Thue-Morse sequences of squares in compact groups

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joint work with Michael Drmota

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Summary

★ Thue-Morse sequence
★ Generalized Thue-Morse sequence and main result
★ Sketch of the proof
★ Applications
  ★ La somme des chiffres des carrés
  ★ Automatic sequences
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

0
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

01
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

0110
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

01101001
Thue-Morse sequence $(t_n)_{n \geq 0}$:

0110100110010110
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

01101001100101101001011001101001
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

\[
01101001100101101001011001101001
\cdots
\]

\[
t_0 = 0, \quad t_{2^n + k} = 1 - t_k \quad (0 \leq k < 2^n)
\]
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

\[
011010011001011010010110011010011001011001101 \ldots
\]

\[
t_0 = 0, \quad t_{2^n+k} = 1 - t_k \quad (0 \leq k < 2^n)
\]

\[
t_n = s_2(n) \mod 2
\]

\[
n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \quad \varepsilon_i(n) \in \{0, 1, \ldots, q - 1\}, \quad s_q(n) = \sum_{i=0}^{\ell-1} \varepsilon_i(n)
\]
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011010011001011010010110011010011001011001101 \ldots
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\]
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\]

\[
\# \{0 \leq n < N : t_n = 0\} \approx \frac{N}{2}
\]
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

\[
\begin{align*}
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & \ldots
\end{align*}
\]

\[
\# \left\{ 0 \leq n < N : t_n = 0 \right\} \approx \frac{N}{2}
\]
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\[
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\[
\# \{0 \leq n < N : t_{3n} = 0\} \approx \frac{N}{2}
\]
Thue-Morse sequence

Thue-Morse sequence \( (t_n)_{n \geq 0} \):

\[
\begin{array}{cccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots
\end{array}
\]

\[
\# \{0 \leq n < N : t_n = 0\} \approx \frac{N}{2}
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\# \{0 \leq p < N, p \text{ prime} : t_p = 0\}
Thue-Morse sequence

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\[01101001100101101001011010011010011001011001101 \ldots\]

Mauduit and Rivat (2009):

\[
\# \{0 \leq p < N, p \text{ prime : } t_p = 0\} \approx \frac{\pi(N)}{2}
\]
Thue-Morse sequence

Thue-Morse sequence \((t_n)_{n \geq 0}\):

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\[
\# \{0 \leq n < N : t_{n^2} = 0\}
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Mauduit and Rivat (2005+):

$$\# \{0 \leq n < N : t_{n^2} = 0 \} \approx \frac{N}{2}$$
Generalized Thue-Morse sequences and main results

- $H$ ... compact (Hausdorff) group
- $q \geq 2$ and $g_0, g_1, \ldots, g_{q-1} \in H$ with $g_0 = e$ (identity element)
- $G \leq H$ ... closure of the subgroup generated by $g_0, g_1, \ldots, g_{q-1}$
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$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n)q^i$$
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Generalized Thue-Morse sequence:

$$T(n) := g_{\varepsilon_0(n)} g_{\varepsilon_1(n)} \cdots g_{\varepsilon_{\ell-1}(n)}$$
Generalized Thue-Morse sequences and main results

- $H$ ... compact (Hausdorff) group
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Generalized Thue-Morse sequence:

$$T(n) := g_{\varepsilon_0(n)} g_{\varepsilon_1(n)} \cdots g_{\varepsilon_{\ell-1}(n)}$$

$H = \langle \mathbb{Z}/2\mathbb{Z}, + \rangle$, $q = 2$, $g_0 = 0$, $g_1 = 1$ : $T(n) = s_2(n) \mod 2 = t_n$
Theorem (Drmota and Morgenbesser, 2010)

There exists a positive integer \( m = m(q, g_0, \ldots, g_{q-1}) \) such that the following holds: Set

\[
d_{\nu} = \sum_{v=0}^{m} 1_{g_v U} \cdot Q(v, m) \, d\mu,
\]

where

- \( \mu \) \ldots Haar measure on \( G \),
- \( U = \text{cl}(\{ T(mn) : n \geq 0 \}) \) \ldots normal subgroup of \( G \) of index \( m \),
- \( Q(v, m) = \#\{ 0 \leq n < m : n^2 \equiv v \mod m \} \).

Then \( (T(n^2))_{n \geq 0} \) is \( \nu \)-uniformly distributed in \( G \), that is,

\[
\frac{1}{N} \sum_{n=0}^{N-1} \delta_{T(n^2)} \to \nu.
\]
Theorem (Drmota and Morgenbesser, 2010)

There exists a positive integer \( m = m(q, g_0, \ldots, g_{q-1}) \) such that the following holds: Set

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- \( Q(\nu, m) = \# \{ 0 \leq n < m : n^2 \equiv \nu \mod m \} \).

Then \( (T(n^2))_{n \geq 0} \) is \( \nu \)-uniformly distributed in \( G \), that is,

\[
\frac{1}{N} \sum_{n=0}^{N-1} f(T(n^2)) \to \int_G f \, d\nu.
\]

for all continuous functions \( f : G \to \mathbb{C} \).
A unitary group representation is a continuous homomorphism $D : G \rightarrow U_n$ for some $n \geq 1$.

$U_n$ is the group of unitary $n \times n$ matrices over $\mathbb{C}$.

$D$ is irreducible if there is no proper subspace $W$ of $\mathbb{C}^n$ with $D(x)W \subseteq W$ for all $x \in G$. 
A **unitary group representation** is a continuous homomorphism $D : G \rightarrow U_n$ for some $n \geq 1$.

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$D$ is irreducible if there is no proper subspace $W$ of $\mathbb{C}^n$ with $D(x)W \subseteq W$ for all $x \in G$

**Lemma**

Let $G$ be a compact group and $\nu$ a regular normed Borel measure on $G$. Then a sequence $(x_n)_{n \geq 0}$ is $\nu$-uniformly distributed in $G$ iff

$$\frac{1}{N} \sum_{n=0}^{N-1} D(x_n) \rightarrow \int_G D \, d\nu$$

holds for all irreducible unitary representations $D$ of $G$.
Remarks:

- The *characteristic integer* $m$ is the largest integer such that $m \mid q - 1$ and such that there exists a representation $D$ of $G$ with

$$D_1(g_u) = e^{-2\pi i \frac{u}{m}} \quad \text{for all } u \in \{0, 1, \ldots, q - 1\}.$$ 

- $(T(n^2))_{n \geq 0}$ is uniformly distributed in $G$ (i.e., $\nu = \mu$) iff $m \leq 2$.

- If $G$ is connected, then $T(n^2)$ is uniformly distributed in $G$. 
Sketch of the proof

\[
\frac{1}{N} \sum_{0 \leq n < N} D(T(n^2))
\]
Sketch of the proof

\[ \frac{1}{N} \sum_{0 \leq n < N} D(T(n^2)) \]

- \( D_0, \ldots, D_{m-1} \):
  \[ D_k(g_u) = e^{-2\pi i \frac{k}{m}u} \quad \text{for all } 0 \leq u < q \text{ and } 0 \leq k < m \]

- all other irreducible unitary representations
Van der Corput type inequality:

\[ \left\| \sum_{0 \leq n < N} D(T(n^2)) \right\|_F \leq \left( \frac{dN}{R} \sum_{|r| < R} \left( 1 - \frac{|r|}{R} \right) \right) \sum_{0 \leq n \leq B} \sum_{0 \leq n+r \leq B} D(T(n+r)^2)D(T(n^2))^H \right\|_F \right)^{1/2} + \frac{f}{2} R \]
Van der Corput type inequality:

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\]

\[
T(n) = g_{\varepsilon_0}(n) g_{\varepsilon_1}(n) \cdots g_{\varepsilon_{\lambda-1}}(n) g_{\varepsilon_\lambda}(n) \cdots g_{\varepsilon_{\ell-1}}(n)
\]
Van der Corput type inequality:

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\[T_\lambda(n) = g_{\varepsilon_0}(n) g_{\varepsilon_1}(n) \cdots g_{\varepsilon_{\lambda-1}}(n)\]
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T_\lambda(n) = g_{\varepsilon_0(n)} g_{\varepsilon_1(n)} \cdots g_{\varepsilon_{\lambda-1}(n)}
\]

\[
(n + r)^2 = (\varepsilon_{\ell - 1} \varepsilon_{\ell - 2} \cdots \varepsilon_\lambda \cdots) q,
\quad
n^2 = (\varepsilon_{\ell - 1} \varepsilon_{\ell - 2} \cdots \varepsilon_\lambda \cdots) q
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\]

\[
D(T((n + r)^2)) D(T(n^2))^H
\]

\[
= D(T_\lambda((n + r)^2)) D(g_{\varepsilon_\lambda}) \cdots D(g_{\varepsilon_{\ell-1}}) D(g_{\varepsilon_{\ell-1}})^H \cdots D(g_{\varepsilon_\lambda})^H D(T_\lambda(n^2))^H
\]

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Generalized Thue-Morse sequences
Van der Corput type inequality:

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\]

\[
T_\lambda(n) = g_{\varepsilon_0(n)} g_{\varepsilon_1(n)} \cdots g_{\varepsilon_{\lambda-1}(n)}
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(n + r)^2 = (\varepsilon_{\ell-1} \varepsilon_{\ell-2} \cdots \varepsilon_\lambda \cdots)q, \quad n^2 = (\varepsilon_{\ell-1} \varepsilon_{\ell-2} \cdots \varepsilon_\lambda \cdots)q
\]

\[
D(T((n + r)^2))D(T(n^2))^H
\]

\[
= D(T_\lambda((n + r)^2))D(g_{\varepsilon_\lambda}) \cdots \left\{ D(g_{\varepsilon_{\ell-1}})D(g_{\varepsilon_{\ell-1}})^H \right\} \cdots D(g_{\varepsilon_\lambda})^H D(T_\lambda(n^2))^H
\]

\[
= D(T_\lambda((n + r)^2))D(T_\lambda(n^2))^H
\]
\[ T_{\lambda}(n) = g_{\varepsilon_0}(n) g_{\varepsilon_1}(n) \cdots g_{\varepsilon_{\mu-1}}(n) g_{\varepsilon_{\mu}}(n) \cdots g_{\varepsilon_{\lambda-1}}(n) \]

Fourier terms:

\[ F_{\lambda}(h) = \frac{1}{q^{\lambda}} \sum_{0 \leq u < q^{\lambda}} e^{-2\pi i \frac{h u}{q^{\lambda}}} D(T_{\lambda}(u)) \]
\[ T_{\mu,\lambda}(n) = g_{\varepsilon_{\mu}}(n) \cdots g_{\varepsilon_{\lambda-1}}(n) \]

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A subtle Fourier analysis of the double truncated sum (following the ideas of Mauduit and Rivat) leads to the following expression:
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\[
\frac{2}{\pi} \log \left( \frac{4e^{\pi/2}q^{\lambda}}{\pi} \right) q^{\lambda/2} \max_{0 \leq \ell < q^{\lambda}} \sum_{d \mid q^{\lambda}} d^{1/2} \cdot \sum_{0 \leq h_1, h_2, h_3, h_4 < q^{\lambda}} \| F_{\mu,\lambda}(h_1) \|_F \| F_{\mu,\lambda}(h_2) \|_F \| F_{\mu,\lambda}(h_3) \|_F \| F_{\mu,\lambda}(h_4) \|_F \\
\text{s.t. } d \mid 2r(h_1 + h_2) + 2sq^{\mu}(h_2 + h_3) + \ell
\]
A subtle Fourier analysis of the double truncated sum (following the ideas of Mauduit and Rivat) leads to the following expression:

$$\frac{2}{\pi} \log \left( \frac{4e^{\pi/2} q^\lambda}{\pi} \right) q^{\lambda/2} \max_{0 \leq \ell < q^\lambda} \sum_{d \mid q^\lambda} d^{1/2} \cdot \sum_{0 \leq h_1, h_2, h_3, h_4 < q^\lambda} \| F_{\mu, \lambda}(h_1) \|_F \| F_{\mu, \lambda}(h_2) \|_F \| F_{\mu, \lambda}(h_3) \|_F \| F_{\mu, \lambda}(h_4) \|_F$$

The analogue of this expression appears in Mauduit and Rivat's work and can be estimated as in their case.
Applications

La somme des chiffres des carrés:

Theorem (Mauduit and Rivat, 2009)

Let \( q, r \geq 2 \) and set \( m = \gcd(q - 1, r) \). Then we have

(i) \( \lim_{x \to \infty} \frac{1}{x} \# \{ n \leq x : s_q(n^2) \equiv a \mod r \} = \frac{1}{r} Q(a, m) \),

(ii) \( (\alpha s_q(n^2))_{n \in \mathbb{N}} \) is uniformly distributed modulo 1 iff \( \alpha \) is irrational.
Applications

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\]

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**Proof:**

(i) \[
H = \mathbb{Z}/r\mathbb{Z}, \quad g_j = j \mod r, \quad 0 \leq j \leq q - 1.
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Proof:

(i) $H = \mathbb{Z}/r\mathbb{Z}, \quad g_j = j \mod r, 0 \leq j \leq q - 1.$

$\quad \rightarrow$ Then $G = H, T(n) = s_q(n) \mod r.$
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\( \Rightarrow \) Then $G = H$, $T(n) = s_q(n) \mod r$.

\( \chi_k(u) = e^{2\pi iuk/r} \) $0 \leq k < r$ \( \ldots \) representations of $G$
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Proof:

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- $H = \mathbb{Z}/r\mathbb{Z}, \quad g_j = j \mod r, 0 \leq j \leq q - 1.$
- Then $G = H, \ T(n) = s_q(n) \mod r.$
- $\chi_k(u) = e^{2\pi iuk/r} \ 0 \leq k < r \ \ldots \ \text{representations of } G$
- $m = \gcd(q - 1, r), \quad D_1(u) := \chi_r/m(u) = e^{2\pi iu/m}.$
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$\triangleright$ $m = \gcd(q - 1, r)$, $D_1(u) := \chi_r/m(u) = e^{2\pi iu/m}$. 

$\blacksquare$
Theorem (Mauduit and Rivat, 2009)

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Proof:

(ii) $H = \mathbb{R}/\mathbb{Z}$, $g_j = \alpha j \mod 1$, $0 \leq j \leq q - 1$. 
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(ii)  

$H = \mathbb{R}/\mathbb{Z}, \quad g_j = \alpha j \mod 1, \ 0 \leq j \leq q - 1.$

Then $G = H$ iff $\alpha$ irrational
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$\triangleright \ T(n) = \alpha s_q(n) \mod 1$
Applications

La somme des chiffres des carrés:

Theorem (Mauduit and Rivat, 2009)

Let $q, r \geq 2$ and set $m = \gcd(q - 1, r)$. Then we have

(i) \[ \lim_{x \to \infty} \frac{1}{x} \# \{ n \leq x : s_q(n^2) \equiv a \pmod{r} \} = \frac{1}{r} Q(a, m), \]

(ii) $(\alpha s_q(n^2))_{n \in \mathbb{N}}$ is uniformly distributed modulo 1 iff $\alpha$ is irrational.

Proof:

(ii)

- $H = \mathbb{R}/\mathbb{Z}$, $g_j = \alpha j \mod 1$, $0 \leq j \leq q - 1$.
- Then $G = H$ iff $\alpha$ irrational
- $T(n) = \alpha s_q(n) \mod 1$
Applications

Automatic sequences:

Definition
A sequence \((u_n)_{n \geq 0}\) is called a \(q\)-automatic sequence, if \(u_n\) is the output of an automaton when the input is the \(q\)-ary expansion of \(n\).
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\[ 32 = (1012)_3 \]
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\[32 = (1012)_3 \quad u_{32} = a, \quad 61 = (2021)_3\]
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\[
32 = (1012)_3 \quad u_{32} = a, \quad 61 = (2021)_3
\]

\[
s_1 / a \quad s_2 / a \quad s_3 / b
\]

\[
\text{Johannes Morgenbesser} \quad \text{Generalized Thue-Morse sequences}
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\begin{align*}
32 &= (1012)_3 & u_{32} &= a, \\
61 &= (2021)_3 & u_{61} &= b
\end{align*}
\]

\((u_n)_{n \geq 0} : aaaaaabaabaaabbaaabbbbaaaabaaaabaaabaaabaaaaaababa\ldots\)
Generalized Thue-Morse sequences
\[ M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
\[ M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]
$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$32 = (1012)_3 :$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
\[
M_0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad M_1 = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad M_2 = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

\[32 = (1012)_3 : \quad M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\]
\[ M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

\[ 32 = (1012)_3 : \quad M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \]
\[
M_0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
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1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
32 = (1012)_3 : \quad M_1 \circ M_0 \circ M_1 \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]
$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$32 = (1012)_3 : \quad M_2 \circ M_1 \circ M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
\[M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\]

\[S(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}\]

\[u_n = f\left(S(n) \mathbf{e}_1\right), \quad \mathbf{e}_j^T S(n) \mathbf{e}_1 = \begin{cases} 1, & \text{output state } s_j \\ 0, & \text{otherwise.} \end{cases}\]
A $q$-automatic sequence is called *invertible* if there exists an automaton such that all transition matrices are invertible and $M_0$ is the identity matrix.
$(u_n)_{n\geq 0} : aaaaaabaabaabaaabbaaabaabaaabbaaabbaaabaabaaabbaaabaabaaabbaaabaabaaabbaaabaabaabaaabbaaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaabaaabbaaabaabaaba...
\[(u_n)_{n \geq 0} : aaaaababaabaaabbaaaaaaaaaba\ldots\]

**Frequency of** \(a\) **in** \((u_{3n})_{n \geq 0} : \lim_{N \to \infty} \frac{1}{N} \#\{0 \leq n < N : u_{3n} = a\}\)**
\[(u_n)_{n \geq 0} : \text{aaaaabbaaabaaabbaaabaaabbaaabbaaaabbaaaabba\ldots}\]

**Frequency of a in \((u_{n^2})_{n \geq 0} :** \[\lim_{N \to \infty} \frac{1}{N} \#\{0 \leq n < N : u_{n^2} = a\} = ?\]
(u_n)_{n \geq 0} : aaaaaabaababaaabbaaabaabbaaabaaabbaaabaabbaaaabaaba\ldots

Frequency of \( a \) in \((u_{n^2})_{n \geq 0} : \lim_{N \to \infty} \frac{1}{N} \# \{0 \leq n < N : u_{n^2} = a \} = ?

H \ldots \text{group of permutation matrices, } g_i = M_i,
$(u_n)_{n \geq 0}: aaaaaabaabaaabaaabbaaaabaaabbaaaabaaabbaaaabaaabbaaaaaaba\ldots$

**Frequency of $a$ in $(u_{n^2})_{n \geq 0}:$**

$$\lim_{N \to \infty} \frac{1}{N} \# \{ 0 \leq n < N : u_{n^2} = a \} = ?$$

$H\ldots$ group of permutation matrices, $g_i = M_i,$

$T(n) = S(n),$$
\[(u_n)_{n \geq 0} : \text{aaaaabaabaabaaabbaaabaaabbaaabaaabbaaabbaaaaaba...} \]

Frequency of \(a\) in \((u_{n^2})_{n \geq 0} : \lim_{N \to \infty} \frac{1}{N} \# \{0 \leq n < N : u_{n^2} = a\} = ?

\[H \ldots \text{group of permutation matrices, } g_i = M_i,\]

\[T(n) = S(n), \quad \frac{1}{N} \# \{0 \leq n < N : u_{n^2} = s_j\} = \frac{1}{N} \sum_{n=0}^{N-1} e_j^T T(n^2) e_1\]
$(u_n)_{n \geq 0} : \text{aaaaabaabaaabaaabbaaaabaaabbaaabaaabbaaaaaaba...}$

**Frequency of** $a$ **in** $(u_{n^2})_{n \geq 0} : \lim_{N \to \infty} \frac{1}{N} \# \{0 \leq n < N : u_{n^2} = a\} = ?$

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**Theorem (Drmota and Morgenbesser, 2010)**

Let $q \geq 2$ and $(u_n)_{n \geq 0}$ be an invertible $q$-automatic sequence. Then the frequency of each letter of the subsequence $(u_{n^2})_{n \geq 0}$ exists.
Thank you!

1 LAUWERENS KUIPERS AND HARALD NIEDERREITER: *Uniform Distribution of Sequences*. Wiley-Interscience Publication, 1974
5 MICHAEL DRMOTA AND JOHANNES F. MORGENBESSER: *Generalized Thue-Morse Sequence of Squares*. submitted
Examples of automatic sequences

Thue-Morse sequence:

\[
\begin{align*}
    s_1 & / 0 \\
    s_2 & / 1
\end{align*}
\]

Rudin-Shapiro sequence:

\[
\begin{align*}
    s_1 & / 1 \\
    s_2 & / 1 \\
    s_3 & / -1 \\
    s_4 & / -1
\end{align*}
\]