Finiteness and periodicity of beta expansions - number theoretical and dynamical open problems

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Rényi's f-expansion ([23])

Let $y = f(x)$ be a positive real function. Consider a digital expansion of a real number x in a form:

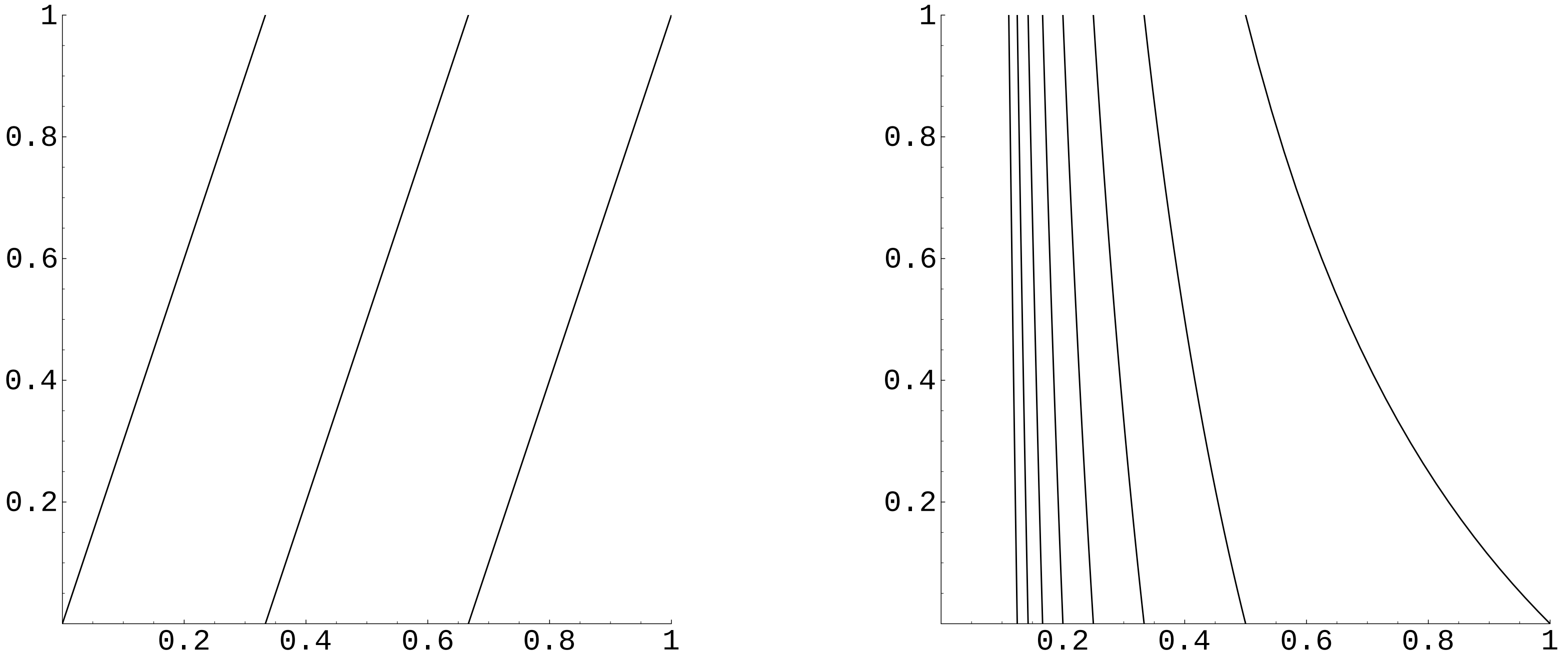
$$x = \varepsilon_0 + f(\varepsilon_1 + f(\varepsilon_2 + f(\varepsilon_3 + \dots.$$

given by an algorithm with $\varepsilon_0 = \lfloor x \rfloor, r_0 = x - \lfloor x \rfloor$ and

$$\varepsilon_{n+1} = \lfloor f^{-1}(r_n) \rfloor, r_{n+1} = f^{-1}(r_n) - \lfloor f^{-1}(r_n) \rfloor$$

$f(x) = x/b$: b-adic expansion

$f(x) = 1/x$: regular continued fraction



(a) 3-adic expansion

(b) Continued fraction

Figure 1: The graph of $f^{-1} \bmod 1$

Rényi studied ergodic properties of f-expansion for monotone function f with mild growth condition.

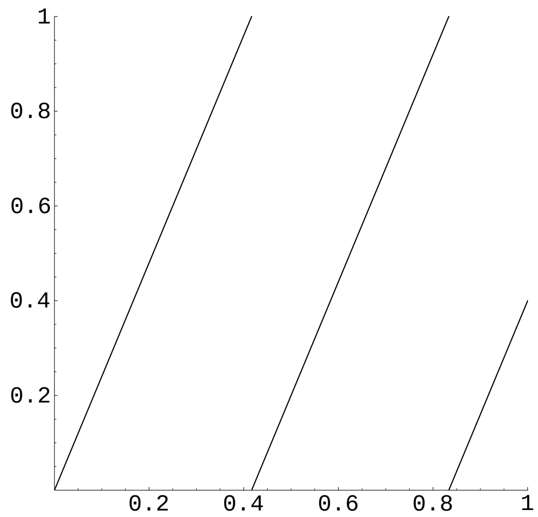


Figure 2: The graph for $b = 2.4$.

'independent digits' $\Longrightarrow$ strong mixing

We start with our fundamental question:

Problem 1. Find a good f-expansion.

What do you mean by ‘good’ ?

There are several aspects to be studied.

- Explicit invariant measure
- Nice ergodic behavior (Mixing, intrinsically ergodic)
- Certain Markovian property (associated to SFT)
- Periodic and finite orbits are characterized
- Natural extension in a nice small space
- Kamae's number system ([14]; additive and multiplicative)
- Relation to Diophantine approximation

β-expansion: Definition

Fix a real number $\beta > 1$ and consider a map $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$ from $[0, 1)$ to itself. The trajectory of x is written as

$$x \xrightarrow{a_1} T_\beta(x) \xrightarrow{a_2} T_\beta^2(x) \xrightarrow{a_3} T_\beta^3(x) \xrightarrow{a_4} \dots$$

with

$$a_i = \left\lfloor \beta T_\beta^{i-1}(x) \right\rfloor \in \mathcal{A} := \mathbb{Z} \cap [0, \beta).$$

Then $x \in [0, 1)$ is uniquely expanded as:

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{a_3}{\beta^3} + \dots$$

through greedy algorithm.

β-expansion: ergodic properties

- Ergodic with an absolute continuous invariant measure (Rényi [23]). Weak mixing. Intrinsically ergodic ([27], [11]).
- Derivative of the measure is made explicit (Parry [22]).
- Exact (Rohlin [24]), therefore mixing of all degree.
- Natural extension is a Bernoulli shift. (Smorodinski [26] , Fischer [9], Ito-Takahashi [13])
- T_β itself is **not** Bernoulli except when β is an integer (Kubo-Murata-Totoki [16]).

Orbit of discontinuity: expansion of 1

Beta expansion defines a map $d_\beta : [0,1) \to \mathcal{A}^{\mathbb{N}}$. This d_β is not surjective when $\beta \notin \mathbb{Z}$. The **expansion of** 1 is defined by $d_\beta(1-) = \lim_{\varepsilon \downarrow 0} d_\beta(1-\varepsilon)$. For $x = a_1 a_2 a_3 \cdots \in \mathcal{A}^{\mathbb{N}}$, we have

$$x \in d_\beta([0,1)) \Longleftrightarrow \sigma^n(x) <_{\mathrm{lex}} d_\beta(1-) \quad (n = 0, 1, 2, \dots)$$

where $\sigma(a_1 a_2 \dots) = a_2 a_3 \dots$ (shift operator) and $<_{\mathrm{lex}}$ is the lexicographical order (Parry [22], Ito-Takahashi [13]).

X_β is the closure of $d_\beta([0,1))$ in $\mathcal{A}^{\mathbb{N}}$. This set is closed and $\sigma(X_\beta) = X_\beta$. This defines the **beta shift** (X_β, σ). The element in X_β is characterized by $\sigma^n(x) \leq_{\mathrm{lex}} d_\beta(1-)$ $(\forall n)$.

Examples

Let $\beta = (1+\sqrt{5})/2$. Then $d_\beta(1-) = (10)^\infty$ holds. $d_\beta([0,1))$ is the set of infinite words on $\{0,1\}$ that 11 and the tail $(10)^\infty$ are forbidden. X_β is exactly the set of infinite words on $\{0,1\}$ without 11.

If $\beta > 1$ is the root of $x^3 - x^2 - x - 1$, then $d_\beta(1-) = (110)^\infty$ holds. $d_\beta([0,1))$ is the set of infinite words on $\{0,1\}$ that 111 and the tail $(110)^\infty$ are forbidden. X_β is exactly the set of infinite words on $\{0,1\}$ without 111.

Symbolic dynamical classification of β

$d_\beta(1-)$ is purely periodic (Simple Parry number) $\Longleftrightarrow$ X_β is SFT, subshift of finite type.

$d_\beta(1-)$ is eventually periodic (Parry number) $\Longleftrightarrow$ X_β is sofic (i.e. factor of SFT).

$d_\beta(1-)$ has bounded run of 0's (Delone number) $\Longleftrightarrow$ X_β has specified property.

Many questions on this classification remain open (Blanchard [6]).

Problem 2. (Salem Periodicity Problem 1) Is there a non-Parry Salem number ? How about $x^6 - 3x^5 - x^4 - 7x^3 - x^2 - 3x + 1$ (Boyd [7]) ?

Problem 3. Is there an algebraic Delone number except Parry numbers ? What about $3/2$ or $\sqrt{2}$?

Periodic expansion

If β is a Pisot number, then $d_\beta(x)$ is eventually periodic for $x \in \mathbb{Q}(\beta) \cap [0,1)$. Conversely if $d_\beta(x)$ is eventually periodic for $x \in \mathbb{Q} \cap [0,1)$ then β must be a Pisot or Salem number (Schmidt [25]). Therefore 'Pisot' implies 'Parry'.

Problem 4. (Salem Periodicity Problem 2) For Salem β, is $d_\beta(x)$ eventually periodic for all $x \in \mathbb{Q}(\beta) \cap [0,1)$?

Possible counter-examples (Boyd [7, 8]) in degree ≥ 6.

Problem 5. For Salem β, is there a way to prove that $d_\beta(x)$ is non-periodic for a fixed x ?

Finite expansion

We say that the expansion of x is **finite**, if $d_\beta(x)$ ends up in 0^∞. Clearly in this case $x \in \mathbb{Z}[1/\beta]$. Frougny-Solomyak [10] studied the property

(F) $d_\beta(x)$ is finite for all $x \in \mathbb{Z}[1/\beta] \cap [0,1)$.

This implies that β is a Pisot number. Converse is not true. For e.g, if β has a positive conjugate $\neq \beta$, then it can not satisfy (F).

The problem to characterize β's with (F) is open for degree ≥ 3, and it is transformed into a 'shift radix system' problem.

Shift Radix System

The vector $(r_0, r_1, \ldots, r_{d-1}) \in \mathbb{R}^d$ is a shift radix system, **SRS** if the integer sequence defined by

$$0 \leq r_0 a_n + r_1 a_{n+1} + \cdots + r_{d-1} a_{n+d-1} + a_{n+d} < 1$$

always falls into a trivial cycle $0^d \to 0^d$. For e.g, $(1/2, 1)$ gives a shift radix system:

$$(-5,3) \to (3,0) \to (0,-1) \to (-1,1) \to (1,0) \to (0,0) \to (0,0)$$

by the recurrence:

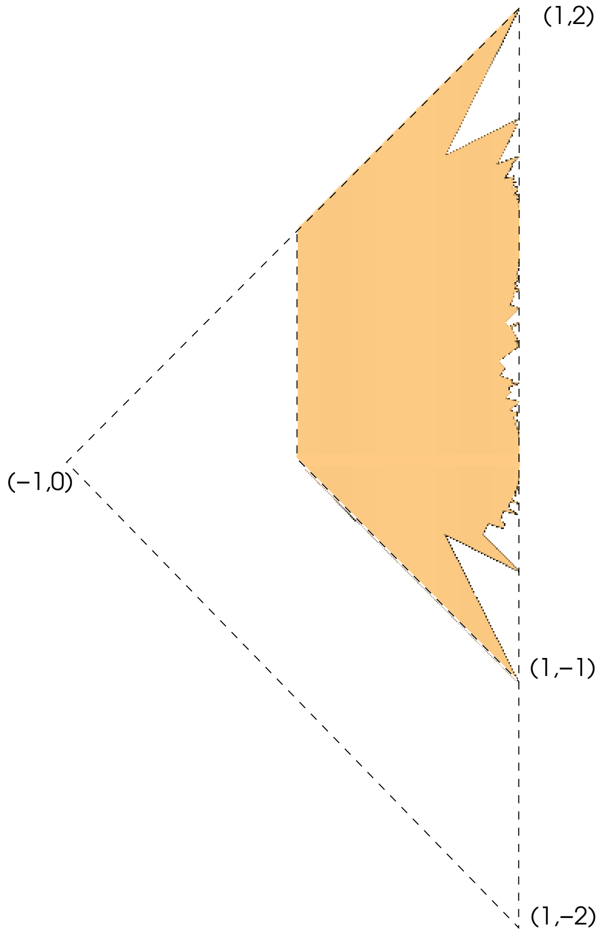
$$0 \leq a_n/2 + a_{n+1} + a_{n+2} < 1.$$

A necessary condition for $(r_0, r_1, \dots, r_{d-1})$ is that $x^d + r_{d-1}x^{d-1} + \cdots + r_0$ is **semi-contractive** (|all roots| ≤ 1). When **contractive** (|all roots| < 1), the orbits must be eventually periodic.

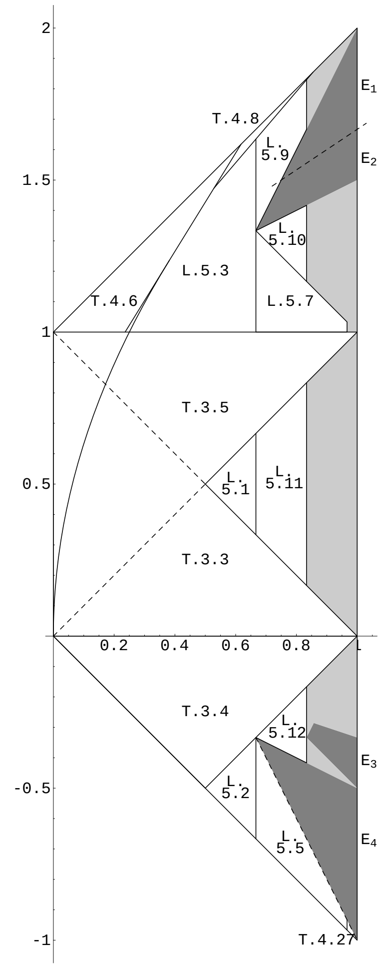
Take a Pisot number β and its minimal polynomial $p(x)$ which is factorized into

$$p(x) = (x - \beta)(x^{d-1} + r_{d-1}x^{d-1} + \cdots + r_1 x + r_0)$$

in $\mathbb{C}$. β has property (F) if and only if $(r_0, r_1, \dots, r_{d-1})$ gives a SRS. Semi-contractive cases correspond to the Salem periodicity problem.



(a) $2d$ SRS Approximation



(b) Known regions

Problem 6. Are SRS polynomials contractive? (True for $d \leq 2$.)

Problem 7. Is SRS region connected ? (Unknown for $d = 2$.)

P.Surer found a cut point $(\frac{40}{41}, \frac{30}{41})$ for $d = 2$!

Problem 8. Prove (or disprove) that $\{(x, y) \mid 0 < 2x < y < x+1\}$ is a SRS region.

Problem 9. Let $|\gamma| < 2$. Prove that each integer sequence $\{a_n\}_n$ satisfying $0 \leq a_n + \gamma a_{n+1} + a_{n+2} < 1$ is eventually periodic.

This 'Salem type periodicity' problem is related to the orbit of piecewise isometry in special cases (Kouptsov-Lowenstein-Vivaldi [15], Akiyama-Brunotte-Pethő-Steiner [3]).

Dual tiling due to Thurston

To extend β-expansion to the left direction, we introduce a set of β-integers:

$$\mathbb{Z}_\beta = \left\{ \sum_{i=0}^{n} a_i \beta^i \;\middle|\; n \in \mathbb{N}, \quad a_n a_{n-1} \ldots a_0 \in d_\beta([0,1)) \right\}$$

Let β be a Pisot number of degree d with r_1 real conjugates and $2r_2$ complex conjugates. We assume that β is a **unit**. Consider an embedding

$$\Phi : \mathbb{Q}(\beta) \to \mathbb{R}^{r_1 - 1} \times \mathbb{C}^{r_2} \simeq \mathbb{R}^{d-1}$$

defined by $x \mapsto (x^{(2)}, \ldots, x^{(r_1)}, x^{(r_1+1)}, \ldots, x^{(r_1+r_2)})$ where $x^{(i)}$ are the non trivial Galois conjugates of x. Since β is Pisot, the set $\overline{\Phi(\mathbb{Z}_\beta)}$ is compact, which is called the **central tile**.

Example: Denote by $\tau' = (1 - \sqrt{5})/2 = -1/\tau$.

$$\begin{aligned} \overline{\Phi(\mathbb{Z}_\tau)} &= \overline{\left\{ \sum_{i \geq 0} a_i (\tau')^i \middle| a_i \in \{0,1\}, a_i a_{i+1} = 0 \right\}} \\ &= [-1, \tau] := A \end{aligned}$$

gives an interval of length τ^2. Dividing by τ', we have

$$\begin{aligned} \beta'^{-1} A &= \left[-\tau^2, -1\right] \cup [-1, \tau] \\ &= BA \end{aligned}$$

with B, an interval of length τ. This growing rule naturally leads us to define an **anti homomorphism** on the word monoid generated by A, B:

$$\sigma(A) = BA, \quad \sigma(B) = A$$

with $\sigma(xy) = \sigma(y)\sigma(x)$ for any words x, y.

$$
\begin{array}{rl}
B & A \\
B & AA \\
BAB & AA \\
BAB & AABAA \\
BABAABAB & AABAA \\
BABAABAB & AABAABABAABAA
\end{array}
$$

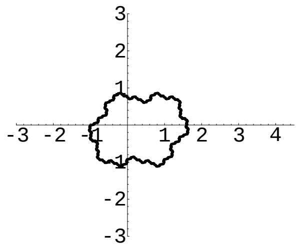
We see $\sigma^\infty(A)$ gives a tiling of $\mathbb{R}$.

Let us do the same game with $A = \overline{\Phi(\mathbb{Z}_\theta)}$, that is,

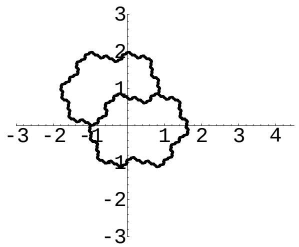
$$\overline{\left\{ \sum_{i \geq 0} a_i (\theta')^i \;\middle|\; a_i \in \{0, 1\}, a_i a_{i+1} a_{i+2} = 0 \right\}}.$$

which is a central tile. In this case, it is a compact set in $\mathbb{C}$. The substitution rule becomes two dimensional.

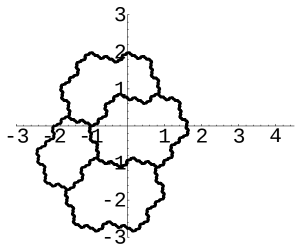
A



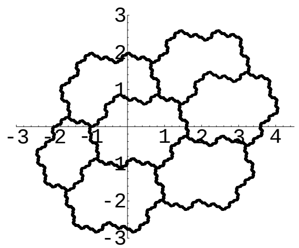
$\theta'^{-1}A$



$\theta'^{-2}A$



$\theta'^{-3}A$



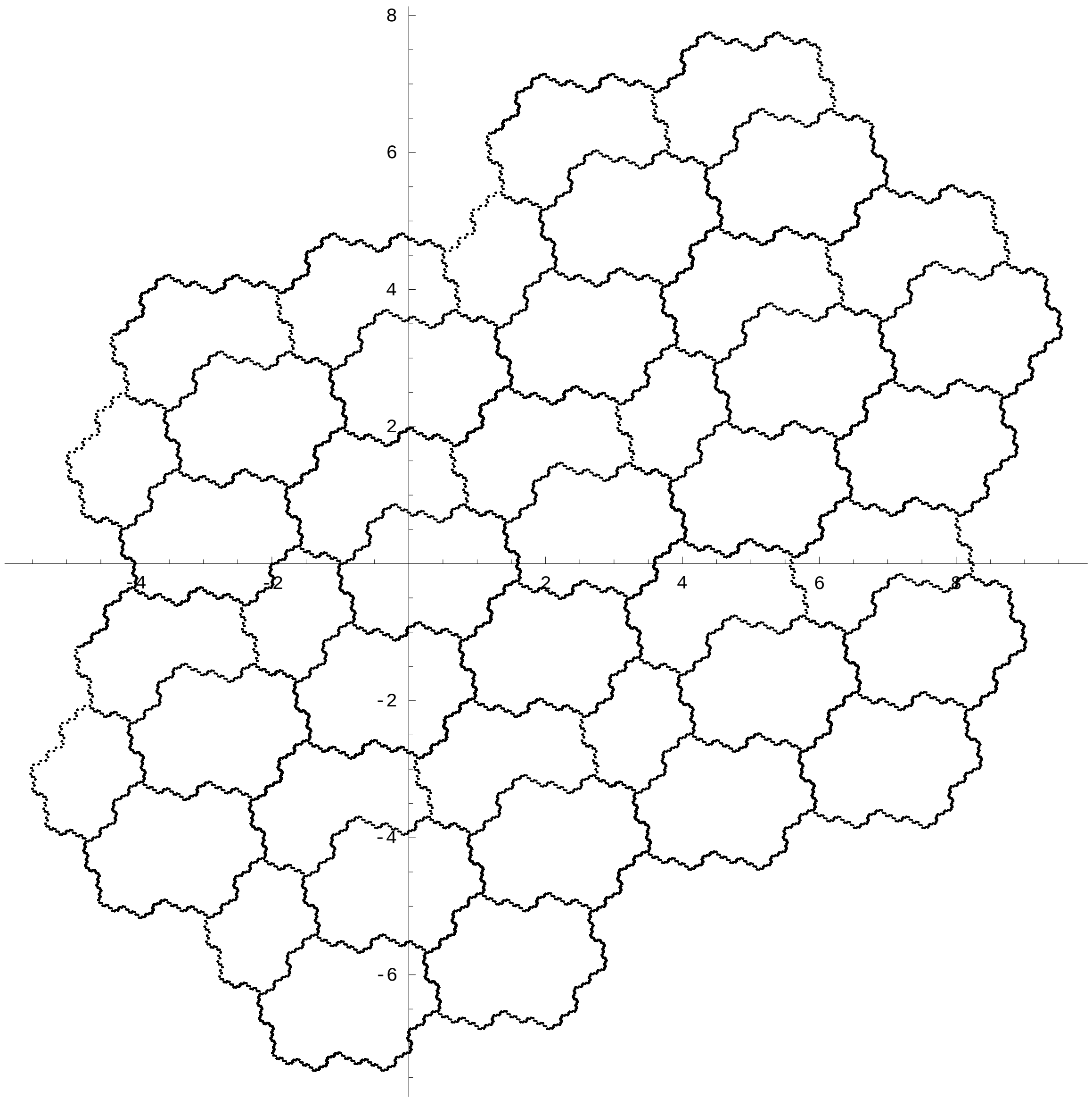


Figure 3: $[110]^\infty$: Tribonacci Case

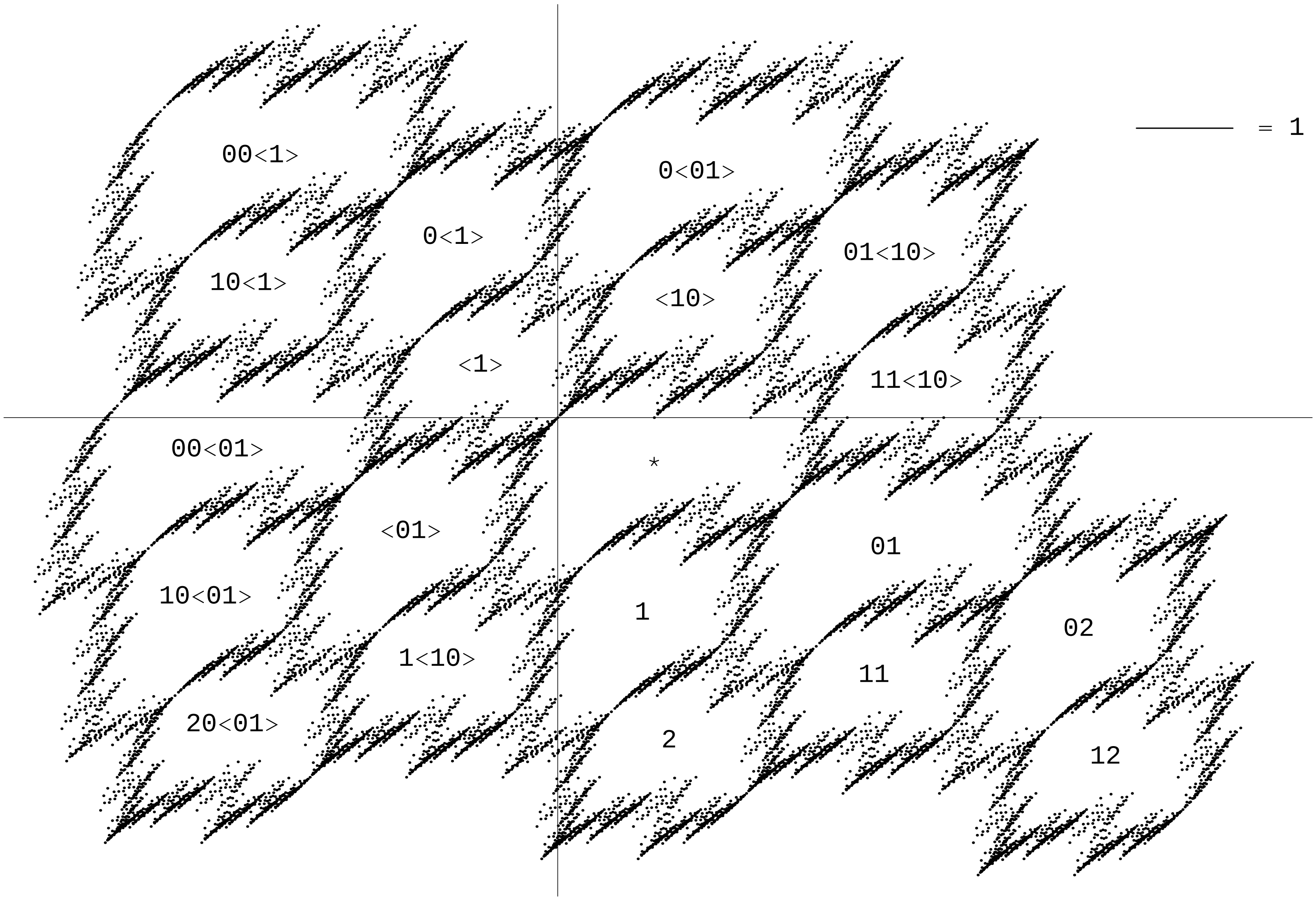


Figure 4: $2[01]^{\infty}$

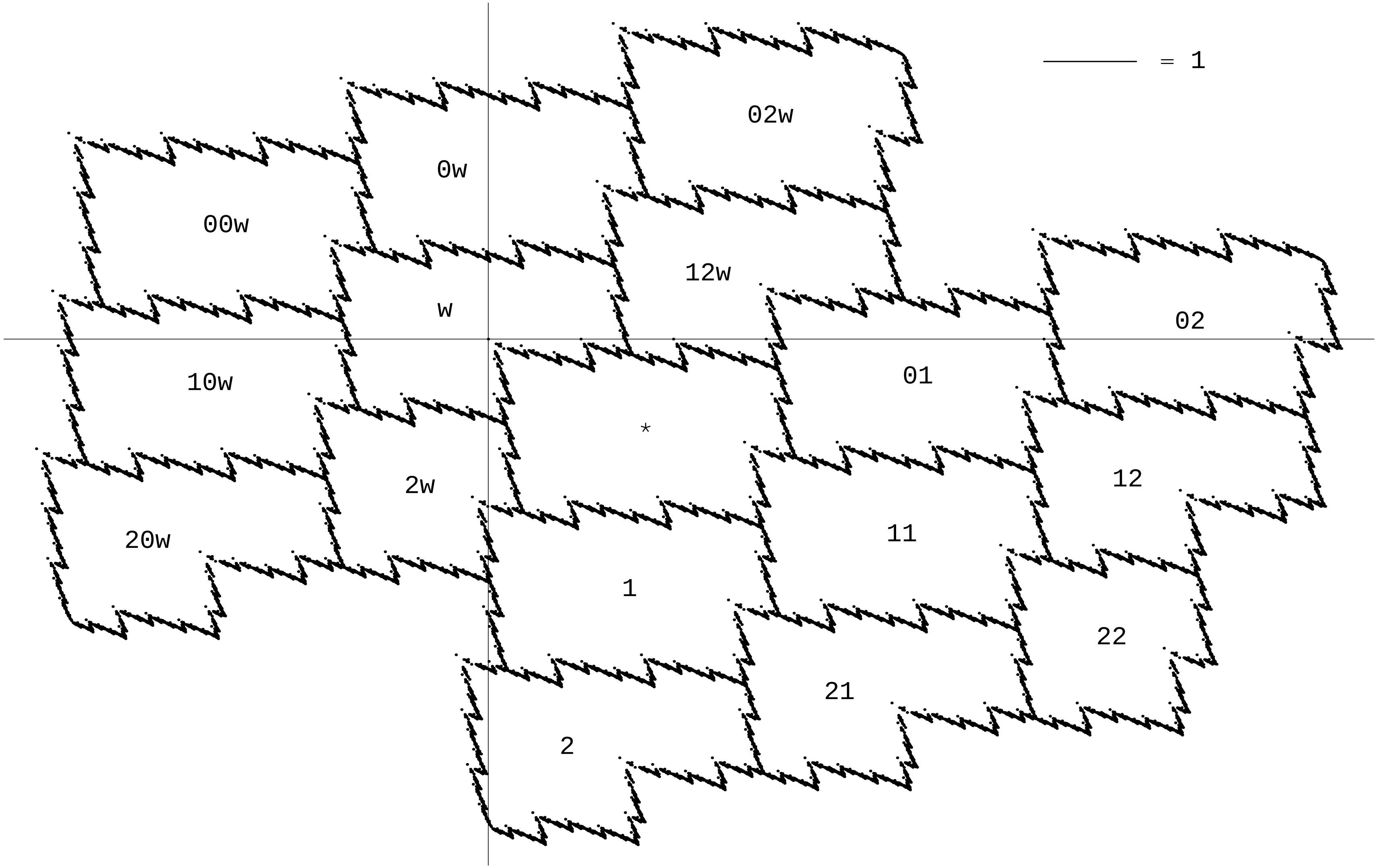


Figure 5: $22[1]^{\infty}$

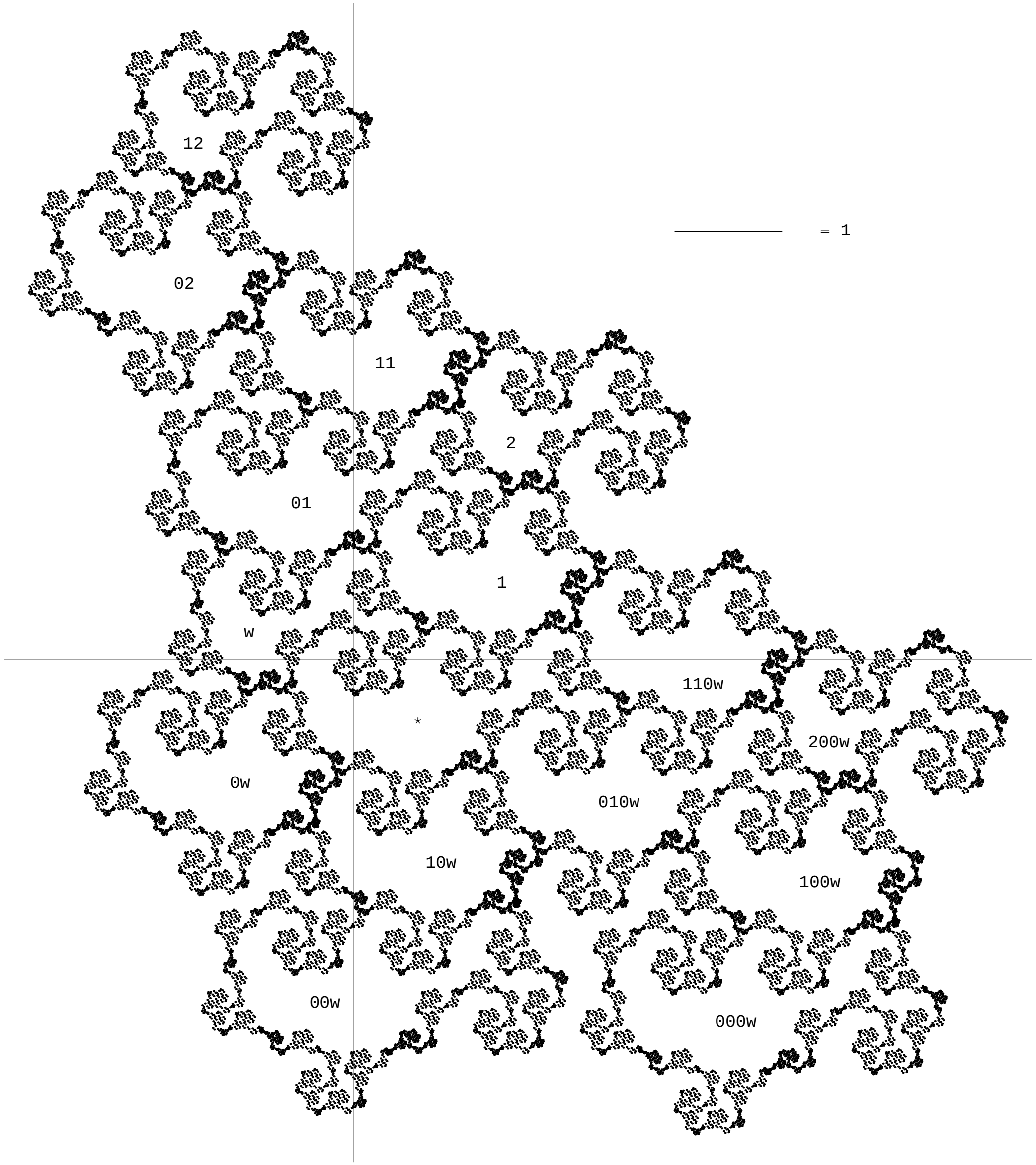


Figure 6: $20[1]^\infty$

Tiling condition: Weak finiteness

Since β is Pisot, the corresponding symbolic dynamics is sofic. Under the **weak finite** condition

(W) For any $z \in \mathbb{Z}[\beta] \cap [0, 1)$, there are x, y with finite expansion such that $z = x - y$,

it is a tiling of $\mathbb{R}^{d-1}$ by finitely many tiles up to translation, which is a geometric realization of a sofic shift X_β ([2]). The condition (F) is clearly stronger than (W). Under (F), the origin is an inner point of the central tile. This condition (W) seems to hold for all Pisot numbers. If so, the converse is true, i.e., the origin is an inner point then (F) holds.

This (W) is understood as a special form of **Pisot conjecture** in substitutive dynamical system. An **arithmetic natural extension** of $([0,1), T_\beta)$ in $\mathbb{R}^d$ is constructed by dual tiles:

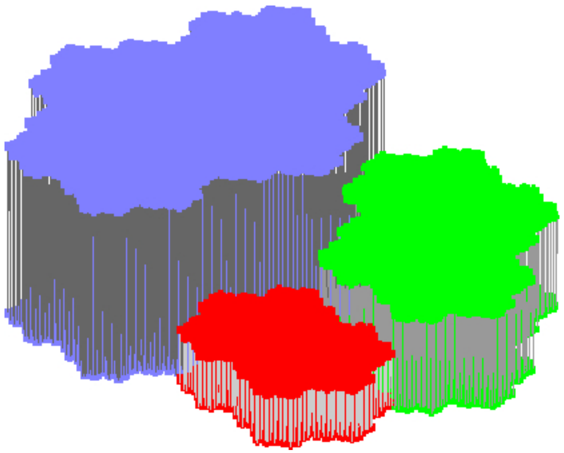


Figure 7: Natural extension

Periodic expansions are characterized by this region (Ito-Rao [12]). For non-unit β, see Berthé-Siegel [5].

Let τ be an irreducible Pisot substitution on $\{0, 1, \dots, d-1\}$. **Pisot conjecture** states that the dynamical system generated by shift closure of a fix point of τ has pure discrete spectrum. Many equivalent **coincidence conditions** are known. Here is one of them (a joint work with J.Y. Lee):

Problem 10. In the fixed point of τ, is there a non empty word w and a **relatively dense** patch of the form:

$$wx_1wx_2wx_3\dots$$

where x_i are arbitrary words satisfying $|wx_1wx_2\dots wx_m| = |\tau^{m-1}(wy)|$ for some y ?

One can show this criterion from (W) for β-substitution.

Height reducing problem

Problem 11. Let β be a Pisot number and put $\mathcal{B} = \{0, \pm 1, \pm 2, \dots, \pm \lfloor \beta \rfloor\}$. Prove $\mathbb{Z}[1/\beta] \cap [0,1) = \mathcal{B}[1/\beta] \cap [0,1)$.

This is a weaker statement than (W). Compare it with the same type of question in expanding case:

Problem 12. Let α be an expanding algebraic integer and $\mathcal{B} = \{0, \pm 1, \pm 2, \dots, \pm(N(\alpha) - 1)\}$ where $N(\alpha)$ is the absolute norm of α over $\mathbb{Q}$. Find an easy proof of $\mathbb{Z}[\alpha] = \mathcal{B}[\alpha]$.

To prove tiling property for different setting, Lagarias-Wang [20, 19, 21] gave an indirect proof of this fact using Wavelet analysis. Class number problems are related ([17, 18]).

Dynamical norm conjecture: Connectedness of tiles

Let β be a Parry number, i.e.,

$$d_\beta(1-) = a_1 a_2 \dots a_m [a_{m+1} a_{m+2} \dots a_{m+\ell}]^\infty \quad \text{with } a_m \neq a_{m+\ell}.$$

Problem 13. (Dynamical Norm Conjecture) Prove that $N(\beta) = |a_m - a_{m+\ell}|$.

This is supported by extensive numerical computation [1]. When β is a simple-Parry Pisot unit, this conjecture implies that the associated tiles are connected. Here the simpleness is necessary since tiles could be disconnected in degree 4 (Akiyama-Gjini [4]) for non-simple Parry cases.

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