REGULARITIES OF THE DISTRIBUTION OF ABSTRACT VAN DER CORPUT SEQUENCES

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Abstract. Similarly to β-adic van der Corput sequences, abstract van der Corput sequences can be defined for abstract numeration systems. Under some assumptions, these sequences are low discrepancy sequences. The discrepancy function is computed explicitely, and a characterization of bounded remainder sets of the form $[0, y)$ is provided.

1. Introduction

Let $(x_n)_{n\geq 0}$ be a sequence with $x_n \in [0, 1)$ for all $n \geq 0$, and

$$D(N, I) = \#\{0 \leq n < N : x_n \in I\} - N\lambda(I)$$

its *discrepancy function* on the interval I, where $\lambda(I)$ denotes the length of I. Then $(x_n)_{n\geq 0}$ is a *low discrepancy sequence* if $\sup_I D(N, I) = \mathcal{O}(\log N)$, where the supremum is taken over all intervals $I \subseteq [0, 1)$. If $D(N, I)$ is bounded in N, then I is called a *bounded remainder set*. For details on the discrepancy, we refer to [KN] and [DT]. References to results on bounded remainder sets can be found in the introduction of [St].

In [Ni], β-adic van der Corput sequences are defined, and it is shown that they are low discrepancy sequences if β is a Pisot number with irreducible β-polynomial. Recall that a *Pisot number* is an algebraic integer greater than 1 with all its conjugates lying in the interior of the unit disk. For these low discrepancy sequences, the interval $[0, y)$, $0 \leq y \leq 1$, is a bounded remainder set if and only if the β-expansion of y is finite or its tail is the same as that of the expansion of 1, see [St].

If β is a Pisot number, then the language of β-expansions is regular, which means that it is recognized by a finite automaton. Therefore these β-expansions are special cases of abstract numeration systems as defined in [LR], see Section 2. This article is devoted to the study of van der Corput sequences defined by more general abstract numeration systems.

2. Definitions

Let $(A, <)$ be a finite and totally ordered alphabet. Denote by A^* the free monoid generated by A for the concatenation product, i.e., the set of finite words with letters in A. The length of a word $w \in A^*$ is denoted $|w|$. Extend the order on A to A^* by the *shortlex* (or genalogical) *order*, which means that $v < w$ if either $|v| < |w|$ or $|v| = |w|$ and there exist $p, v', w' \in A^*$, $a, b \in A$ such that $v = pav'$, $w = pbw'$ and $a < b$.

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The triple $S = (L, A, <)$ is an *abstract numeration system* if L is an infinite regular language over A and the *numerical value* of a word $w \in L$ is defined by

$$\mathrm{val}_S(w) = \#\{v \in L : \ v < w\}.$$

If $\mathrm{val}_S(w) = n$, then we say that w is the *representation* of n and write $\mathrm{rep}_S(n) = w$.

Assume that the language L grows exponentially, with

$$\lim_{m\to\infty} \frac{\log \#\{v \in L : \ |v| \le m\}}{m} = \log \beta.$$

Then real numbers are represented by infinite words which are limits of sequences of words in L. The value of an infinite word $u = \lim_{j\to\infty} w^{(j)}$, $w^{(j)} \in L$, is

$$\mathrm{val}_S^\omega(u) = \lim_{j\to\infty} \frac{\mathrm{val}_S(w^{(j)})}{\#\{v \in L : \ |v| \le |w^{(j)}|\}}.$$

Let L_ω the set of these words u. Since $\mathrm{val}_S^\omega(u) \in [1/\beta, 1]$, we define the *normalized value*

$$\langle u \rangle = \frac{\beta\, \mathrm{val}_S^\omega(u) - 1}{\beta - 1} \in [0, 1].$$

We extend this definition to finite words $w \in L$ which are prefixes of words in L_ω by setting $\langle w \rangle = \langle u \rangle$, where u is the lexicographically smallest word in L_ω with prefix w. Since we want to define a sequence without multiple occurrences of the same value, we set

$$L' = \{w \in L : \ \langle w \rangle \ne \langle v \rangle \text{ for all } v \in L \text{ with } v < w\}.$$

The *mirror image* of a word $w = w_1 w_2 \cdots w_k$, $w_j \in A$, is $\widetilde{w} = w_k \cdots w_2 w_1$. The mirror image of a language L is $\widetilde{L} = \{\widetilde{w} : \ w \in L\}$.

Assume that every $w \in L$ is the prefix of some $u \in L_\omega$. Then we define the *abstract van der Corput sequence* corresponding to S by setting

$$x_n = \langle w \rangle \ \text{ with } \ \widetilde{w} = \mathrm{rep}_{\widetilde{S}'}(n),$$

where $\widetilde{S}' = (\widetilde{L}', A, <)$. This means that $\{x_n : \ n \ge 0\} = \{\langle w \rangle : \ w \in L\} = \{\langle w \rangle : \ w \in L'\}$, where the $w \in L'$ are ordered by the shortlex order on their mirror images.

Let $\mathcal{A}_L = (Q, q_0, A, \tau, F)$ be a (complete) deterministic finite automaton recognizing L, with set of states Q, initial state q_0, transition function $\tau : \ Q \times A \to Q$ and set of final states F. The transition function is extended to words, $\tau : \ Q \times A^* \to Q$, by setting $\tau(q, \varepsilon) = q$ for the empty word ε and $\tau(q, wa) = \tau(\tau(q, w), a)$. A word $w \in A^*$ is accepted by $\mathcal{A}_L$, and thus in L, if and only if $\tau(q, w) \in F$.

Assume that there exists an ordering of the states such that

- the maximal state is the initial state,
- all states except the minimal state are final,
- $\tau(q, a) < \tau(r, a)$ for some $q, r \in Q$, $a \in A$ implies $q < r$,
- $\tau(s, a) = s$ for the minimal state s and all $a \in A$.

An automaton satisfying this property will be called *automaton with ordered states.*

From now on, all automata will be automata with ordered states with set of states $Q = \{0, 1, \ldots, d\}$, thus initial state $q_0 = d$ and set of final states $F = \{1, \ldots, d\}$.

Lemma 1. *If L is recognized by an automaton with ordered states $\mathcal{A}_L = (Q, d, A, \tau, Q\setminus\{0\})$, $Q = \{0, 1, \ldots, d\}$, then $\widetilde{L}$ is recognized by $\mathcal{A}_{\widetilde{L}} = (Q, d, A, \widetilde{\tau}, Q \setminus \{0\})$, where*

$$\widetilde{\tau}(r, a) = \#\{q \in Q : \tau(q, a) + r > d\} \quad \text{for all} \quad r \in Q,\, a \in A.$$

$\mathcal{A}_{\widetilde{L}}$ is an aumaton with ordered states as well.

Proof. A deterministic automaton $\mathcal{A}'$ recognizing $\widetilde{L}$ is obtained by choosing the set of final states in $\mathcal{A}_L$ as initial state of $\mathcal{A}'$ and setting recursively $\tau'(r, a) = \{q \in Q : \tau(q, a) \in r\}$. The initial state of $\mathcal{A}'$ is thus $\{1, \ldots, d\}$. Because of the ordering of the states, $\tau(q, a) = 0$ implies that $\tau(q', a) = 0$ for all $q' \le q$, hence $\tau'(\{1, \ldots, d\}, a) = \{d-r+1, \ldots, d\}$ for some $r \in Q$ (with $r = 0$ corresponding to the empty set). Similarly, we obtain for all $r \in Q$, $a \in A$, that $\tau'(\{d-r+1, \ldots, d\}, a) = \{d-r'+1, \ldots, d\}$ for some $r' \in Q$, with

$$r' = \#\{q \in Q : \tau(q, a) \in \{d-r+1, \ldots, d\}\} = \#\{q \in Q : \tau(q, a) + r > d\}.$$

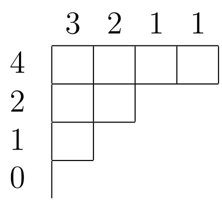
The final states of $\mathcal{A}'$ are all sets containing the initial state d of $\mathcal{A}$: $\{d-r+1, \ldots, d\}$, $1 \le r \le d$. If we label the states by r instead of $\{d-r+1, \ldots, d\}$, then we obtain $\mathcal{A}_{\widetilde{L}}$, which is easily seen to be an automaton with ordered states. $\square$

The next lemma provides a fundamental characterization of the words in a language recognized by an automaton with ordered states.

Lemma 2. *Let $w_1 \cdots w_k \in A^*$. For any $j \in \{0, 1, \ldots, k\}$, we have $w_1 \cdots w_k \in L$ if and only if $\tau(d, w_1 \cdots w_j) + \widetilde{\tau}(d, w_k \cdots w_{j+1}) > d$.*

Proof. By the proof of Lemma 1, $\widetilde{\tau}(d, w_k \cdots w_{j+1}) = r$ means that $\tau(q, w_{j+1} \cdots w_k) > 0$ if and only if $q > d-r$. Therefore we have $\tau(d, w_1 \cdots w_k) = \tau(\tau(d, w_1 \cdots w_j), w_{j+1} \cdots w_k) > 0$ if and only if $\tau(d, w_1 \cdots w_j) + \widetilde{\tau}(d, w_k \cdots w_{j+1}) > d$. $\square$

Remark. If $\tau(d, a) + \cdots + \tau(1, a)$ is considered as a partition, then $\widetilde{\tau}(d, a) + \cdots + \widetilde{\tau}(1, a)$ is the conjugate partition. E.g., if $(\tau(4, a), \ldots, \tau(1, a)) = (4, 2, 1, 0)$, then $(\widetilde{\tau}(4, a), \ldots, \widetilde{\tau}(1, a)) = (3, 2, 1, 1)$, and the corresponding Ferrers diagram is



Let $M_L = (\#\{a \in A : \tau(q, a) = r\})_{d \ge q, r \ge 1}$ be the incidence matrix of the co-accessible part of $\mathcal{A}_L$. (A state q is co-accessible if $\tau(q, w) \in F$ for some $w \in A^*$.) Assume that M_L is primitive, let $\beta > 1$ be its Perron-Frobenius eigenvalue and $(\eta_d, \ldots, \eta_1)^t$ be the corresponding right eigenvector of M_L with $\eta_d = 1$. Set $\eta_0 = 0$ and $\epsilon_q(b) = \sum_{a<b} \eta_{\tau(q,a)}$ for $b \in A$, $q \in Q$.

For an infinite word $u = u_1u_2\cdots$ with $u_1\cdots u_j \in L$ for every $j \geq 1$, let $q_0q_1\cdots$ be the corresponding sequence of states defined by $\tau(q_{j-1}, u_j) = q_j$ for all $j \geq 1$. It was shown in [LR] that $\mathrm{val}_S^\omega(u) = 1/\beta + (\beta-1)\sum_{j=1}^\infty \epsilon_{q_{j-1}}(u_j)\beta^{-j}$ and thus

$$\langle u_1u_2\cdots\rangle = \sum_{j=1}^\infty \epsilon_{q_{j-1}}(u_j)\beta^{-j}.$$

We clearly have $\langle u_1u_2\cdots\rangle \leq \langle u_1'u_2'\cdots\rangle$ if $u_1u_2\cdots$ is lexicographically smaller than $u_1'u_2'\cdots$. The primitivity of M_L implies that $\eta_q > 0$ for all $q > 0$, thus $\epsilon_q(a) < \epsilon_q(b)$ if $a < b$ and $\tau(q,a) > 0$. Therefore we have $\langle v\rangle = \langle w\rangle$ for $v, w \in L$ if and only if v is a prefix of w and no right extension of v with length $|w|$ is lexicographically smaller than w.

Let a_0 be the smallest letter of A and assume that $\widetilde{\tau}(d, a_0) = d$, i.e., $\tau(q, a_0) > 0$ for all $q > 0$. Then we have $va_0^k \in L$ for all $v \in L$, $k \geq 0$, where a_0^k means that the letter a_0 is repeated k times. This implies that $\langle v\rangle = \langle w\rangle$ with $v < w$ if and only if $w = va_0^{|w|-|v|}$, hence L' consists exactly of those words in L which do not end with a_0.

Example. Let $\mathcal{A}_L$ be an automaton with ordered states on the alphabet $A = \{0, 1, \ldots, B\}$ with integers $b_q \in A$, $q \in \{1, \ldots, d\}$, such that $\tau(q,a) = d$ for all $a < b_q$ and $\tau(q,a) = 0$ for all $a > b_q$. Assume that M_L is primitive and let β be its Perron-Frobenius eigenvalue. Then we have

$$\langle u_1u_2\cdots\rangle = \sum_{j=1}^\infty \sum_{a<u_j} \eta_{\tau(q_{j-1},a)}\beta^{-j} = \sum_{j=1}^\infty \sum_{a<u_j} \eta_d\beta^{-j} = \sum_{j=1}^\infty u_j\beta^{-j} \text{ for all } u_1u_2\cdots \in L_\omega.$$

Let $t_1t_2\cdots$ be the lexicographically maximal sequence in L_ω and $q_0q_1\cdots$ the corresponding sequence of states, i.e., $t_j = b_{q_{j-1}}$ and $q_j = \tau(q_{j-1}, b_{q_{j-1}})$ for all $j \geq 1$. Since $\mathcal{A}_L$ is an automaton with ordered states, $q_j < q_k$ implies $b_{q_j} < b_{q_k}$ or $b_{q_j} = b_{q_k}$, $q_{j+1} < q_{k+1}$, thus $t_{j+1}t_{j+2}\cdots \leq t_{k+1}t_{k+2}\cdots$ (with the lexicographical ordering). In particular, we have $t_{j+1}t_{j+2}\cdots \leq t_1t_2\cdots$. Since $\sum_{j=1}^\infty t_j\beta^{-j} = 1$, the sequence $t_1t_2\cdots$ is the expansion of 1 with respect to β if $t_{j+1}t_{j+2}\cdots < t_1t_2\cdots$ for all $j \geq 1$, cf. [Pa, St]. Otherwise, the expansion of 1 is $t_1\cdots t_{j-1}(t_j+1)00\cdots$, where j is the minimal positive integer with $t_{j+1}t_{j+2}\cdots = t_1t_2\cdots$. (The sequence $t_1t_2\cdots$ is sometimes called quasi-greedy or infinite expansion of 1.)

If $u_1u_2\cdots \in L_\omega$, then we have either $u_1u_2\cdots = t_1t_2\cdots$ or some $k \geq 1$ such that $u_1\cdots u_{k-1} = t_1\cdots t_{k-1}$, $u_k < t_k$. Since $q_k = d$ in the latter case, we obtain $u_1u_2\cdots \in L_\omega$ if and only if $u_ju_{j+1}\cdots \leq t_1t_2\cdots$ for all $j \geq 1$. Therefore, $u_1u_2\cdots$ is either the (greedy) β-expansion of $\langle u_1u_2\cdots\rangle$ or its quasi-greedy expansion. Since the abstract van der Corput sequence is defined by finite words $u_1\cdots u_k \in L$, and $u_1\cdots u_k00\cdots$ is always a (greedy) β-expansion, we obtain exactly the β-adic van der Corput sequence defined in [Ni, St]. Therefore we call $\mathcal{A}_L$ a *β-automaton.*

3. Discrepancy function

Let $C(N,I) = \#\{x_n \in I:\ 0 \le n < N\}$. Then we have, for $y = \langle u_1 u_2 \cdots \rangle$,

$$D(N,[0,y)) = \sum_{j=1}^{\infty} \Big(C\big(N, [\langle u_1 \cdots u_{j-1}\rangle, \langle u_1 \cdots u_{j-1} u_j\rangle)\big) - N \epsilon_{q_{j-1}}(u_j) \beta^{-j} \Big).$$

If we set $\mathrm{rep}_{\widetilde{S}'}(N) = w_\ell \cdots w_1$, $w_j = a_0$ for $j > \ell$, $r_j = \widetilde{\tau}(d, w_\ell \cdots w_{j+1})$ for $0 \le j \le \ell$, $r_j = d$ for $j > \ell$, and

$$L_{q,r}^k = \{v_1 \cdots v_k \in A^k : \tau(q, v_1 \cdots v_k) + r > d\} = \{v_1 \cdots v_k \in A^k : q + \widetilde{\tau}(r, v_k \cdots {}_1) > d\}$$

for all $q, r \in Q$, $k \ge 0$, then we obtain the following lemma.

Lemma 3. *If L is recognized by an automaton with ordered states and $\widetilde{\tau}(d, a_0) = d$, then*

$$C\big(N, [\langle u_1 \cdots u_{j-1}\rangle, \langle u_1 \cdots u_{j-1} u_j\rangle)\big) = \sum_{k=j+1}^{\ell} \sum_{a<u_j} \sum_{b<w_k} \# L^{k-j-1}_{\tau(q_{j-1},a), \widetilde{\tau}(r_k,b)} + \mu_j(N,y),$$

$$\mu_j(N,y) = \begin{cases} \#\{a < u_j : \tau(q_{j-1},a) + r_j > d\} & \text{if } u_j \le w_j, \\ \#\{a < w_j : \tau(q_{j-1},a) + r_j > d\} & \text{if } u_j > w_j,\ u_{j-1} \cdots u_1 \ge w_{j-1} \cdots w_1, \\ \#\{a \le w_j : \tau(q_{j-1},a) + r_j > d\} & \text{if } u_j > w_j,\ u_{j-1} \cdots u_1 < w_{j-1} \cdots w_1. \end{cases}$$

Proof. We have to consider the words $v \in L'$ with $\widetilde{v} = \mathrm{rep}_{\widetilde{S}'}(n)$ for some $n < N$. Since $\widetilde{v} < w_\ell \cdots w_1$ if and only if $a_0^{\ell-|v|} \widetilde{v} < w_\ell \cdots w_1$, and $v a_0^{\ell-|v|} \in L$ because of $\widetilde{\tau}(d, a_0) = d$, we can consider the words $v_1 \cdots v_\ell \in L$ with $v_\ell \cdots v_1 < w_\ell \cdots w_1$ instead. The set of these words can be written as $\bigcup_{k=1}^{\ell} \bigcup_{b<w_k} \{v_1 \cdots v_{k-1} b w_{k+1} \cdots w_\ell :\ \widetilde{\tau}(r_k, b v_{k-1} \cdots v_1) > 0\}$.

If $j \le \ell$, then $\langle v_1 \cdots v_\ell \rangle$ is in $\big[\langle u_1 \cdots u_{j-1}\rangle, \langle u_1 \cdots u_{j-1} u_j\rangle\big)$ if and only if $v_1 \cdots v_{j-1} = u_1 \cdots u_{j-1}$ and $v_j < u_j$. For $j < k \le \ell$, every word in $u_1 \cdots u_{j-1} a L^{k-j-1}_{\tau(q_{j-1},a),\widetilde{\tau}(r_k,b)} b w_{k+1} \cdots w_\ell$ with $a < u_j$, $b < w_k$ provides therefore some x_n in the given interval, which proves the main part of the formula. It remains to count the words $u_1 \cdots u_{j-1} a w_{j+1} \cdots w_\ell \in L$ with $a < u_j$ and $a u_{j-1} \cdots u_1 < w_j \cdots w_1$, which provides $\mu_j(N,y)$.

If $j > \ell$, then we have $\langle v_1 \cdots v_\ell \rangle = \langle v_1 \cdots v_\ell a_0^{j-\ell} \rangle \in \big[\langle u_1 \cdots u_{j-1}\rangle, \langle u_1 \cdots u_{j-1} u_j\rangle\big)$ if and only if $u_j > a_0 = w_j$, $u_{j-1} \cdots u_{\ell+1} = a_0 \cdots a_0 = w_{j-1} \cdots w_{\ell+1}$, and $u_\ell \cdots u_1 = v_\ell \cdots v_1 < w_\ell \cdots w_1$. Since $\tau(q_{j-1}, a_0)$ is positive and $r_j = d$, we have $\tau(q_{j-1}, w_j) + r_j > d$. $\square$

Assume that the characteristic polynomial of M_L is irreducible and let $\beta_2, \ldots, \beta_d$ be the conjugates of $\beta_1 = \beta$. Then the characteristic polynomial of $M_{\widetilde{L}}$ is equal to that of M_L. Since $\# L_{q,d}^k = \#\{v \in A^k :\ \tau(q,v) > 0\}$ and $\# L_{d,r}^k = \#\{v \in A^k :\ \widetilde{\tau}(r,v) > 0\}$, we have

$$\# L_{q,r}^k = \sum_{i=1}^{d} \eta_q^{(i)} \theta_r^{(i)} \beta_i^k,$$

where $(\theta_d, \ldots, \theta_1)^t$ is a right eigenvector of $M_{\widetilde{L}}$ to the eigenvalue β, $\theta_0 = 0$ and $z^{(i)}$ denotes the image of z by the isomorphism from $\mathbb{Q}(\beta)$ to $\mathbb{Q}(\beta_i)$ mapping β^j to β_i^j.

Assume furthermore that β is a Pisot number, i.e., that its conjugates satisfy $|\beta_i| \le \rho$ for some $\rho < 1$. With $N = \sum_{k=1}^{\ell} \sum_{b<w_k} \# L_{d,\widetilde{\tau}(r_k,b)}^{k-1}$, we obtain

$$\begin{aligned}
D(N,[0,y)) &= \sum_{j=1}^{\infty} \Bigg(\sum_{k=j+1}^{\ell} \sum_{a<u_j} \sum_{b<w_k} \# L_{\tau(q_{j-1},a),\widetilde{\tau}(r_k,b)}^{k-j-1} + \mu_j(N,y) - \sum_{k=1}^{\ell} \sum_{b<w_k} \# L_{d,\widetilde{\tau}(r_k,b)}^{k-1} \frac{\epsilon_{q_{j-1}}(u_j)}{\beta^j} \Bigg) \\
&= \sum_{j=1}^{\infty} \Bigg(\sum_{a<u_j} \sum_{k=j+1}^{\ell} \sum_{b<w_k} \sum_{i=2}^{d} \theta_{\widetilde{\tau}(r_k,b)}^{(i)} \Big(\eta_{\tau(q_{j-1},a)}^{(i)} \beta_i^{k-j-1} - \eta_{\tau(q_{j-1},a)} \beta_i^{k-1} \beta^{-j} \Big) + \mu_j(N,y) \\
&\qquad - \sum_{a<u_j} \sum_{k=1}^{j} \sum_{b<w_k} \sum_{i=1}^{d} \theta_{\widetilde{\tau}(r_k,b)}^{(i)} \eta_{\tau(q_{j-1},a)} \beta_i^{k-1} \beta^{-j} \Bigg) = \sum_{j=1}^{\infty} \sum_{a<u_j} \mathcal{O}(1).
\end{aligned} \tag{1}$$

Changing the order of summation gives

$$\begin{aligned}
D(N,[0,y)) &= \sum_{k=1}^{\ell} \sum_{b<w_k} \Bigg(\sum_{j=1}^{k-1} \sum_{i=2}^{d} \theta_{\widetilde{\tau}(r_k,b)}^{(i)} \Big(\epsilon_{q_{j-1}}^{(i)}(u_j) \beta_i^{k-j-1} - \epsilon_{q_{j-1}}(u_j) \beta_i^{k-1} \beta^{-j} \Big) \\
&\qquad - \sum_{j=k}^{\infty} \sum_{i=1}^{d} \theta_{\widetilde{\tau}(r_k,b)}^{(i)} \epsilon_{q_{j-1}}(u_j) \beta_i^{k-1} \beta^{-j} \Bigg) + \sum_{j=1}^{\infty} \mu_j(N,y) \\
&= \sum_{k=1}^{\ell} \Bigg(\mu_k(N,y) - \sum_{b<w_k} \theta_{\widetilde{\tau}(r_k,b)} \sum_{j=k}^{\infty} \frac{\epsilon_{q_{j-1}}(u_j)}{\beta^{j-k+1}} + \sum_{b<w_k} \sum_{i=2}^{d} \theta_{\widetilde{\tau}(r_k,b)}^{(i)} \sum_{j=1}^{k-1} \epsilon_{q_{j-1}}^{(i)}(u_j) \beta_i^{k-j-1} \Bigg) + \mathcal{O}(1) \\
&= \sum_{k=1}^{\ell} \sum_{b<w_k} \mathcal{O}(1) + \mathcal{O}(1) = \mathcal{O}(\log N).
\end{aligned}$$

An automaton satisfying the above assumptions that the incidence matrix of the co-accessible part has one simple eigenvalue $\beta > 1$ and all other eigenvalues in the interior of the unit disk will be called a *Pisot automaton.* The above calculations prove the following theorem.

Theorem 4. *Let $S = (L, A, <)$ be an abstract numeration system where L is recognized by a Pisot automaton with ordered states and $\widetilde{\tau}(d, a_0) = d$ for the minimal letter $a_0 \in A$. Then the corresponding abstract van der Corput sequence is a low discrepancy sequence.*

This theorem is a generalization of Ninomiya's result for β-adic van der Corput sequences: If β is a Pisot number, then the infinite expansion of 1 is eventually periodic, i.e., $t_1 t_2 \cdots = t_1 \cdots t_m (t_{m+1} \cdots t_d)^\omega$ with $d > m \ge 0$. If d is chosen minimally and the β-polynomial $(x^d - t_1 x^{d-1} - \cdots - t_d) - (x^m - t_1 x^{m-1} - \cdots - t_m)$ is irreducible, then the β-automaton with $b_q = t_j$ for $q = \#\{k \le d :\ t_k t_{k+1} \cdots \le t_j t_{j+1} \cdots\}$ satisfies the assumptions of this theorem.

Now assume $y \in \mathbb{Q}(\beta)$, which is equivalent to $u_1u_2\cdots$ being eventually periodic, see [RS]. If we set $y_k = \sum_{j=k}^{\infty} \epsilon_{q_{j-1}(u_j)}\beta^{k+1-j}$, and $u_1u_2\cdots$, $q_0q_1\cdots$ have period length p, then

$$\sum_{j=1}^{k-1} \epsilon^{(i)}_{q_{j-1}}(u_j)\beta_i^{k-j-1} = \big(\epsilon^{(i)}_{q_{k-p-1}}(u_{k-p})\beta_i^{p-1} + \cdots + \epsilon^{(i)}_{q_{k-2}}(u_{k-1})\big)\big(1+\beta_i^p+\beta_i^{2p}+\cdots\big) + \mathcal{O}(\beta_i^k)$$
$$= \Big(\frac{\epsilon_{q_{k-1}}(u_j)\beta^{p-1} + \cdots + \epsilon_{q_{k+p-2}}(u_{k+p-1})}{1-\beta^p}\Big)^{(i)} + \mathcal{O}(\beta_i^k) = -y_k^{(i)} + \mathcal{O}(\beta_i^k)$$

for $2 \le i \le d$. This gives

$$D(N,[0,y)) = \sum_{k=1}^{\ell}\Bigg(\mu_k(N,y) - \sum_{b<w_k}\sum_{i=1}^{d}\theta^{(i)}_{\widetilde{\tau}(r_k,b)}y_k^{(i)}\Bigg) + \mathcal{O}(1).$$

If we set $\zeta_r(x) = \sum_{i=1}^{d}\theta_r^{(i)}x^{(i)}$ for $x \in \mathbb{Q}(\beta)$, $r \in Q$, then

$$D(N,[0,y)) = \sum_{k=1}^{\ell}\Bigg(\mu_k(N,y) - \sum_{b<w_k}\zeta_{\widetilde{\tau}(r_k,b)}(y_k)\Bigg) + \mathcal{O}(1). \tag{2}$$

4. Bounded remainder sets

In this section, we prove the following theorem.

Theorem 5. *Let $(x_n)_{n\ge 0}$ be an abstract van der Corput sequence defined by $(L,A,<)$ where L is recognized by a Pisot automaton with ordered states and $\widetilde{\tau}(q,a_0^{d-1}) = d$ for the minimal letter $a_0 \in A$ and all $q > 0$. Let $y = \langle u_1u_2\cdots\rangle$, $y_k = \sum_{j=k}^{\infty}\epsilon_{q_{j-1}}(u_j)\beta^{k+1-j}$ for $k \ge 1$.*

Then $D(N,[0,y))$ is bounded in N if and only if $y \in \mathbb{Q}(\beta)$ and there exists some $m \ge 1$ such that either $y_m = 0$ or

$$\zeta_{\widetilde{\tau}(d,v_\ell\cdots v_k)}(y_k) = \begin{cases} 1 & \text{if } v_k\cdots v_\ell \le u_k\cdots u_\ell \text{ and } u_1\cdots u_{k-1}v_k\cdots v_\ell \in L \\ 0 & \text{else} \end{cases} \tag{3}$$

for all $v_k\cdots v_\ell \in A^*$, $m \le k \le \ell$.

The proof of the theorem is split up into three propositions. Note that the conditions for Propositions 6 and 7 are weaker than those for Theorem 5.

Proposition 6. *Let $(x_n)_{n\ge 0}$ be an abstract van der Corput sequence defined by $(L,A,<)$ where L is recognized by an automaton with ordered states, $\widetilde{\tau}(d,a_0) = d$, and the incidence matrix M_L is primitive with Perron-Frobenius eigenvalue β.*

If $D(N,I)$ is bounded, then $\lambda(I) \in \mathbb{Q}(\beta)$.

Proof. Proposition 6 is proved in the same way as Theorem 1 in [St]. Define a substitution $q \mapsto \widetilde{\tau}(q,a_1)\cdots\widetilde{\tau}(q,a_m)$, with $\{a_1,\ldots,a_m\} = \{a \in A : \widetilde{\tau}(q,a) > 0\}$ and $a_1 < \cdots < a_m$, $1 \le q \le d$, which plays the role of the substitution τ in [St]. Since $\widetilde{\tau}(d,a_0) = d$, we have $d \mapsto dw$ for some $w \in A^*$. Then a continuous successor function on L_ω (with the usual topology on right infinite words) satisfying $\mathrm{rep}_{\widetilde{S}'}(n)a_0^\omega \mapsto \mathrm{rep}_{\widetilde{S}'}(n+1)a_0^\omega$ is topologically conjugate to the successor function on $\mathcal{D}$ defined in [St]. $\square$

Proposition 7. *Let $(x_n)_{n\ge 0}$ be an abstract van der Corput sequence defined by $(L, A, <)$ where L is recognized by a Pisot automaton with ordered states and $\widetilde{\tau}(d, a_0) = d$.*

If $y \in \mathbb{Q}(\beta)$ and there exists some $m \ge 1$ such that either $y_m = 0$ or (3) holds for all $v_k \cdots v_\ell \in A^*$, $m \le k \le \ell$, then $D(N, [0, y))$ is bounded.

Proof. Let $\mathrm{rep}_{\widetilde{S}'}(N) = w_\ell \cdots w_1$ and $r_k = \widetilde{\tau}(d, w_\ell \cdots w_{k+1})$, $0 \le k \le \ell$. If $y_m = 0$ for some $m \ge 1$, then $u_j = a_0$ for all $j \ge m$ and the result follows from (1). Otherwise, we have

$$\zeta_{\widetilde{\tau}(r_k,b)}(y_k) = \zeta_{\widetilde{\tau}(d,w_\ell\cdots w_{k+1}b)}(y_k) = \begin{cases} 1 & \text{if } bw_{k+1}\cdots w_\ell \le u_k \cdots u_\ell,\ \tau(q_{k-1}, b) + r_k > d, \\ 0 & \text{else,} \end{cases}$$

for $k \ge m$, thus

$$\mu_k(N, y) - \sum_{b<w_k} \zeta_{\widetilde{\tau}(r_k,b)}(y_k) = \begin{cases} 1 & \text{if } u_k > w_k,\ u_{k-1}\cdots u_1 < w_{k-1}\cdots w_1,\ q_{k-1} + r_{k-1} > d, \\ -1 & \text{if } u_k < w_k,\ u_{k+1}\cdots u_\ell \ge w_{k+1}\cdots w_\ell,\ q_k + r_k > d, \\ 0 & \text{else.} \end{cases}$$

Denote this difference by Δ_k. If $\Delta_k = -1$, $m \le k \le \ell$, and $u_{k+1}\cdots u_\ell > w_{k+1}\cdots w_\ell$, then we have $\Delta_j = 1$, where $j > k$ is defined by $u_{k+1}\cdots u_{j-1} = w_{k+1}\cdots u_{j-1}$, $u_j > w_j$, and $\Delta_{k+1} = \cdots = \Delta_{j-1} = 0$. If $\Delta_k = 1$, $m \le k \le \ell$, then let $u_{k-1}\cdots u_{j+1} = w_{k-1}\cdots w_{j+1}$, $u_j < w_j$, $j < k$, and we obtain $\Delta_j = -1$ if $j \ge m$. Therefore the 1's and (-1)'s alternate in $\Delta_m \cdots \Delta_\ell$, hence $\sum_{k=1}^{\ell} \Delta_k$ is bounded and $D(N, [0, y)) = \mathcal{O}(1)$ by (2). $\square$

Proposition 8. *Let $(x_n)_{n\ge 0}$ be as in Theorem 5. If $D(N, [0, y))$ is bounded, then $y \in \mathbb{Q}(\beta)$ and there exists $m \ge 1$ such that either $y_m = 0$ or (3) holds for all $v_k \cdots v_\ell \in A^*$, $m \le k \le \ell$.*

Proof. By Proposition 6, we have $y \in \mathbb{Q}(\beta)$ and thus $u_1 u_2 \cdots = u_1 \cdots u_{m'} (u_{m'+1} \cdots u_{m'+p})^\omega$, $q_0 q_1 \cdots = q_0 \cdots q_{m'-1} (q_{m'} \cdots q_{m'+p-1})^\omega$ for some $m' \ge 0$, $p \ge 1$, by [RS]. We can assume $u_{m'+1} \cdots u_{m'+p} > a_0 \cdots a_0$ since $y_{m'+1} = 0$ otherwise.

Let $m = m' + \max(d, p+1)$ and $v_k \cdots v_\ell \in A^*$, $m \le k \le \ell$. If $v_k \cdots v_\ell \notin L$, then (3) holds since $\zeta_{\widetilde{\tau}(d,v_\ell\cdots v_k)}(y_k) = \zeta_0(y_k) = 0$. If $v_k \cdots v_\ell \in L \setminus a_0^*$, then assume w.l.o.g. $v_\ell > a_0$, since $\widetilde{\tau}(d, a_0^{d-1}) = d$ implies $\widetilde{\tau}(d, a_0) = d$, thus (3) holds for $a_0 v_{\ell-1} \cdots v_k$ if it holds for $v_{\ell-1} \cdots v_k$.

Let $J \ge 1$ be such that $Jp \ge \ell - k + d$. Then $\widetilde{\tau}(v_\ell \cdots v_k a_0^{Jp-\ell+k-1}) = d$, and we define

$$N_K = \mathrm{val}_{\widetilde{S}'}\big((v_\ell \cdots v_k a_0^{Jp-\ell+k-1})^K v_\ell \cdots v_k a_0^{k-1}\big)$$

for $K \ge 0$. If furthermore $Jp > \ell - k + p$, then $\mu_j(N_0, y) = \mu_{j+hJp}(N_K, y)$ for $k \le j < k + Jp$, $0 \le h \le K$. With (2), we obtain

$$D(N_K, [0, y)) = (K+1)\Bigg(\sum_{j=k}^{k+Jp-1} \mu_j(N_0, y) - \sum_{j=k}^{\ell} \sum_{b<v_j} \zeta_{\widetilde{\tau}(d,v_\ell\cdots v_{j+1}b)}(y_j)\Bigg) + \mathcal{O}(1).$$

Therefore $D(N, [0, y)) = \mathcal{O}(1)$ implies

$$\sum_{j=k}^{\infty} \mu_j(N_0, y) = \sum_{j=k}^{\infty} \sum_{b<v_j} \zeta_{\widetilde{\tau}(r_j,b)}(y_j), \tag{4}$$

where $r_j = \widetilde{\tau}(d, v_\ell \cdots v_{j+1})$ for $k \le j \le \ell$, $r_j = d$ and $v_j = a_0$ for $j > \ell$.

Assume first that there exists some $v_k' > v_k$ such that $v_k' v_{k+1} \cdots v_\ell \in L$, and consider $N_0' = \mathrm{val}_{\widetilde{S}'}(v_\ell \cdots v_{k+1} v_k' a_0^{k-1})$. Then we have

$$\sum_{j=k+1}^{\infty} \mu_j(N_0', y) - \sum_{j=k+1}^{\infty} \mu_j(N_0, y) = \begin{cases} 1 & \text{if } v_k \le u_k < v_k',\ u_{k+1} \cdots u_{j-1} = v_{k+1} \cdots v_{j-1}, \\ & \quad u_j > v_j,\ \tau(q_{j-1}, v_j) + r_j > d \text{ for some } j > k, \\ 0 & \text{else.} \end{cases}$$
$$= \begin{cases} 1 & \text{if } v_k \le u_k < v_k',\ v_{k+1} \cdots v_\ell \le u_{k+1} \cdots u_\ell,\ q_k + r_k > d, \\ 0 & \text{else.} \end{cases}$$

If $\widetilde{\tau}(r_k, b) = 0$ for $v_k < b < v_k'$, then we have furthermore

$$\sum_{b < v_k'} \zeta_{\widetilde{\tau}(r_k, b)}(y_k) - \sum_{b < v_k} \zeta_{\widetilde{\tau}(r_k, b)}(y_k) = \zeta_{\widetilde{\tau}(r_k, v_k)}(y_k),$$
$$\mu_k(N_0', y) - \mu_k(N_0, y) = \begin{cases} 1 & \text{if } v_k < u_k,\ q_{k-1} + r_{k-1} > d, \\ 0 & \text{else.} \end{cases}$$

By using (4) for N_0 and N_0', we obtain

$$\zeta_{\widetilde{\tau}(d, v_\ell \cdots v_k)}(y_k) = \zeta_{\widetilde{\tau}(r_k, v_k)}(y_k) = \begin{cases} 1 & \text{if } v_k \cdots v_\ell \le u_k \cdots u_\ell,\ q_{k-1} + r_{k-1} > d, \\ 0 & \text{else,} \end{cases}$$

since $v_k < u_k < v_k'$ implies $\widetilde{\tau}(r_k, u_k) = 0$, thus $q_k + r_k \le d$, and $v_k = u_k$ implies that $q_k + r_k > d$ is equivalent with $q_{k-1} + r_{k-1} > d$. Thus (3) holds in this case.

If $v_k \cdots v_\ell = a_0 \cdots a_0$, then similar arguments apply, hence $\zeta_d(y_k) = 1$ unless $y_k = 0$.

Assume now $\widetilde{\tau}(r_k, b) = 0$ for all $b > v_k$, and consider

$$\zeta_{r_k}(y_{k+1}) = \sum_{i=1}^{d} \theta_{r_k}^{(i)} \big(\beta y_k - \epsilon_{q_{k-1}}(u_k)\big)^{(i)} = \sum_{i=1}^{d} \beta_i \theta_{r_k}^{(i)} y_k^{(i)} - \sum_{i=1}^{d} \theta_{r_k}^{(i)} \sum_{b < u_k} \eta_{\tau(q_{k-1}, b)}^{(i)}.$$

Using $\beta_i \theta_{r_k}^{(i)} = \sum_{b \in A} \theta_{\widetilde{\tau}(r_k, a)}^{(i)}$ since $(\theta_d^{(i)}, \ldots, \theta_1^{(i)})^t$ is an eigenvector of $M_{\widetilde{L}}$, and

$$\sum_{i=1}^{d} \eta_{\tau(q_{k-1}, b)}^{(i)} \theta_{r_k}^{(i)} = \# L_{\tau(q_{k-1}, b), r_k}^{0} = \begin{cases} 1 & \text{if } \tau(q_{k-1}, b) + r_k > d, \\ 0 & \text{else,} \end{cases}$$

we obtain

$$\zeta_{r_k}(y_{k+1}) = \sum_{b \in A} \zeta_{\widetilde{\tau}(r_k, b)}(y_k) - \#\{b < u_k : q_{k-1} + \widetilde{\tau}(r_k, b) > d\}.$$

We already know that (3) holds for $\zeta_{\widetilde{\tau}(r_k, b)}(y_k) = \zeta_{\widetilde{\tau}(d, v_\ell \cdots v_{k+1} b)}(y_k)$, $b < v_k$, hence

$$\zeta_{r_k}(y_{k+1}) = \zeta_{\widetilde{\tau}(r_k, v_k)}(y_k) + \begin{cases} -1 & \text{if } v_k < u_k,\ q_{k-1} + r_{k-1} > d, \\ 1 & \text{if } v_k > u_k,\ v_{k+1} \cdots v_\ell \le u_{k+1} \cdots u_\ell,\ q_k + r_k > d, \\ 0 & \text{else.} \end{cases}$$

If $v_{k+1} < \max\{b \in A : \widetilde{\tau}(r_{k+1}, b) > 0\}$, then (3) holds for $\zeta_{r_k}(y_{k+1}) = \zeta_{\widetilde{\tau}(d, v_\ell \cdots v_{k+1})}(y_{k+1})$. If $\zeta_{r_k}(y_{k+1}) = \zeta_{\widetilde{\tau}(r_k, v_k)}(y_k) - 1$, then $v_k < u_k$ implies $u_k v_{k+1} \cdots v_\ell \notin L$, hence $\zeta_{r_k}(y_{k+1}) = 0$.

Since $\zeta_{r_k}(y_{k+1}) = \zeta_{\widetilde{\tau}(r_k,v_k)}(y_k) + 1$ implies $\zeta_{r_k}(y_{k+1}) = 1$, we get

$$\zeta_{\widetilde{\tau}(r_k,v_k)}(y_k) = \begin{cases} 1 & \text{if } 0 = \zeta_{r_k}(y_{k+1}) = \zeta_{\widetilde{\tau}(r_k,v_k)}(y_k) - 1 \text{ or } 1 = \zeta_{r_k}(y_{k+1}) = \zeta_{\widetilde{\tau}(r_k,v_k)}(y_k), \\ 0 & \text{else,} \end{cases}$$

hence (3) holds for $\zeta_{\widetilde{\tau}(d,v_\ell\cdots v_k)}(y_k)$ in this case as well.

Finally, there exists some $j \geq k$ such that $v_j < \max\{b \in A : \widetilde{\tau}(r_j, b) > 0\}$. Then we obtain inductively that (3) holds for $\zeta_{\widetilde{\tau}(d,v_\ell\cdots v_j)}(y_j), \ldots, \zeta_{\widetilde{\tau}(d,v_\ell\cdots v_k)}(y_k)$. □

In the case of β-adic van der Corput sequences, the bounded remainder sets $[0, y)$ are characterized by the fact that $y_m = \eta_q$ for some $m \geq 1$, $q \in Q$. In the more general case, we have the following partial characterization.

Proposition 9. *Let $(x_n)_{n\geq 0}$ be an abstract van der Corput sequence defined by $(L, A, <)$ where L is recognized by a Pisot automaton with ordered states and $\widetilde{\tau}(d, a_0) = d$.*

If there exists some $m \geq 1$ and some $s \in Q$ such that, for all $k \geq m$, $\epsilon_{q_{k-1}}(u_k) = \epsilon_{\tau(s,u_m\cdots u_{k-1})}(u_k)$ and $\tau(s, u_m \cdots u_{k-1}b) = 0$ for all $b > u_k$, then $D(N, [0, y))$ is bounded.

Proof. Let $s_k = \tau(s, u_m \cdots u_{k-1})$ for $k \geq m$. If $s_k = 0$ for some $k \geq m$, then we have $y_k = 0$ and $D(N, [0, y))$ is bounded. Therefore we can assume $s_k > 0$ for all $k \geq m$. Then $u_m u_{m+1} \cdots$ is the lexicographically maximal sequence accepted from s, which implies $y_m = \eta_s$, in particular $y \in \mathbb{Q}(\beta)$. We provide two differents ways to complete the proof.

First, assume w.l.o.g. $m \geq d$. Then the primitivity of M_L implies $\tau(d, v_1 \cdots v_{m-1}) = s$ for some $v_1 \cdots v_{m-1} \in L$. Let $z = \langle v_1 \cdots v_{m-1} u_m u_{m+1} \cdots \rangle$. If $v_1 \cdots v_{m-1}$ is the maximal word of length $m-1$ in L, then $v_1 \cdots v_{m-1} u_m u_{m+1} \cdots$ is the lexicographically maximal sequence in L_ω, hence $z = 1$ and $D(N, [0, z)) = 0$. Otherwise, we have $z = \langle w_1 \cdots w_{m-1} a_0 a_0 \cdots \rangle$, where $w_1 \cdots w_{m-1}$ is the successor of $v_1 \cdots v_{m-1}$ in L, thus $D(N, [0, z))$ is bounded as well. We have $y_k = z_k$ for all $k \geq m$ and $\mu_k(N, y) = \mu_k(N, z)$ for almost all $k \geq m$, thus

$$D(N, [0, y)) = \sum_{k=m}^{\ell} \Bigg(\mu_k(N, y) - \sum_{b<w_k} \zeta_{\widetilde{\tau}(r_k,b)}(y_k) \Bigg) + \mathcal{O}(1) = D(N, [0, z)) + \mathcal{O}(1) = \mathcal{O}(1).$$

The second proof uses Proposition 7. Since $y_k = \eta_{s_k}$ for all $k \geq m$, we have $\zeta_r(y_k) = \#L^0_{s_k,r}$ for all $r \in Q$. By the ordering of the states and the primitivity of the matrix, $q > r$ implies $\eta_q > \eta_r$, and $\epsilon_q(a) = \epsilon_r(a)$ implies therefore $\tau(q, b) = \tau(r, b)$ for all $b < a$. In case $v_k \cdots v_\ell < u_k \cdots u_\ell$, we have thus $\tau(q_{k-1}, v_k \cdots v_\ell) = \tau(s_k, v_k \cdots v_\ell)$, which means that $u_1 \cdots u_{k-1} v_k \cdots v_\ell \in L$ is equivalent with $s_k + \widetilde{\tau}(d, v_\ell \cdots v_k) > d$, which is equivalent with $\zeta_{\widetilde{\tau}(d,v_\ell\cdots v_k)} = 1$. For $v_k \cdots v_\ell = u_k \cdots u_\ell$, we have $\tau(s_k, v_k \cdots v_\ell) > 0$, hence $\zeta_{\widetilde{\tau}(d,v_\ell\cdots v_k)} = 1$. In case $v_k \cdots v_\ell > u_k \cdots u_\ell$, we have $\tau(s_k, v_k \cdots v_\ell) = 0$ and thus $\widetilde{\tau}(d, v_\ell \cdots v_k) + s_k \leq d$, $\zeta_{\widetilde{\tau}(d,v_\ell\cdots v_k)} = 0$. Therefore, (3) holds for all $v_k \cdots v_\ell \in A^*$, $m \leq k \leq \ell$. □

We conclude with an example which shows that there might be bounded remainder sets $[0, y)$, where y does not satisfy the conditions of Proposition 9.

Example. Let $A = Q = \{0, 1, \ldots, 4\}$, and $\mathcal{A}_L$ be given by the transition table

$$(\tau(q,a))_{4\ge q\ge 1,\, 0\le a\le 4} = \begin{pmatrix} 4 & 4 & 4 & 4 & 2 \\ 4 & 4 & 4 & 3 & 1 \\ 3 & 3 & 2 & 2 & 1 \\ 3 & 3 & 2 & 1 & 0 \end{pmatrix}, \text{ hence } (\widetilde{\tau}(q,a))_{4\ge q\ge 0,\, 0\le a\le 3} = \begin{pmatrix} 4 & 4 & 4 & 4 & 3 \\ 4 & 4 & 4 & 3 & 1 \\ 4 & 4 & 2 & 2 & 0 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

(Remember that the columns $\widetilde{\tau}(.,a)$ are obtained by conjugating the Ferrers diagram corresponding to $\tau(.,a)$.) If $y = \langle 4033\cdots\rangle$, then $q_0 q_1 \cdots = 4233\cdots$. For $k \ge 3$, we have thus $\epsilon_{q_{k-1}}(u_k) = 3\eta_4$, $y_k = \eta_3 - \eta_2 + \eta_1$, which implies $\zeta_4(y_k) = \zeta_2(y_k) = 1$, $\zeta_3(y_k) = \zeta_1(y_k) = 0$. It can be easily verified that (3) holds for all $v_k \cdots v_\ell \in A^*$, $3 \le k \le \ell$, but the conditions on y of Proposition 9 are not satisfied. However, $\mathcal{A}_L$ is not a Pisot automaton.

It is an open question whether there exists an abstract van der Corput sequence with a bounded remainder set $[0, y)$ such that $y_m \ne \eta_q$ for all $m \ge 1$, $q \in Q$.

We conclude by the remark that the boundedness of $D(N, I)$ is not invariant under translation of the interval, i.e., $D(N, [z, y + z))$ can be unbounded if $[0, y)$ is a bounded remainder set and vice versa, see [St].

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