Resemblance and difference between beta-integers and ordinary integers

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Outline

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- Let $\beta > 1$ and $x \geq 0$, any series

$$x = \sum_{i=-\infty}^{k} x_i \beta^i =: x_k x_{k-1} \ldots x_0 \bullet x_{-1} \ldots, \qquad x_i \in \mathbb{N},$$

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  in $\{0, 1, \dots, \lfloor \beta \rfloor\}$ for $\beta \notin \mathbb{N}$
- $\mathbb{Z}_\beta := \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k x_{k-1} \dots x_0 \bullet\}$

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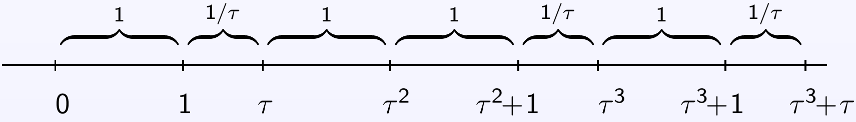
$$\mathbb{Z}_\tau = \left\{ \pm \sum_{i=0}^{k} x_i \tau^i \;\middle|\; x_i \in \{0,1\},\ x_i x_{i+1} = 0 \right\}$$

$$\mathbb{Z}_\tau = \pm\{0, 1, \tau, \tau^2, \tau^2+1, \tau^3, \tau^3+1, \tau^3+\tau, \tau^4, \tau^4+1, \dots\}$$



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Pisot numbers $\subset$ Parry numbers $\subset$ Perron numbers

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- **Example**: $u_\tau = \varphi(u_\tau)$ for φ defined by

$$\varphi(0) = 01, \quad \varphi(1) = 0$$

Simple Parry numbers

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If φ, $u_\beta = \varphi(u_\beta)$, of the form

$$\begin{aligned} \varphi(0) &= 0^{t_1}1 \\ \varphi(1) &= 0^{t_2}2 \\ &\vdots \\ \varphi(m-2) &= 0^{t_{m-1}}(m-1) \\ \varphi(m-1) &= 0^{t_m} \end{aligned}$$

with $t_i \in \mathbb{N}$ and $t_j t_{j+1} \cdots t_m \prec t_1 t_2 \cdots t_m$ for every $1 < j \leq m$ and $t_m \neq 0$, then

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Parry polynomial of β:

$$p(x) = x^m - t_1 x^{m-1} - t_2 x^{m-2} - \cdots - t_m$$

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\varphi(m-2) &= 0^{t_{m-1}}m-1 \\
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\end{aligned}$$

with $p \in \mathbb{N}$, $p \neq 0$, $t_i \in \mathbb{N}$ and any suffix of $t_1 \dots t_p(t_{p+1} \dots t_m)^\omega$ lexicographically smaller than the sequence itself, then

β is a *non-simple Parry number*

Asymptotic behavior of beta-integers

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Theorem

For any Pisot number such that its Parry and minimal polynomial coincide, $(b_n - c_\beta n)_{n \in \mathbb{N}}$ is a bounded sequence.

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Conjecture

If β is a Parry number, then $(b_n - c_\beta n)_{n\in\mathbb{N}}$ is a bounded sequence if and only if β is a Pisot number such that its Parry and minimal polynomial coincide.

Repetitions in beta-integers

- quadratic non-simple Parry case
- simple Parry case

Powers of words

Definition

Let w be a nonempty word, $r \in \mathbb{Q}$, then v is r-th power of w if v is a prefix of w^ω and $r = \frac{|v|}{|w|}$, i.e.,

$$v = w^r := w^{\lfloor r \rfloor} w',$$

where w' is a proper prefix of w.

Example

Let $w = 123$ and $v = 12312312312 = (123)^3 12$, then $r = \frac{|v|}{|w|} = \frac{11}{3} = 3 + \frac{2}{3}$ and so v is $\frac{11}{3}$-power of w.

Index of finite word

Definition

Let $u = (u_n)_{n \geq 1}$ be an infinite word and w its nonempty factor. Then the index of w in u is given by

$$\mathrm{ind}(w) = \sup\{r \in \mathbb{Q} \,|\, w^r \text{ is a factor of } u\}.$$

Remark

$\mathrm{ind}(w) = \max\{r \in \mathbb{Q} \,|\, w^r \text{ is a factor of } u\}$ or $\mathrm{ind}(w) = \infty$

Example

$u = 12(121)^\omega = 12\,121\,121\,121\,\cdots$, then

$$\mathrm{ind}(121) = \infty, \ \mathrm{ind}(12) = 2 + \frac{1}{2}$$

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Definition

A factor v of an infinite word u is bispecial if v has two distinct left extensions and two distinct right extensions in u, i.e., av, bv, vc, vd are factors of u for some letters $a \neq b$, $c \neq d$.

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Lemma

Let w be a factor of u_β and let w have the maximal index among factors of the same length. Put $k := \lfloor ind(w) \rfloor$ and denote w' the prefix of w such that

$$w^{ind(w)} = w^k w'.$$

Then all the following factors are bispecial:

$$w', ww', \ldots, w^{k-1}w'.$$

Quadratic non-simple Parry case

$u_\beta = \varphi(u_\beta)$, where

$$\begin{aligned} \varphi(0) &= 0^p 1, \\ \varphi(1) &= 0^q 1, \\ & \quad p, q \in \mathbb{N},\ p > q \geq 1 \end{aligned}$$

Bispecial factors in u_β

Lemma

Let v be a bispecial factor of u_β containing at least one letter 1. Then there exists a unique bispecial factor $\tilde{v}$ such that

$$v = 0^q 1 \varphi(\tilde{v}) 0^q =: T(\tilde{v}),$$

i.e.,

each bispecial factor is either $0^s, s = 1, 2, \ldots, p-1$, or it is equal to the T-image of another bispecial factor or of the empty word.

Index of u_β

Corollary

The index of u_β is given by the following formula

$$ind(u_\beta) = \sup\{ind(w^{(n)}) \mid n \in \mathbb{N}\},$$

where

$$w^{(1)} = 0, \qquad w^{(n+1)} = 0^q 1 \varphi(w^{(n)})(0^q 1)^{-1}.$$

Moreover, the maximal power

- *of* 0 *is* 0^p,
- *of* $w^{(n+1)}$ *is* $v^{(n+1)} = 0^q 1 \varphi(v^{(n)})(0^q 1)^{-1}$.

Calculation of ind$(w^{(n)})$

Substitution matrix $M_\varphi = \begin{pmatrix} |\varphi(0)|_0 & |\varphi(0)|_1 \\ |\varphi(1)|_0 & |\varphi(1)|_1 \end{pmatrix} = \begin{pmatrix} p & 1 \\ q & 1 \end{pmatrix}$

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$$\mathrm{ind}(w^{(n)}) = \frac{|v^{(n)}|}{|w^{(n)}|} =$$

$$= \frac{(p+1,0)M_\varphi^n \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) + (0, \frac{2q+1-p}{q}) M_\varphi^n \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) - (1, \frac{2q+1-p}{q}) \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}{(1,0)M_\varphi^n \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}$$

$$= p + 1 + \frac{\frac{2q+1-p}{q}\left((1-\beta')\beta^{n+1} - (1-\beta)\beta'^{n+1}\right) - \frac{3q+1-p}{q}(\beta-\beta')}{\beta^{n+1} - \beta'^{n+1}}$$

Result for quadratic non-simple Parry case

Theorem

If $p \leq 3q + 1$,

$$ind(u_\beta) = p + 1 + \frac{2q + 1 - p}{\beta - 1},$$

otherwise

$$ind(u_\beta) = ind(w^{(n_0)}) > p + 1 + \frac{2q + 1 - p}{\beta - 1}$$

for certain $n_0 \in \mathbb{N}$.

Simple Parry case

$$\mathrm{ind}(u_\beta) = \sup\{\mathrm{ind}(\varphi^n(0)) \mid n \in \mathbb{N}\}$$

- Recursive construction of the sequence $(v^{(n)})_{n\in\mathbb{N}}$ of the maximal powers of $\varphi^n(0)$
- Explicit form of $v^{(n)}$ for u_β satisfying
  1. $t_1 > \max\{t_2, \ldots, t_{m-1}\}$

$$\mathrm{ind}(u_\beta) \geq \beta + t_m + \frac{\beta^{m-1} - t_m}{\beta^{m-1} - 1} > t_1 + t_m$$

  2. $t_1 = \cdots = t_{m-1} =: t$

$$\mathrm{ind}(u_\beta) \geq t + t_m + \frac{t}{\beta - 1}$$