Additive functions and number systems

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Outline

Number systems and additive functions

Arithmetical properties

Asymptotic distribution

Normal numbers

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Number systems

Let $\mathcal{R}$ be an integral domain,
$b \in \mathcal{R}$, and $\mathcal{N} = \{n_1, \dots, n_m\} \subset \mathcal{R}$.
Then we call the pair $(b, \mathcal{N})$ a *number system* in $\mathcal{R}$ if every $g \in \mathcal{R}$ admits a unique and finite representation of the form

$$g = \sum_{j=0}^{h} a_j(g) b^j \quad \text{with} \quad a_i(g) \in \mathcal{N} \quad \text{for} \quad i = 0, \dots, h \tag{1}$$

and $a_h(g) \neq 0$ if $h \neq 0$. We call b the base and $\mathcal{N}$ the set of digits.

Examples for number systems

- $b \in \mathbb{Z}$, $b \leq -2$ and $\mathcal{N} := \{0, 1, \dots, |b| - 1\}$, then $(b, \mathcal{N})$ is a number system in $\mathbb{Z}$.
- $B \in \mathbb{F}_q[X]$ a polynomial, $\deg B > 1$, $\mathcal{N} := \{P \in \mathbb{F}_q[X] : \deg P < \deg B\}$. then $(B, \mathcal{N})$ is a number system in $\mathbb{F}_q[X]$.
- Let β be an algebraic integer over $\mathbb{Z}$. Furthermore let $b \in \mathbb{Z}[\beta]$ and $\mathcal{N} := \{0, 1, \dots, \mathrm{N}(b) - 1\}$. Then under certain circumstances the pair $(b, \mathcal{N})$ is a number system in $\mathbb{Z}[\beta]$.

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Additive functions

Let $\mathcal{R}$ be an integral domain and $(b, \mathcal{N})$ be a number system in this domain.
Then we call a function $f : \mathcal{R} \to \mathbb{R}$ *b-additive*, if for g as in (1) we have that

$$f(g) = \sum_{k=0}^{h} f(a_k b^k).$$

Moreover we call it *strictly b-additive*, if for g as in (1) we have that

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Moreover we call it strictly b-additive, if for g as in (1) we have that

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The sum-of-digits function

A very simple example of a strictly b-additive function is the sum-of-digits function s_b, which is defined by

$$s_b(g) = \sum_{k=0}^{h} a_k$$

for g as in (1).

Outline

Delange's Result

Theorem Delange (1975)

$$\sum_{n\leq x} s_q(n) = \frac{q-1}{2} N \log_q N + NF\left(\log_q N\right),$$

where $\log_q$ *is the logarithm to base* q *and* F *is a* 1*-periodic, continuous and nowhere differentiable function.*

Peter's Result

Theorem Peter (2002)

There are $c \in \mathbb{R}$ *and* $\varepsilon > 0$ *such that*

$$\sum_{n \leq N} s_q(\lfloor \alpha n^k \rfloor) = \frac{q-1}{2} N \log_q(\alpha N^k) + cN + NF\left(\log_q(\alpha N^k)\right) + \mathcal{O}(N^{1-\varepsilon})$$

where $\lfloor x \rfloor$ *is the greatest integer less than* x*,* F *a* 1*-periodic function and* $\alpha = 1$ *or* $\alpha > 0$ *an irrational of finite type.*

Pseudo polynomial

Let $\alpha_0, \beta_0, \dots, \alpha_d, \beta_d \in \mathbb{R}$, $\alpha_0 > 0$ and $\beta_0 > \beta_1 > \cdots > \beta_d \geq 1$, where at least one $\beta_i \notin \mathbb{Z}$. Then we define a *pseudo* polynomial p as

$$p(x) := \alpha_0 x^{\beta_0} + \cdots + \alpha_d x^{\beta_d}.$$

Over a pseudo-polynomial sequence

Theorem Nakai and Shiokawa (1990)

Let p be a pseudo polynomial. Then

$$\sum_{n\leq N} s_q(\lfloor p(n)\rfloor) = \frac{q-1}{2} N \log_q p(N) + \mathcal{O}(N)$$

where $\log_q$ denotes the logarithm to base q.

Arbitrary additive functions

Theorem M (201?)

Let $q \in \mathbb{N} \setminus \{1\}$ *and* f *be a strictly* q*-additive function with* $f(0) = 0$*. If* p *is a pseudo polynomial, then there exists* $\varepsilon > 0$ *such that*

$$\sum_{n \le N} f\left(\lfloor p(n) \rfloor\right) = \mu_f N \log_q(p(N)) + NF\left(\log_q(p(N))\right) + \mathcal{O}\left(N^{1-\varepsilon}\right).$$

Outline

Asymptotic distribution in $\mathbb{Z}$

Let f be a q-additive function such that $f(aq^k) = \mathcal{O}(1)$ as $k \to \infty$ and $a \in \mathcal{N}$. Furthermore let

$$m_{k,q} := \frac{1}{q} \sum_{a \in \mathcal{N}} f(aq^k), \quad \sigma_{k,q}^2 := \frac{1}{q} \sum_{a \in \mathcal{N}} f^2(aq^k) - m_{k,q}^2,$$

and

$$M_q(x) := \sum_{k=0}^{N} m_{k,q}, \quad D_q^2(x) = \sum_{k=0}^{N} \sigma_{k,q}^2$$

with $N = [\log_q x]$.

Asymptotic distribution in $\mathbb{Z}$

Theorem Bassily and Katái (1995)

Assume that $D_q(x)/(\log x)^{1/3} \to \infty$ as $x \to \infty$ and let $p(x)$ be a polynomial with integer coefficients, degree d and positive leading term. Then, as $x \to \infty$,

$$\frac{1}{x}\#\left\{n < x \middle| \frac{f(p(n)) - M_q(x^d)}{D_q(x^d)} < y\right\} \to \Phi(y),$$

where Φ is the normal distribution function.

Length of expansion

Theorem Kovacs and Pethő (1992)

Let $\ell(\gamma)$ be the length of the expansion of γ to the base b. Then

$$\left|\ell(\gamma) - \max_{1\leq i\leq n} \frac{\log\left|\gamma^{(i)}\right|}{\log\left|b^{(i)}\right|}\right| \leq C.$$

Area of interest

We fix a T and set T_i for $1 \leq i \leq n$ such that

$$\log T_i = \log T \frac{\log \left| b^{(i)} \right|^n}{\log |\mathrm{N}(b)|}.$$

Furthermore we will write

$$N(\mathbf{T}) = N(T_1, \ldots, T_r) := \left\{ \lambda \in R : \left| \lambda^{(i)} \right| \leq T_i, 1 \leq i \leq r \right\}.$$

Asymptotic distribution in $\mathbb{Z}[\beta]$

Theorem M (2009)

Assume that there exists an $\varepsilon > 0$ such that $D_b(x)/(\log x)^\varepsilon \to \infty$ as $x \to \infty$ and let p be a polynomial of degree d. Then, as $T \to \infty$,

$$\frac{1}{\# N(\mathbf{T})} \# \left\{ z \in N(\mathbf{T}) \middle| \frac{f(\lfloor p(z) \rfloor) - M_b(T^d)}{D_b(T^d)} < y \right\} \to \Phi(y),$$

where Φ is the normal distribution function.

Some remarks

- It should suffices that
$$D_b(x) \to \infty \quad \text{for} \quad x \to \infty.$$
(The reason for that will follow in the last section.)
- One can replace $p(n)$ by $\lfloor p(n) \rfloor$. Also shifting of the "decimal" dot is possible.

Outline

Continuation

We extend our number system onto $\mathcal{K}_\infty$ the completion of the field of quotients $\mathcal{K}$ of $\mathcal{R}$. Then we get that every $\gamma \in \mathcal{K}_\infty$ has a (not necessarily unique) representation of the shape

$$\gamma = \sum_{j=-\infty}^{\ell(\gamma)} a_j(\gamma) b^j \quad (a_j(\gamma) \in \mathcal{N}).$$

Fundamental domain

In this context the *fundamental domain* $\mathcal{F}$ indicates the properties of this extension. It is defined as all numbers with zero in the integer part of their b-ary representation, *i.e.*,

$$\mathcal{F} := \left\{ \gamma \in \mathcal{K}_\infty \middle| \gamma = \sum_{j \geq 1} a_j b^{-j}, a_j \in \mathcal{N} \right\}.$$

Block count

Let $\theta \in \mathcal{K}_\infty$ be such that

$$\theta = \sum_{j \geq 1} a_j b^{-j}.$$

Then for $d_1 \ldots d_k \in \mathcal{N}^k$ being a block of digits of length ℓ we denote by $\mathcal{N}(\theta; d_1 \ldots d_k; N)$ the number of occurrences of this block in the first N digits of θ. Thus

$$\mathcal{N}(\theta; d_1 \ldots d_r; N) := \#\{0 \leq n < N : d_1 = a_{n+1}, \ldots, d_r = a_{n+r}\}.$$

Normal number

Now we call θ *normal in* $(b, \mathcal{N})$ if for every $k \geq 1$ we have that

$$\mathcal{R}_N(\theta) = \mathcal{R}_{N,r}(\theta) := \sup_{d_1 \ldots d_r} \left| \frac{1}{N} \mathcal{N}(\theta; d_1 \ldots d_r; N) - \frac{1}{|\mathcal{N}|^r} \right| = o(1)$$

where the supremum is taken over all possible blocks $d_1 \ldots d_r \in \mathcal{N}^r$ of length r.

Construction of normal numbers

In order to construct a normal number we often take a strictly increasing sequence $(s_n)_{n\geq 1}$ of real numbers and concatenate its values. Thus we define

$$\theta((s_n)_{n\geq 1}) := 0.\lfloor s_1\rfloor\lfloor s_2\rfloor\lfloor s_3\rfloor\lfloor s_4\rfloor\lfloor s_5\rfloor\dots$$

Constructions of normal numbers

Theorem Champernowne (1933)

$\theta((n)_{n\geq 1})$ is normal.

Theorem Copeland and Erdős (1946)

Let $s_n \in \mathbb{N}$. If

$$\forall\delta > 0 \exists N \in \mathbb{N} : \#\{s_n : s_n \leq N\} \geq N^\delta,$$

then $\theta((s_n)_{n\geq 1})$ is normal.

Construction of normal numbers

Theorem Nakai and Shiokawa (1992)

Let f be a polynomial with real coefficients. Then $\theta((f(n))_{n\geq 1})$ is normal.

Theorem M, Thuswaldner and Tichy (2007)

Let f be an entire function of bounded logarithmic order. Then $\theta((f(n))_{n\geq 1})$ and $\theta((f(p))_{p\in\mathbb{P}})$ are normal.

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Block counting

For proving that one of the constructions above really yields a normal number one counts the number of occurrences of a pattern within the expansion and ignores the number occurring between two expansions.

Counting the patterns

In order to prove the arithmetic or asymptotic behaviour one might consider the following generalisation of the above block counting function.

$$\begin{aligned}\mathcal{N}((s_n)_{n\geq 1}; (d_1,\ell_1),\dots,(d_k,\ell_k); N) \\ = \#\{(n,\ell) : 1 \leq n \leq N, 0 \leq \ell < \ell(s_n) \\ , a_{\ell+\ell_1}(s_n) = d_1,\dots,a_{\ell+\ell_k}(s_n) = d_k\}.\end{aligned}$$

Connections

- **Arithmetic summation:**

$$\mathcal{N}((n)_{n\geq 1}; (d,0); N) = N\log N + N\Phi(\log N) + \mathcal{O}\left(N^{1-\varepsilon}\right)$$

- **Normal number:**

$$\mathcal{N}((s_n)_{n\geq 1}; (d_1,0), \ldots (d_k, k-1); N) = N\log N + \mathcal{O}\left(N\right)$$

- **Asymptotic distribution:**

$$\mathcal{N}((s_n)_{n\geq 1}; (d_1,\ell_1), \ldots, (d_k, \ell_k); N) = N\log N + \mathcal{O}\left(N\right)$$

Indicator function

$$\mathcal{N}((n)_{n\geq 1};(d_1,\ell_1)\dots(d_k,\ell_k);N)-\frac{1}{q^k}N\log(s_N)$$
$$=\sum_{n\leq N}\sum_{0\leq\ell<\ell(s_N)}\prod_{j=1}^{k}\left(\mathcal{I}_{\ell+\ell_j,d_j}\left(\lfloor s_n\rfloor\right)-\frac{1}{q}\right)+\mathcal{O}(1).$$

with

$$\mathcal{I}_{\ell,d}(x)=\begin{cases}1 & \text{if } a_\ell(x)=d,\\ 0 & \text{else.}\end{cases}$$

Fourier transform

$$\sum_{n\leq N}\left(\mathcal{I}_{\ell,d}\left(\lfloor s_n\rfloor\right)-\frac{1}{q}\right)$$

$$\ll \frac{N}{\delta}+\sum_{\nu=1}^{\infty}\min\left(\frac{\delta}{\nu^2},\frac{1}{\nu}\right)\left|\sum_{n\leq N}e\left(\frac{\nu}{q^{\ell+1}}s_n\right)\right|.$$

Diophantine approximation

Since in most of the examples above we used polynomials we write

$$p(x) = \alpha_k x^k + \cdots + \alpha_1 x + \alpha_0.$$

Then we are interested in the size of b_i for

$$\left| \frac{\nu}{q^{\ell+1}} \alpha_i - \frac{a_i}{b_i} \right| \leq \frac{(\log N)^H}{N^k}.$$

Division of the expansion

Since in our case the coefficients look like

$$\frac{\nu}{q^{\ell+1}}\alpha_i.$$

the Diophantine approximation leads us to a division of the expansion according to the position of the digit within the expansion.

- Most significant digits.
- Middle digits.
- Least significant digits.