$

By the above formulae, we easily see that $D(N,[0,y))$ is bounded if $y_k > 0$ for only finitely many $k \ge 1$. Now we consider $y \in \mathbb{Q}(\beta)$. Bertrand [1] and K. Schmidt [21] proved independently that the elements $y \in \mathbb{Q}(\beta)$ are exactly those who have eventually periodic β-expansion. (See Rigo and Steiner [20] for an alternative proof including number systems defined by substitutions.) Furthermore, by the above formulae, a finite number of digits of the β-expansion of y as well as a shift of digits has no influence on the boundedness of $D(N,[0,y))$. Therefore we may assume that the β-expansion of y is purely periodic.

For $y = .(y_1 \dots y_q)^\infty$, we have

$$\sum_{k=\ell}^{\infty} y_k\beta^{\ell-k-1} = \frac{y_\ell\beta^{p-1} + \cdots + y_{\ell+p-1}}{\beta^p - 1} = s_{\ell,d-1}\beta^{d-1} + \cdots + s_{\ell,0}\beta^0 = P_\ell(\beta)$$

for some $s_{\ell,j} \in \mathbb{Q}$. If we set $y_k = y_{k+q}$ for $k \le 0$, then we obtain

$$\sum_{k=-\infty}^{\ell-1} y_k\beta_i^{\ell-k-1} = \frac{y_{\ell-p}\beta_i^{p-1} + \cdots + y_{\ell-1}}{1-\beta_i^p} = -P_\ell(\beta_i),$$

$$\gamma_\ell(b) = c_{b,1}\sum_{k=\ell}^{\infty} y_k\beta^{\ell-k-1} - \sum_{i=2}^{d} c_{b,i}\sum_{k=-\infty}^{\ell-1} y_k\beta_i^{\ell-k-1} = s_{\ell,d-1}|\tau^{d-1}(b)| + \cdots + s_{\ell,0}|\tau^0(b)|$$

and

$$D(N,[0,y)) = \sum_{\ell=1}^{\infty} \Big(\mu_\ell(N,y) - \gamma_\ell(m_{\ell,\tau}(N))\Big) + \mathcal{O}(1)$$

by extending γ_ℓ naturally on words, $\gamma_\ell(w) = \sum_{b=1}^{d} |w|_b \gamma_\ell(b)$.

We split the remaining part of the proof into two lemmata.

Lemma 3. *If β is a Pisot number with irreducible β-polynomial, then $D(N,[0,.(a_{d-p+1}\dots a_d)^\infty)$ is bounded.*

Proof. We have

$$.y_\ell y_{\ell+1}\dots = .a_{d-p+\ell}a_{d-p+\ell+1}\dots = \beta^{d-p+\ell-1} - a_1\beta^{d-p+\ell-2} - \dots - a_{d-p+\ell-1}$$

and, by Proposition 1, we easily see

$$|\tau^k(b)| = a_1|\tau^{k-1}(b)| + \dots + a_k|\tau^0(b)| + \begin{cases} 1 & \text{if } a_1\dots a_k t_b < a_1a_2\dots \\ 0 & \text{else} \end{cases}$$

for all $k > 0$, hence

$$\gamma_\ell(b) = \begin{cases} 1 & \text{if } t_b < a_{d-p+\ell}a_{d-p+\ell+1}\dots \\ 0 & \text{else.} \end{cases}$$

By definition, we have $t_{u_j(b_{\ell+1})} \le jt_{b_{\ell+1}} < t_{u_j(b_{\ell+1})+1}$, therefore

$$\gamma_\ell(u_j(b_{\ell+1})) = \begin{cases} 1 & \text{if } jt_{b_{\ell+1}} < a_{d-p+\ell}a_{d-p+\ell+1}\dots \\ 0 & \text{else.} \end{cases}$$

With $m_{\ell,\tau}(N) = u_0(b_{\ell+1})\dots u_{|m_{\ell,\tau}(N)|-1}(b_{\ell+1})$, we obtain

$$\gamma_\ell(m_{\ell,\tau}(N)) = \begin{cases} |m_{\ell,\tau}(N)| & \text{if } |m_{\ell,\tau}(N)| \le a_{d-p+\ell} \\ a_{d-p+\ell} & \text{if } |m_{\ell,\tau}(N)| > a_{d-p+\ell}, \\ & t_{b_{\ell+1}} \ge a_{d-p+\ell+1}a_{d-p+\ell+2}\dots \\ a_{d-p+\ell}+1 & \text{else} \end{cases}$$

and

$$\begin{aligned} \Delta_\ell &= \mu_\ell(N,.(a_{d-p+1}\dots a_d)^\infty) - \gamma_\ell(m_{\ell,\tau}(N)) \\ &= \begin{cases} -1 & \text{if } |m_{\ell,\tau}(N)| > a_{d-p+\ell}, t_{b_{\ell+1}} < a_{d-p+\ell+1}a_{d-p+\ell+2}\dots \\ 1 & \text{if } |m_{\ell,\tau}(N)| < a_{d-p+\ell}, \\ & |m_{\ell-1,\tau}(N)|\dots|m_{1,\tau}(N)| > a_{d-p+\ell-1}\dots a_{d-p+1} \\ 0 & \text{else.} \end{cases} \end{aligned}$$

If $\Delta_\ell = -1$, then $t_{b_{\ell+1}} < a_{d-p+\ell+1}a_{d-p+\ell+2}\dots$ and

$$t_{b_{\ell+1}} \le |m_{\ell+1,\tau}(N)|t_{b_{\ell+2}} < t_{b_{\ell+1}+1} \le a_{d-p+\ell+1}a_{d-p+\ell+2}\dots$$

implies either $|m_{\ell+1,\tau}(N)| < a_{d-p+\ell+1}$, thus $\Delta_{\ell+1} = 1$, or

$$|m_{\ell+1,\tau}(N)| = a_{d-p+\ell+1}, t_{b_{\ell+2}} < a_{d-p+\ell+2}a_{d-p+\ell+3}\dots \text{ and } \Delta_{\ell+1} = 0.$$

Inductively, we obtain some $k > \ell$ such that $\Delta_{\ell+1} = \dots = \Delta_{k-1} = 0$ and $\Delta_k = 1$.

If $\Delta_\ell = 1$, then $|m_{\ell-1,\tau}(N)| \dots |m_{1,\tau}(N)| > a_{d-p+\ell-1} \dots a_{d-p+1}$ implies either

$$|m_{\ell-1,\tau}(N)| > a_{d-p+\ell-1} \text{ and } t_{b_\ell} \le |m_{\ell,\tau}(N)| t_{b_{\ell+1}} < a_{d-p+\ell},$$

thus $\Delta_{\ell-1} = -1$, or

$$|m_{\ell-1,\tau}(N)| = a_{d-p+\ell-1}, |m_{\ell-2,\tau}(N)| \dots |m_{1,\tau}(N)| > a_{d-p+\ell-2} \dots a_{d-p+1}$$

and $\Delta_{\ell-1} = 0$. Inductively, we obtain some $k < \ell$ such that $\Delta_k = -1$ and $\Delta_{k+1} = \cdots = \Delta_{\ell-1} = 0$.

Therefore we have $\sum_{\ell=1}^{\infty} \Delta_\ell = 0$ and the discrepancy function is bounded. □

$D(N, [0, .(a_{d-p+j} \dots a_d a_{d-p+1} \dots a_{d-p+j-1})^\infty)$, $1 < j \le p$, is bounded as well because a shift of digits does not change the boundedness.

Lemma 4. *If $D(N, [0, y))$ is bounded and $y \ne 0$ has purely periodic β-expansion, then the expansion of 1 is eventually periodic and $y = .a_L a_{L+1} \dots$ for some $L > d - p$.*

Proof. Let the β-expansion of y be $.y_1 y_2 \dots = .(y_1 \dots y_q)^\infty$. Consider sequences of integers N_K given by

$$(m_{1,\tau}(N_K), m_{2,\tau}(N_K), \dots) = ((m_1, \dots, m_{Jq})^K, \varepsilon, \varepsilon, \dots)$$

with $m_{\ell+1} = \cdots = m_{Jq} = \varepsilon$ for some $\ell \ge 1$, $J \ge 1$ such that $b_{\ell+1} = 1$ and $y_{\ell+1} \dots y_{Jq} > 0 \dots 0$. For these sequences, we have

$$\mu_{j+kJq}(N_K, y) = \mu_j(N_K, y),\ \gamma_{j+kJq}(m_{j+kJq,\tau}(N_K)) = \gamma_j(m_j)$$

for all $j \le Jq$, $k < K$. Thus $D(N_K, [0, y))$ is bounded if and only if

$$\sum_{j=1}^{Jq} (\mu_j(N_1, y) - \gamma_j(m_j)) = 0$$

Let furthermore $m_1 = \cdots = m_{k-1} = \varepsilon$ for some $k \in \{1, \dots, \ell\}$, hence $\mu_j(N_1, y) = \gamma_j(m_j)$ for all $j < k$. Consider simultaneously integers N'_K with $m'_k = \varepsilon$ and $m'_j = m_j$ for all $j \ne k$. Then we have $\mu_j(N'_1, y) = \gamma_j(m'_j) = 0$ for all $j < k$, $\gamma_j(m'_j) = \gamma_j(m_j)$ for all $j > k$ and

$$\sum_{j=k+1}^{Jq} \mu_j(N_1, y) = \sum_{j=k+1}^{Jq} \mu_j(N'_1, y) + \begin{cases} 1 & \text{if } |m_k| > y_k, \\ & |m_{k+1}| \dots |m_{Jq}| < y_{k+1} \dots y_{Jq} \\ 0 & \text{else,} \end{cases}$$

thus

$$\begin{aligned} \gamma_k(m_k) - \mu_k(N_1, y) &= \sum_{j=k+1}^{Jq} (\mu_j(N_1, y) - \gamma_j(m_j)) \\ &= \begin{cases} 1 & \text{if } |m_k| > y_k, |m_{k+1}| \dots |m_\ell| \le y_{k+1} \dots y_\ell \\ 0 & \text{else} \end{cases} \end{aligned}$$

and

$$\gamma_k(m_k) = \begin{cases} |m_k| & \text{if } |m_k| \le y_k \\ y_k & \text{if } |m_k| > y_k, |m_{k+1}| \dots |m_\ell| > y_{k+1} \dots y_\ell \\ y_k + 1 & \text{else.} \end{cases}$$

If $m_k b_k <_p \tau(b_{k+1})$, then $m_\ell, \ldots, m_{k+1}, m_k b_k$ is a τ-1-admissible sequence and we obtain

$$\gamma_k(b_k) = \gamma_k(m_k b_k) - \gamma_k(m_k) = \begin{cases} 1 & \text{if } |m_k| \ldots |m_\ell| \le y_k \ldots y_\ell \\ 0 & \text{else,} \end{cases} \tag{4}$$

in particular $\gamma_k(1) = 1$ for all $k \ge 1$ (with $k = \ell, m_k = \varepsilon$).

If $m_k b_k = \tau(b_{k+1})$, consider

$$.y_{k+1} y_{k+2} \ldots = \beta \times .y_k y_{k+1} \ldots - y_k = s_{k,d-1}\beta^d + \cdots + s_{k,0}\beta - y_k,$$

hence

$$\begin{aligned} \gamma_{k+1}(b_{k+1}) &= s_{k,d-1}|\tau^d(b_{k+1})| + \cdots + s_{k,0}|\tau(b_{k+1})| - y_k \\ &= s_{k,d-1}|\tau^{d-1}(m_k b_k)| + \cdots + s_{1,0}|m_k b_k| - y_k = \gamma_k(m_k) + \gamma_k(b_k) - y_k \\ &= \gamma_k(b_k) + \begin{cases} -1 & \text{if } |m_k| < y_k \text{ (i.e. } |m_k| = a_1 - 1, y_k = a_1) \\ 0 & \text{if } |m_k| = y_k \text{ or } |m_k| > y_k, |m_{k+1}| \ldots |m_\ell| > y_{k+1} \ldots y_\ell \\ 1 & \text{else.} \end{cases} \end{aligned}$$

In case $|m_k| = |\tau(b_{k+1})| - 1 = a_1 - 1$, $y_k = a_1$, we have $a_1 t_{b_{k+1}} \ge a_1 a_2 \ldots$, $y_{k+1} y_{k+2} \ldots < a_2 a_3 \ldots$ and $t_{b_{k+1}} \le |m_{k+1}| t_{b_{k+2}} \le \cdots \le |m_{k+1}| \ldots |m_\ell| 0^\infty$, hence $|m_{k+1}| \ldots |m_\ell| \ge a_2 \ldots a_{\ell-k+1} \ge y_{k+1} \ldots y_\ell$. One of these inequalities is strict because $t_{b_{k+1}} = |m_{k+1}| \ldots |m_\ell| 0^\infty = a_2 \ldots a_{\ell-k+1} 0^\infty$ implies $|m_{k+1}| \ldots |m_\ell| = a_2 \ldots a_d 0^{\ell-k-d+1} > y_{k+1} \ldots y_\ell$. Therefore we have, for all b_k, b_{k+1},

$$\gamma_k(b_k) - \gamma_{k+1}(b_{k+1}) = \begin{cases} 1 & \text{if } |m_k| \ldots |m_\ell| \le y_k \ldots y_\ell, |m_{k+1}| \ldots |m_\ell| > y_{k+1} \ldots y_\ell \\ -1 & \text{if } |m_k| \ldots |m_\ell| > y_k \ldots y_\ell, |m_{k+1}| \ldots |m_\ell| \le y_{k+1} \ldots y_\ell \\ 0 & \text{else.} \end{cases}$$

and, with $\gamma_{\ell+1}(b_{\ell+1}) = \gamma_{\ell+1}(1) = 1$, (4) holds for all m_k, b_k.

Now, let $k = 1$ and $m_\ell, \ldots, m_1$ and $m'_\ell, \ldots, m'_1$ be τ-1-admissible sequences with companion sequences $b_\ell, \ldots, b_1$ and $b'_\ell, \ldots, b'_1$. If $b_1 < b'_1$, then we have $|m_1| t_{b_2} < t_{b_1+1} \le t_{b'_1} \le |m'_1| t_{b'_2}$, thus either $|m_1| < |m'_1|$ or $|m_1| = |m'_1|$, $b_2 < b'_2$. Inductively, we obtain $|m_1| \ldots |m_\ell| < |m'_1| \ldots |m'_\ell|$ and $\gamma_1(b_1) \ge \gamma_1(b'_1)$. Therefore we have some $b' \ge 2$ such that

$$\gamma_1(b) = \begin{cases} 1 & \text{if } b < b' \\ 0 & \text{else.} \end{cases}$$

Finally, consider the system of linear equations

$$s_{1,d-1}|\tau^{d-1}(b)| + \cdots + s_{1,0}|\tau^0(b)| = \begin{cases} 1 & \text{if } b < b' \\ 0 & \text{else} \end{cases}$$

for $1 \le b \le d$. We have $t_{b'} = a_L a_{L+1} \ldots$ for some $L \ge 2$. Then, by the proof of Lemma 3, $(s_{1,d-1}, \ldots, s_{1,0}) = (0, \ldots, 0, 1, -a_1, \ldots, -a_{L-1})$ is a solution of this system, i.e. $y = .a_L a_{L+1} \ldots$ To show that these solutions are unique, consider linear combinations of the column vectors $(|\tau^\ell(1)|, \ldots, |\tau^\ell(d)|)^T$ (over $\mathbb{Q}$). We have, with $\beta_1 = \beta$,

$$\sum_{\ell=0}^{d-1} r_\ell \begin{pmatrix} |\tau^\ell(1)| \\ \vdots \\ |\tau^\ell(d)| \end{pmatrix} = \sum_{\ell=0}^{d-1} r_\ell M^\ell \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{\ell=0}^{d-1} r_\ell \sum_{j=1}^{d} v_j \beta_j^\ell \mathbf{e}_j = \sum_{j=1}^{d} v_j \mathbf{e}_j \sum_{\ell=0}^{d-1} r_\ell \beta_j^\ell,$$

where M is the incidence matrix of τ, $M = (|\tau(b)|_c)_{1 \le b,c \le d}$, and the $\mathbf{e}_j$, $1 \le j \le d$, are right eigenvectors of M to the eigenvalues β_j. If $r_\ell \in \mathbb{Q}$, then all r_ℓ must be zero, hence the vectors $(|\tau^\ell(1)|, \ldots, |\tau^\ell(d)|)$, $0 \le \ell < d$, are linearly independent and the system of linear equations has a unique solution.

To conclude the proof of the lemma, note that $a_L a_{L+1} \ldots$ is purely periodic if and only if $L > d - p$. □

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