Discrete geometry and numeration

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Numeration: Mathematics and Computer Science–Mars 09
A classical problem in Diophantine approximation

How to approximate a line in $\mathbb{R}^3$ by points in $\mathbb{Z}^3$?

How to define a discrete line in $\mathbb{R}^3$?
... and its dual version

How to approximate a plane in $\mathbb{R}^3$ by points in $\mathbb{Z}^3$?
Numeration/representation systems in discrete geometry

- **Continued fractions** (regular, multidimensional unimodular)
- **S-adic systems** (infinite composition of a finite number of substitutions)

Continued fractions/ Ostrowski numeration

Jacobi-Perron, Brun c.f., generalized substitutions
Let $a, b, c$ be strictly positive real numbers.

The (standard) arithmetic discrete plane $\mathcal{P}((a,b,c),h)$ is defined as

$$
\mathcal{P}((a,b,c),h) = \{(p,q,r) \in \mathbb{Z}^3 \mid 0 \leq ap + bq + cr + h < a + b + c\}.
$$

We consider the stepped surface $\mathcal{P}((a,b,c),h)$ defined as the union of the facets of integer translates of unit cubes whose set of integer vertices equals $\mathcal{P}((a,b,c),h)$. 
A first dynamical description

Let $E_1$, $E_2$, and $E_3$ be the three following faces:

A point $(p, q, r) \in \mathbb{Z}^3$ is the distinguished vertex of a face in $\mathcal{P}_{(a,b,c),h}$ of type

1. if and only if $ap + bq + cr + h \in [0, a[$
2. if and only if $ap + bq + cr + h \in [a, a + b[$
3. if and only if $ap + bq + cr + h \in [a + b, a + b + c[$
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- 2 if and only if \(ap + bq + cr + h \in [a, a + b]\)
- 3 if and only if \(ap + bq + cr + h \in [a + b, a + b + c]\)

The triple of strictly positive numbers \((a, b, c)\) being fixed, let:

\[
R_a : [0, a + b + c] \rightarrow [0, a + b + c], \quad x \mapsto x + a \mod a + b + c,
\]

\[
R_b : [0, a + b + c] \rightarrow [0, a + b + c], \quad x \mapsto x + b \mod a + b + c.
\]

Discrete planes are codings of \(\mathbb{Z}^2\)-actions
A first dynamical description

A point \((p, q, r) \in \mathbb{Z}^3\) is the distinguished vertex of a face in \(P((a,b,c),h)\) of type

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We can deduce information on

- possible local configurations/factors
- density of factors

Arithmetic discrete planes with the same normal vector have the same language
Toward an arithmetic representation

We can deduce information on

- possible local configurations/factors
- density of factors

We would like to be able to

- **Generate** discrete planes
- **Recognize** discrete planes: given a set of points in $\mathbb{Z}^3$, is it contained in an arithmetic discrete plane?
How to describe a discrete plane?

We will use a classical strategy based on

- induction/first return map
- substitutions
- continued fractions

\[ \mathcal{P}_{\vec{n},h} = \{ \vec{x} \in \mathbb{Z}^3 \mid 0 \leq \langle \vec{n}, \vec{x} \rangle + h < a + b + c \} \]

We want to describe \( \mathcal{P}_{\vec{n},h} \) w.r. to

- A continued fraction algorithm for the normal vector \( \vec{n} \)
- An Ostrowski-type numeration system associated with the chosen c.f.a. for \( h \)
The induced map $T_A$ of a map $T$ on a subset $A$ is defined by

$$T_A(x) = T^{n_x}(x) \text{ with } n_x = \inf \{ p > 0 | T^p(x) \in A \}.$$ 

What is the meaning for a $\mathbb{Z}^2$-action?

- if $I$ is the induction interval, consider the set of $(m, n)$ such that $R^m_a R^n_b x \in I$.
- This subset is NOT a sublattice of $\mathbb{Z}^2$

We want to reorganize the induced orbit and give a one-to-one correspondence with the original one in terms of substitutions

substitution = reconstruction of the lost information
Connectedness of arithmetic planes

**Question**

Find the smallest width $\omega$ for which the plane $P(\vec{n}, h, \omega)$

$$P_{\vec{n}, h, \omega} = \{ \vec{x} \in \mathbb{Z}^d \mid 0 \leq \langle \vec{x}, \vec{n} \rangle + h < \omega \}$$

is connected.

Rational parameters: [Brimkov-Barneva] [Gérard][Jamet-Toutant]
The **projective** Jacobi-Perron algorithm is defined on the unit square $X = [0, 1) \times [0, 1)$ by:

$$\Phi(\alpha, \beta) = \left( \frac{\beta}{\alpha} - \lfloor \frac{\beta}{\alpha} \rfloor, \frac{1}{\alpha} - \lfloor \frac{1}{\alpha} \rfloor \right) = (\{\beta/\alpha\}, \{1/\alpha\})$$

The **linear** Jacobi-Perron algorithm is defined on the positive cone

$$\{(a, b, c) \in \mathbb{R}^3 | 0 \leq a, b < c\}$$

by:

$$F(a, b, c) = (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a).$$
The linear Jacobi-Perron algorithm is defined on the positive cone

\[ \{(a, b, c) \in \mathbb{R}^3 \mid 0 \leq a, b < c\} \]

by

\[ (a_1, b_1, c_1) = (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a). \]

Set \( B = \lfloor b/a \rfloor a, C = \lfloor c/a \rfloor \)

\[
\begin{pmatrix}
   a_1 \\
   b_1 \\
   c_1
\end{pmatrix} = \begin{pmatrix}
   -B & 1 & 0 \\
   -C & 0 & 1 \\
   1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
   a \\
   b \\
   c
\end{pmatrix}
\]
Jacobi-Perron algorithm

The linear Jacobi-Perron algorithm is defined

\[ \{(a, b, c) \in \mathbb{R}^3 \mid 0 \leq a, b < c\} \]

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\end{pmatrix}
\begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix}
\]

- The matrix \( \begin{pmatrix}
    -B & 1 & 0 \\
    -C & 0 & 1 \\
    1 & 0 & 0
\end{pmatrix} \) belongs to \( SL_3(\mathbb{Z}) \). It is invertible
- The Jacobi-Perron algorithm is unimodular
- The inverse of \( \begin{pmatrix}
    -B & 1 & 0 \\
    -C & 0 & 1 \\
    1 & 0 & 0
\end{pmatrix} \) belongs to \( SL_3(\mathbb{N}) \)
Jacobi-Perron algorithm

The inverse of \[
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c
\end{pmatrix}
\]

We can tile \( I_a \cup I_b \cup I_c \) by intervals \( I_{a_1}, I_{b_1}, I_{c_1} \)

\[
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & B \\
0 & 1 & C
\end{pmatrix} \begin{pmatrix}
a_1 \\
b_1 \\
c_1
\end{pmatrix}
\]
Jacobi-Perron substitution

The equation

\[ ap + bq + cr = a_1 p_1 + b_1 q_1 + c_1 r_1 \]

provides a relation between \((p_1, q_1, r_1)\) and \((p, q, r)\).

We now can go from the induced orbit to the original full orbit under the \(\mathbb{Z}^2\)-action.

\[ P_{\vec{n}} \rightsquigarrow \text{full orbit} \]
\[ P_{\vec{n}_1} \rightsquigarrow \text{induced orbit} \]

We want to reconstruct the full orbit from the induced orbit.

We want to reconstruct \(P_{\vec{n}}\) from \(P_{\vec{n}_1}\).
From a continued fraction algorithm to substitutions

We are given a unimodular matrix \( M \in SL_3(\mathbb{N}) \) which describes a continued fraction algorithm

\[
\bar{n} = M \bar{n}_1
\]

We want

\[
ap + bq + cr = a_1 p_1 + b_1 q_1 + c_1 r_1
\]

\[
\langle \bar{n}, \bar{x} \rangle = \langle \bar{n}_1, \bar{x}_1 \rangle
\]

Hence

\[
\langle \bar{n}, \bar{x} \rangle = \langle M \bar{n}_1, \bar{x} \rangle = \langle \bar{n}_1, t M \bar{x} \rangle
\]

provides

\[
\bar{x} \rightsquigarrow \bar{x}_1 = t M \bar{x}
\]

We go from \( P_{\bar{n}_1} \) to \( P_{\bar{n}} \) by

\[
\bar{x}_1 \mapsto t M^{-1} \bar{x}
\]

We then use a tiling of \( I_a \cup I_b \cup I_c \) by intervals \( I_{a_1}, I_{b_1}, I_{c_1} \)
From a continued fraction algorithm to substitutions

- We go from $\mathcal{P}_{\vec{n}_1}$ to $\mathcal{P}_{\vec{n}}$ by
  $$\vec{x}_1 \mapsto t M^{-1} \vec{x}$$

- The way we tile $I_a \cup I_b \cup I_c$ by intervals $I_{a_1}, I_{b_1}, I_{c_1}$ is noncanonical

- With each such choice is associated a substitution rule

Continued fraction algorithm $\leadsto$ Induction process $\leadsto$ Substitution rule

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**Theorem [Arnoux, B., Ito]**

Let $\sigma$ be a unimodular substitution. Let $\vec{n} \in \mathbb{R}^d_+$ be a positive vector. The substitution rule maps without overlaps the stepped plane $\mathcal{P}_{\vec{n}, h}$ onto $\mathcal{P}_{M \vec{n}, h}$. 
Substitutions

Let $\sigma$ be a substitution on $\mathcal{A}$.  

Example: 

$$\sigma(1) = 12, \quad \sigma(2) = 13, \quad \sigma(3) = 1.$$ 

The incidence matrix $M_\sigma$ of $\sigma$ is defined by 

$$M_\sigma = (|\sigma(j)|_i)_{(i,j) \in \mathcal{A}^2},$$ 

where $|\sigma(j)|_i$ counts the number of occurrences of the letter $i$ in $\sigma(j)$.

## Unimodular substitution

$\det M_\sigma = \pm 1$

## Abelianisation

Let $d$ be the cardinality of $\mathcal{A}$. Let $\mathbf{l} : \mathcal{A}^* \rightarrow \mathbb{N}^d$ be the abelinisation map

$$\mathbf{l}(w) = ^t(|w|_1, |w|_2, \cdots, |w|_d).$$
Generalized substitutions

Let \((x, 1^*), (x, 2^*), (x, 3^*)\) stand for the following faces

Generalized substitution [Arnoux-Ito][Ei]

Let \(\sigma\) be a unimodular morphism of the free group.

\[
E_1^*(\sigma)(x, i^*) = \sum_{k \in A} \sum_{P, \sigma(k) = P \in S} (M_{\sigma^{-1}}^{-1}(x - I(P)), k^*).
\]
Theorem [Arnoux-Ito, Fernique]

Let $\sigma$ be a unimodular substitution. Let $\vec{n} \in \mathbb{R}^d_+$ be a positive vector. The generalized substitution $E_1^*(\sigma)$ maps without overlaps the stepped plane $\mathcal{P}_{\vec{n},h}$ onto $\mathcal{P}_{tM_\sigma \vec{n},h}$. 
Ostrowski’s representation of integers can be extended to real numbers.

The base is given by the sequence \((\theta_n)_{n \geq 0}\), where \(\theta_n = (q_n\alpha - p_n)\).

Every real number \(-\alpha \leq \beta < 1 - \alpha\) can be expanded uniquely in the form

\[
\beta = \sum_{k=1}^{+\infty} c_k \theta_{k-1},
\]

where

\[
\begin{cases} 
 0 \leq c_1 \leq a_1 - 1 \\
 0 \leq c_k \leq a_k \text{ for } k \geq 2 \\
 c_k = 0 \text{ if } c_{k+1} = a_{k+1} \\
 c_k \neq a_k \text{ for infinitely many odd integers.}
\end{cases}
\]
A skew product of the Gauss map

We consider the following skew product of the Gauss map

$$T: (\alpha, \beta) \mapsto (\{1/\alpha\}, \{\beta/\alpha\}) = (1/\alpha - a_1, \beta/\alpha - b_1) = (\alpha_1, \beta_1).$$

We have

$$\beta_1 = \beta/\alpha - b_1 \text{ and thus } \beta = b_1 \alpha + \alpha \beta_1.$$ 

We deduce that

$$\beta = \sum_{k=1}^{+\infty} b_k \alpha_1 \cdots \alpha_{k-1} = \sum_{k=1}^{+\infty} b_k |q_{k-1} \alpha - p_{k-1}|.$$
A skew product of the Gauss map

We consider the following skew product of the Gauss map

\[ T : (\alpha, \beta) \mapsto \left( \{1/\alpha\}, \{\beta/\alpha\} \right) = (1/\alpha - a_1, \beta/\alpha - b_1) = (\alpha_1, \beta_1). \]

We have

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We deduce that

\[ \beta = \sum_{k=1}^{+\infty} b_k \alpha \alpha_1 \cdots \alpha_{k-1} = \sum_{k=1}^{+\infty} b_k |q_{k-1} \alpha - p_{k-1}|. \]

Indeed we use the fact that

\[
\begin{pmatrix} 1 \\ \alpha_n \end{pmatrix} = \frac{1}{\alpha \cdots \alpha_{n-1}} M_{a_1}^{-1} \cdots M_{a_n}^{-1} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \text{ where } M_{a_i}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix}. \]

We deduce that

\[ \alpha \cdots \alpha_{n-1} = \text{ first coordinate of } (M_{a_1} \cdots M_{a_n})^{-1} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \langle l_1^{(n)}, (1, \alpha) \rangle. \]

We conclude by noticing

\[ M_a = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \text{ and } M_{a_1} \cdots M_{a_n} = \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix}. \]
A skew product of the Gauss map

We consider the following skew product of the Gauss map

\[ T: (\alpha, \beta) \mapsto (\{1/\alpha\}, \{\beta/\alpha\}) = (1/\alpha - a_1, \beta/\alpha - b_1) = (\alpha_1, \beta_1). \]

We have

\[ \beta_1 = \beta/\alpha - b_1 \quad \text{and thus} \quad \beta = b_1 \alpha + \alpha \beta_1. \]

We deduce that

\[ \beta = \sum_{k=1}^{+\infty} b_k \alpha_1 \cdots \alpha_{k-1} = \sum_{k=1}^{+\infty} b_k |q_{k-1} \alpha - p_{k-1}|. \]

We similarly consider the following skew product of the Brun map

\[ T(\alpha, \beta, \gamma) = \begin{cases} (\beta/\alpha, 1/\alpha - a_1, \gamma/\alpha - b_1) & \text{if } \beta < \alpha \\ (1/\beta - a_1, \alpha/\beta, \gamma/\beta - b_1) & \text{if } \beta > \alpha \end{cases} \]

or of the Jacobi-Perron map

\[ T(\alpha, \beta, \gamma) = (\{\beta/\alpha\}, \{1/\alpha\}, \{\gamma/\alpha\}). \]
The **linear** Jacobi-Perron algorithm is defined on the positive cone \( \{(a, b, c) \in \mathbb{R}^3 | 0 \leq a, b < c\} \) by the transformation \( F \):

\[
F(a, b, c) = (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a).
\]

Let

\[
B = \lfloor b/a \rfloor \quad \text{and} \quad C = \lfloor c/a \rfloor.
\]

We have as **admissibility conditions**

\[
0 \leq B_n \leq C_n, \quad C_n \geq 1, \quad \text{if} \quad B_n = C_n \quad \text{then} \quad B_{n+1} \neq 0.
\]
Jacobi-Perron substitutions

We denote by $\sigma_{B,C}$ the substitution over the three-letter alphabet $\{1, 2, 3\}$ defined by:

$$\sigma_{B,C}(1) = 3, \quad \sigma_{B,C}(2) = 13^B, \quad \sigma_{B,C}(3) = 23^C.$$ 

Its incidence matrix $M$ equals

$$M = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & B & C
\end{pmatrix}$$

Recall that the linear Jacobi-Perron algorithm is defined on the positive cone $\{(a, b, c) \in \mathbb{R}^3 \mid 0 \leq a, b < c\}$ by the transformation $F$:

$$F(a, b, c) = (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a).$$

If $F(a, b, c) = (a_1, b_1, c_1)$, then

$$(a_1, b_1, c_1) = t M^{-1}(a, b, c) \text{ with } B = \lfloor b/a \rfloor \text{ and } C = \lfloor c/a \rfloor.$$
We denote by $\sigma_{B,C}$ the substitution over the three-letter alphabet $\{1, 2, 3\}$ defined by:

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Jacobi-Perron substitutions

The linear Jacobi-Perron algorithm is defined by

$$F(a, b, c) = (a_1, b_1, c_1) = (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a).$$

We have

$$\sigma_{B, C}(1) = 3, \quad \sigma_{B, C}(2) = 13^B, \quad \sigma_{B, C}(3) = 23^C, \quad M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & B & C \end{pmatrix}$$

We thus have

$$(a, b, c) = ^t M (a_1, b_1, c_1)$$

with $B = \lfloor b/a \rfloor$ and $C = \lfloor c/a \rfloor$.

Hence

$$E_1^*(\sigma)(\mathcal{P}(a_1, b_1, c_1)) = \mathcal{P}(a, b, c).$$
Jacobi-Perron substitutions

We have
\[ E_1^*(\sigma)(\mathcal{P}(a_1,b_1,c_1)) = \mathcal{P}(a,b,c). \]

If we obtain the JP digits \((B_1, C_1) \cdots (B_n, C_n)\), then
\[ E_1^*(\sigma(B_1,C_1)) \circ E_1^*(\sigma(B_2,C_2)) \cdots \circ E_1^*(\sigma(B_n,C_n))\mathcal{P}(a_n,b_n,c_n) = \mathcal{P}(a,b,c). \]

Since the unit cube \(\mathcal{U}\) belongs to every discrete plane, we conclude
\[ (a_n, b_n, c_n) = F^n(a, b, c) \implies E_1^*(\sigma(B_1,C_1)) \cdots E_1^*(\sigma(B_n,C_n))(\mathcal{U}) \subset \mathcal{P}(a,b,c). \]

Question

Does \( E_1^*(\sigma(B_1,C_1)) \cdots E_1^*(\sigma(B_n,C_n))(\mathcal{U}) \subset \mathcal{P}(a,b,c) \) generate the whole plane \(\mathcal{P}(a,b,c)\)?
Consider a discrete plane \( \mathcal{P}_{(a,b,c)} \). Let \( \mathcal{U} \) be the unit cube at the origin. By considering the iterates of \( \mathcal{U} \) under the action of a generalized substitution, are we able to generate the whole discrete plane \( \mathcal{P}_{(a,b,c)} \)?
Geometric Finiteness Property

**Definition**

Let $\sigma$ be a unimodular Pisot substitution. The geometric $(F')$-property is satisfied if and only if

$$\mathcal{P}_\sigma = \bigcup_{k \in \mathbb{N}} (E^*_1(\sigma))^k(U).$$

(F) $d_\beta(x)$ finite for all $x \in \mathbb{Z}[1/\beta] \cap [0, 1)$
# Theorem [Ito-Ohtsuki]

There exists a finite set of faces $\mathcal{V}$ with $\mathcal{U} \subset \mathcal{V}$ s.t. if there exists $n$ such that for all $k$

1. $B_{n+3k} = C_{n+3k}$
2. $C_{n+3k} - B_{n+3k} \geq 1$
3. $B_{n+3k+2} = 0$,

then the sequence of patterns

$$E_1^*(\sigma(B_1, C_1)) \ldots E_1^*(\sigma(B_n, C_n))(\mathcal{V})$$

generates the whole plane $P_{(a, b, c)}$.

Otherwise, the sequence of patterns

$$E_1^*(\sigma(B_1, C_1)) \ldots E_1^*(\sigma(B_n, C_n))(\mathcal{U})$$

generates the whole plane $P_{(a, b, c)}$. 

---

**Jacobi Perron expansions**
Boundary of fundamental domains

The sequence of patterns

\[ E_1^*(\sigma(\mathcal{B}_1, \mathcal{C}_1)) \cdots E_1^*(\sigma(\mathcal{B}_n, \mathcal{C}_n))(\mathcal{U}) \]

generates the whole plane \( \mathcal{P}_{(a,b,c)} \)

What is the shape of these patterns?

**Theorem [Ei]** Let \( \sigma \) be an invertible three-letter substitution. The boundary of \( E_1^*(\sigma)(\mathcal{U}) \) is given by \( \overline{\sigma^{-1}} \), the mirror image of the inverse of \( \sigma \)

**Theorem [B., Lacasse, Paquin, Provençal]** Take any admissible JP expansion. The boundaries of the patterns

\[ E_1^*(\sigma(\mathcal{B}_1, \mathcal{C}_1)) \cdots E_1^*(\sigma(\mathcal{B}_n, \mathcal{C}_n))(\mathcal{U}) \]

are selfavoiding curves.

**Question** Does the renormalization provide a Rauzy fractal with disjoint subpieces?
Multidimensional continued fractions and discrete geometry

- Approximate a direction par nested cones
- Lattice reduction
- Multidimensional Euclid algorithm
- A sequence of best approximations

We would like to get

- Reasonable convergence speed
- Reasonable computation time of the rational approximations with respect to the precision
- Detect rational dependencies
- Characterization of cubic numbers
Convergence

Theorem

There exists $\delta > 0$ s.t. for a.e. $(\alpha, \beta)$, there exists $n_0 = n_0(\alpha, \beta)$ s.t. for all $n \geq n_0$

$$|\alpha - p_n/q_n| < \frac{1}{q_n^{1+\delta}}$$

$$|\beta - r_n/q_n| < \frac{1}{q_n^{1+\delta}},$$

where $p_n, q_n, r_n$ are given by Brun/JP.

Brun [Ito-Fujita-Keane-Ohtsuki '93+'96]; Jacobi-Perron [Broise-Guivarc'h '99]
Brun's transformation is defined on $[0, 1]^d \setminus \{0\}$ by

$$T(\alpha_1, \cdots, \alpha_d) = \left( \frac{\alpha_1}{\alpha_d}, \cdots, \frac{\alpha_{i-1}}{\alpha_i}, \{ \frac{1}{\alpha_i} \}, \frac{\alpha_{i+1}}{\alpha_i}, \cdots, \frac{\alpha_d}{\alpha_i} \right),$$

where

$$i = \min \{ j \mid \alpha_j = \|\alpha\|_\infty \}.$$

The linear version is obtained by subtracting the second largest entry to the largest one.
<table>
<thead>
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<th>Arithmetics</th>
<th>Geometry</th>
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<tbody>
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<td>$d$-uple $\alpha \in [0, 1]^d$</td>
<td>stepped plane $P_{(1, \alpha)}$</td>
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</tr>
<tr>
<td>$d$-uple $\alpha \in [0, 1]^d$</td>
<td>stepped plane $\mathcal{P}_{(1, \alpha)}$</td>
</tr>
<tr>
<td>$(1, \alpha_n) \propto B_n(1, \alpha_{n+1})$</td>
<td>$\mathcal{P}<em>{(1, \alpha_n)} = \Theta</em>{\sigma_n}^*(\mathcal{P}<em>{(1, \alpha</em>{n+1})})$ with $t_B$ incidence matrix of $\sigma_n$</td>
</tr>
<tr>
<td>$(1, \frac{33}{38}, \frac{2}{19}) \propto B_{1,1}(1, \frac{5}{33}, \frac{4}{33})$</td>
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<td>( \mathcal{P}(1, \alpha_n) = \Theta_{\sigma_n}^* (\mathcal{P}(1, \alpha_{n+1})) )</td>
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<td>with ( t_B_n ) incidence matrix of ( \sigma_n )</td>
</tr>
<tr>
<td>((1, \frac{5}{33}, \frac{4}{33}) \propto B_{6,1}(1, \frac{3}{4}, \frac{4}{5}) )</td>
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</tr>
<tr>
<td>( (1, \frac{1}{3}, \frac{1}{3}) \propto B_{3,1}(1, 0, 1) )</td>
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</tbody>
</table>
We want to consider not only a substitution but a sequence of substitutions.

**Definition**

A sequence $u$ is said **S-adic** if there exist

- a finite set of substitutions $S$ over an alphabet $D = \{0, \ldots, d - 1\}$
- a morphism $\varphi$ from $D^*$ to $A^*$
- an infinite sequence of substitutions $(\sigma_n)_{n \geq 1}$ with values in $S$ such that

$$u = \lim_{n \to +\infty} \sigma_1 \sigma_2 \ldots \sigma_n(0).$$