Discrete geometry and numeration

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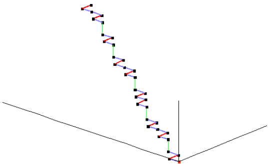
Numeration: Mathematics and Computer Science–Mars 09

A classical problem in Diophantine approximation

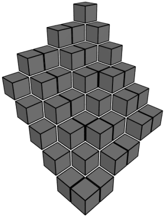
How to approximate a line in $\mathbb{R}^3$ by points in $\mathbb{Z}^3$?



How to define a discrete line in $\mathbb{R}^3$?

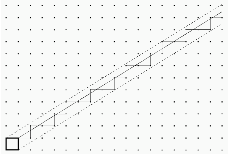
... and its dual version

How to approximate a plane in $\mathbb{R}^3$ by points in $\mathbb{Z}^3$?

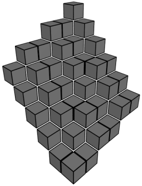


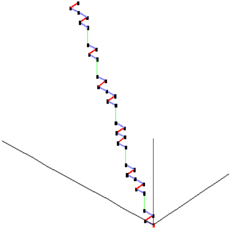
Numeration/representation systems in discrete geometry

- Continued fractions (regular, multidimensional unimodular)
- S-adic systems (infinite composition of a finite number of substitutions)



Continued fractions/ Ostrowski numeration





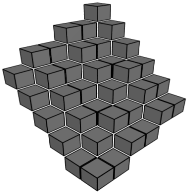
Jacobi-Perron, Brun c.f., generalized substitutions

Arithmetic discrete plane

Let a, b, c be strictly positive real numbers

The (standard) arithmetic discrete plane $\mathcal{P}_{((a,b,c),h)}$ is defined as

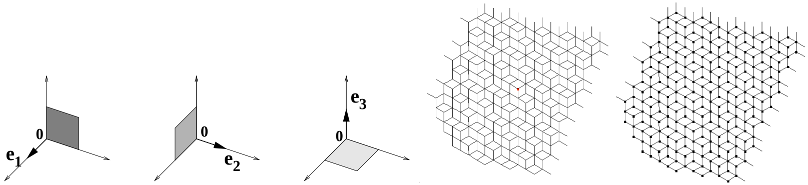
$$\mathcal{P}_{((a,b,c),h)} = \{(p,q,r) \in \mathbb{Z}^3 \mid 0 \leq ap + bq + cr + h < a + b + c\}.$$



We consider the stepped surface $\mathfrak{P}_{((a,b,c),h)}$ defined as the union of the facets of integer translates of unit cubes whose set of integer vertices equals $\mathcal{P}_{((a,b,c),h)}$

A first dynamical description

Let E_1, E_2, and E_3 be the three following faces:



$$\mathfrak{P}_{((a,b,c),h)} = \{(p,q,r) \in \mathbb{Z}^3 \mid 0 \leq ap + bq + cr + h < a + b + c\}.$$

A point $(p,q,r) \in \mathbb{Z}^3$ is the distinguished vertex of a face in $\mathfrak{P}_{((a,b,c),h)}$ of type

- 1 if and only if $ap + bq + cr + h \in [0, a[$
- 2 if and only if $ap + bq + cr + h \in [a, a+b[$
- 3 if and only if $ap + bq + cr + h \in [a+b, a+b+c[$

A first dynamical description

A point $(p, q, r) \in \mathbb{Z}^3$ is the distinguished vertex of a face in $\mathfrak{P}_{((a,b,c),h)}$ of type

- 1 if and only if $ap + bq + cr + h \in [0, a[$
- 2 if and only if $ap + bq + cr + h \in [a, a + b[$
- 3 if and only if $ap + bq + cr + h \in [a + b, a + b + c[$

The triple of strictly positive numbers (a, b, c) being fixed, let:

$$\begin{aligned} R_a : [0, a+b+c[\quad &\rightarrow [0, a+b+c[\qquad x \mapsto x + a \bmod a+b+c, \\ R_b : [0, a+b+c[\quad &\rightarrow [0, a+b+c[\qquad x \mapsto x + b \bmod a+b+c. \end{aligned}$$

Discrete planes are codings of $\mathbb{Z}^2$-actions

A first dynamical description

A point $(p, q, r) \in \mathbb{Z}^3$ is the distinguished vertex of a face in $\mathcal{P}_{((a,b,c),h)}$ of type

- 1 if and only if $ap + bq + cr + h \in [0, a[$
- 2 if and only if $ap + bq + cr + h \in [a, a + b[$
- 3 if and only if $ap + bq + cr + h \in [a + b, a + b + c[$

We can deduce information on

- possible local configurations/factors
- density of factors

Arithmetic discrete planes with the same normal vector have the same language

Toward an arithmetic representation

We can deduce information on

- possible local configurations/factors
- density of factors

We would like to be able to

- Generate discrete planes
- Recognize discrete planes: given a set of points in $\mathbb{Z}^3$, is it contained in an arithmetic discrete plane?

How to describe a discrete plane?

We will use a classical strategy based on

- induction/first return map
- substitutions
- continued fractions

$$\mathcal{P}_{\vec{n},h} = \{\vec{x} \in \mathbb{Z}^3 \mid 0 \leq \langle \vec{n}, \vec{x} \rangle + h < a + b + c\}$$

We want to describe $\mathcal{P}_{\vec{n},h}$ w.r. to

- A continued fraction algorithm for the normal vector $\vec{n}$
- An Ostrowski-type numeration system associated with the chosen c.f.a. for h

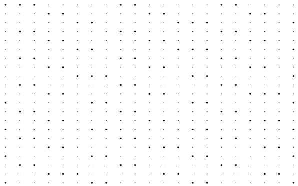
Induction

The induced map T_A of a map T on a subset A is defined by

$$T_A(x) = T^{n_x}(x) \text{ with } n_x = \inf\{p > 0 | T^p(x) \in A\}.$$

What is the meaning for a $\mathbb{Z}^2$-action?

- if I is the induction interval, consider the set of (m, n) such that $R_a^m R_b^n x \in I$.
- This subset is NOT a sublattice of $\mathbb{Z}^2$



We want to reorganize the induced orbit and give a one-to-one correspondence with the original one in terms of substitutions

substitution = reconstruction of the lost information

Connectedness of arithmetic planes

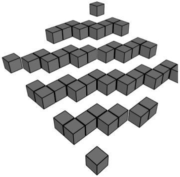
Question

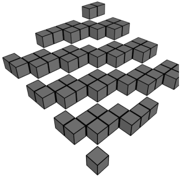
Find the smallest width ω for which the plane $\mathcal{P}(\vec{n}, h, \omega)$

$$\mathcal{P}_{\vec{n},h,\omega} = \{\vec{x} \in \mathbb{Z}^d \mid 0 \leq \langle \vec{x}, \vec{n} \rangle + h < \omega\}$$

is connected.

Rational parameters: [Brimkov-Barneva] [Gérard][Jamet-Toutant]





Jacobi-Perron algorithm

The projective Jacobi-Perron algorithm is defined on the unit square $X = [0,1) \times [0,1)$ by:

$$\Phi(\alpha,\beta) = \left(\frac{\beta}{\alpha} - \lfloor\frac{\beta}{\alpha}\rfloor, \frac{1}{\alpha} - \lfloor\frac{1}{\alpha}\rfloor\right) = (\{\beta/\alpha\}, \{1/\alpha\})$$

The linear Jacobi-Perron algorithm is defined on the positive cone

$$\{(a,b,c) \in \mathbb{R}^3 | 0 \leq a, b < c\}$$

by:

$$F(a,b,c) = (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a).$$

Jacobi-Perron algorithm

The linear Jacobi-Perron algorithm is defined on the positive cone

$$\{(a, b, c) \in \mathbb{R}^3 \mid 0 \le a, b < c\}$$

by

$$(a_1, b_1, c_1) = (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a).$$

Set $B = \lfloor b/a \rfloor a$, $C = \lfloor c/a \rfloor$

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} -B & 1 & 0 \\ -C & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Jacobi-Perron algorithm

The linear Jacobi-Perron algorithm is defined

$$\{(a,b,c) \in \mathbb{R}^3 \,|\, 0 \leq a, b < c\}$$

Set $B = \lfloor b/a \rfloor a$, $C = \lfloor c/a \rfloor$

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} -B & 1 & 0 \\ -C & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

- The matrix $\begin{pmatrix} -B & 1 & 0 \\ -C & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ belongs to $SL_3(\mathbb{Z})$. It is invertible
- The Jacobi-Perron algorithm is unimodular
- The inverse of $\begin{pmatrix} -B & 1 & 0 \\ -C & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ belongs to $SL_3(\mathbb{N})$

Jacobi-Perron algorithm

The inverse of $\begin{pmatrix} -B & 1 & 0 \\ -C & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ belongs to $SL_3(\mathbb{N})$

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} -B & 1 & 0 \\ -C & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We can tile $I_a \cup I_b \cup I_c$ by intervals $I_{a_1}, I_{b_1}, I_{c_1}$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & B \\ 0 & 1 & C \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$$

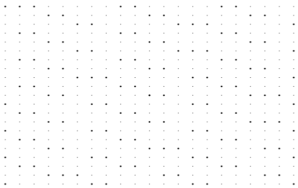
Jacobi-Perron substitution

The equation

$$ap + bq + cr = a_1 p_1 + b_1 q_1 + c_1 r_1$$

provides a relation between (p_1, q_1, r_1) and (p, q, r)

We now can go from the induced orbit to the original full orbit under the $\mathbb{Z}^2$-action



$$\mathcal{P}_{\vec{n}} \rightsquigarrow \text{full orbit}$$

$$\mathcal{P}_{\vec{n}_1} \rightsquigarrow \text{induced orbit}$$

We want to reconstruct the full orbit from the induced orbit

We want to reconstruct $\mathcal{P}_{\vec{n}}$ from $\mathcal{P}_{\vec{n_1}}$

From a continued fraction algorithm to substitutions

We are given a unimodular matrix $M \in SL_3(\mathbb{N})$ which describes a continued fraction algorithm

$$\vec{n} = M\,\vec{n_1}$$

We want

$$ap + bq + cr = a_1p_1 + b_1q_1 + c_1r_1$$

$$\langle \vec{n}, \vec{x} \rangle = \langle \vec{n_1}, \vec{x_1} \rangle$$

Hence

$$\langle \vec{n}, \vec{x} \rangle = \langle M\,\vec{n_1}, \vec{x} \rangle = \langle \vec{n_1}, {}^tM\,\vec{x} \rangle$$

provides

$$\vec{x} \rightsquigarrow \vec{x_1} = {}^tM\,\vec{x}$$

We go from $\mathcal{P}_{\vec{n_1}}$ to $\mathcal{P}_{\vec{n}}$ by

$$\vec{x_1} \mapsto {}^tM^{-1}\,\vec{x}$$

We then use a tiling of $I_a \cup I_b \cup I_c$ by intervals $I_{a_1}, I_{b_1}, I_{c_1}$

From a continued fraction algorithm to substitutions

- We go from $\mathcal{P}_{\vec{n_1}}$ to $\mathcal{P}_{\vec{n}}$ by

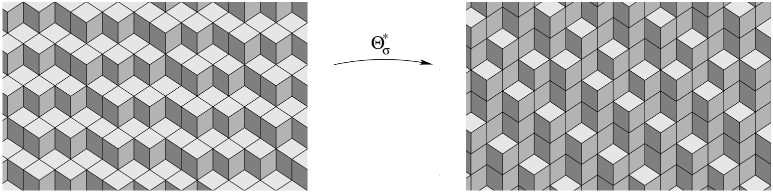
$$\vec{x_1} \mapsto {}^tM^{-1}\,\vec{x}$$

- The way we tile $I_a \cup I_b \cup I_c$ by intervals $I_{a_1}, I_{b_1}, I_{c_1}$ is noncanonical
- With each such choice is associated a substitution rule

Continued fraction algorithm $\rightsquigarrow$ Induction process $\rightsquigarrow$ Substitution rule

Theorem [Arnoux,B.,Ito]

Let σ be a unimodular substitution. Let $\vec{n} \in \mathbb{R}^d_+$ be a positive vector. The substitution rule maps without overlaps the stepped plane $\mathcal{P}_{\vec{n},h}$ onto $\mathcal{P}_{M\,\vec{n},h}$.



Substitutions

Let σ be a substitution on $\mathcal{A}$.
Example:

$$\sigma(1) = 12,\ \sigma(2) = 13,\ \sigma(3) = 1.$$

The incidence matrix $\mathbf{M}_\sigma$ of σ is defined by

$$\mathbf{M}_\sigma = (|\sigma(j)|_i)_{(i,j)\in\mathcal{A}^2},$$

where $|\sigma(j)|_i$ counts the number of occurrences of the letter i in $\sigma(j)$.

Unimodular substitution

$$\det \mathbf{M}_\sigma = \pm 1$$

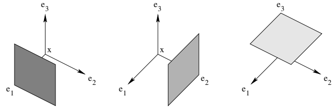
Abelianisation

Let d be the cardinality of $\mathcal{A}$. Let $\mathbf{l} : \mathcal{A}^\star \to \mathbb{N}^d$ be the abelinisation map

$$\mathbf{l}(w) = {}^t(|w|_1, |w|_2, \cdots, |w|_d).$$

Generalized substitutions

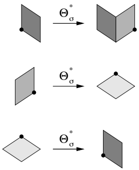
Let $(\mathbf{x}, 1^*)$, $(\mathbf{x}, 2^*)$, $(\mathbf{x}, 3^*)$ stand for the following faces

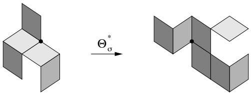


Generalized substitution [Arnoux-Ito][Ei]

Let σ be a unimodular morphism of the free froup.

$$E_1^*(\sigma)(\mathbf{x}, i^*) = \sum_{k \in \mathcal{A}} \sum_{P,\, \sigma(k) = PiS} \left(\mathbf{M}_\sigma^{-1}\left(\mathbf{x} - \mathbf{l}(P)\right), k^*\right).$$

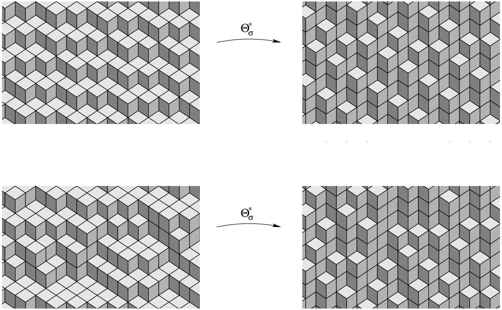




Action on planes and surfaces

Theorem [Arnoux-Ito, Fernique]

Let σ be a unimodular substitution. Let $\vec{n} \in \mathbb{R}^d_+$ be a positive vector. The generalized substitution $E_1^*(\sigma)$ maps without overlaps the stepped plane $\mathcal{P}_{\vec{n},h}$ onto $\mathcal{P}_{{}^tM_\sigma \vec{n},h}$.



Ostrowski expansion of real numbers

Ostrowski's representation of integers can be extended to real numbers.

The base is given by the sequence $(\theta_n)_{n\geq 0}$, where $\theta_n = (q_n\alpha - p_n)$.

Every real number $-\alpha \leq \beta < 1-\alpha$ can be expanded uniquely in the form

$$\beta = \sum_{k=1}^{+\infty} c_k \theta_{k-1},$$

where

$$\begin{cases} 0 \leq c_1 \leq a_1 - 1 \\ 0 \leq c_k \leq a_k \text{ for } k \geq 2 \\ c_k = 0 \text{ if } c_{k+1} = a_{k+1} \\ c_k \neq a_k \text{ for infinitely many odd integers.} \end{cases}$$

A skew product of the Gauss map

We consider the following skew product of the Gauss map

$$T\colon (\alpha,\beta) \mapsto (\{1/\alpha\},\{\beta/\alpha\}) = (1/\alpha - a_1, \beta/\alpha - b_1) = (\alpha_1,\beta_1).$$

We have

$$\beta_1 = \beta/\alpha - b_1 \text{ and thus } \beta = b_1\alpha + \alpha\beta_1.$$

We deduce that

$$\beta = \sum_{k=1}^{+\infty} b_k \alpha\alpha_1\cdots\alpha_{k-1} = \sum_{k=1}^{+\infty} b_k |q_{k-1}\alpha - p_{k-1}|.$$

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$$\beta = \sum_{k=1}^{+\infty} b_k \alpha\alpha_1\cdots\alpha_{k-1} = \sum_{k=1}^{+\infty} b_k |q_{k-1}\alpha - p_{k-1}|.$$

Indeed we use the fact that

$$\begin{pmatrix} 1 \\ \alpha_n \end{pmatrix} = \frac{1}{\alpha\cdots\alpha_{n-1}} \mathbf{M}_{a_n}^{-1}\cdots\mathbf{M}_{a_1}^{-1}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} \text{ where } \mathbf{M}_a^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix}.$$

We deduce that

$$\alpha\cdots\alpha_{n-1} = \text{ first coordinate of } (\mathbf{M}_{a_1}\cdots\mathbf{M}_{a_n})^{-1}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \langle \mathbf{l}_1^{(n)}, (1,\alpha)\rangle.$$

We conclude by noticing

$$\mathbf{M}_a = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{M}_{a_1}\cdots\mathbf{M}_{a_n} = \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix}$$

A skew product of the Gauss map

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We have

$$\beta_1 = \beta/\alpha - b_1 \text{ and thus } \beta = b_1\alpha + \alpha\beta_1.$$

We deduce that

$$\beta = \sum_{k=1}^{+\infty} b_k \alpha\alpha_1\cdots\alpha_{k-1} = \sum_{k=1}^{+\infty} b_k |q_{k-1}\alpha - p_{k-1}|.$$

We similarly consider the following skew product of the Brun map

$$T(\alpha,\beta,\gamma) = \begin{cases} (\beta/\alpha, 1/\alpha - a_1, \gamma/\alpha - b_1) & \text{if } \beta < \alpha \\ (1/\beta - a_1, \alpha/\beta, \gamma/\beta - b_1) & \text{if } \beta > \alpha \end{cases}$$

or of the Jacobi-Perron map

$$T(\alpha,\beta,\gamma) = (\{\beta/\alpha\},\{1/\alpha\},\{\gamma/\alpha\}).$$

Jacobi-Perron algorithm

The linear Jacobi-Perron algorithm is defined on the positive cone $\{(a,b,c) \in \mathbb{R}^3 | 0 \leq a, b < c\}$ by the transformation F:

$$F(a,b,c) = (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a).$$

Let

$$B = \lfloor b/a \rfloor \text{ and } C = \lfloor c/a \rfloor.$$

We have as admissibility conditions

$$0 \leq B_n \leq C_n, \; C_n \geq 1, \quad \text{if } B_n = C_n \text{ then } B_{n+1} \neq 0.$$

Jacobi-Perron substitutions

We denote by $\sigma_{B,C}$ the substitution over the three- letter alphabet $\{1,2,3\}$ defined by:

$$\sigma_{B,C}(1) = 3, \qquad \sigma_{B,C}(2) = 13^B, \qquad \sigma_{B,C}(3) = 23^C.$$

Its incidence matrix $\mathbf{M}$ equals

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & B & C \end{pmatrix}$$

Recall that the linear Jacobi-Perron algorithm is defined on the positive cone $\{(a,b,c) \in \mathbb{R}^3 | 0 \le a, b < c\}$ by the transformation F:

$$F(a,b,c) = (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a).$$

If $F(a,b,c) = (a_1, b_1, c_1)$, then

$$(a_1, b_1, c_1) = {}^t\mathbf{M}^{-1}(a,b,c) \text{ with } B = \lfloor b/a \rfloor \text{ and } C = \lfloor c/a \rfloor.$$

Jacobi Perron generalized substitution

We denote by $\sigma_{B,C}$ the substitution over the three- letter alphabet $\{1,2,3\}$ defined by:

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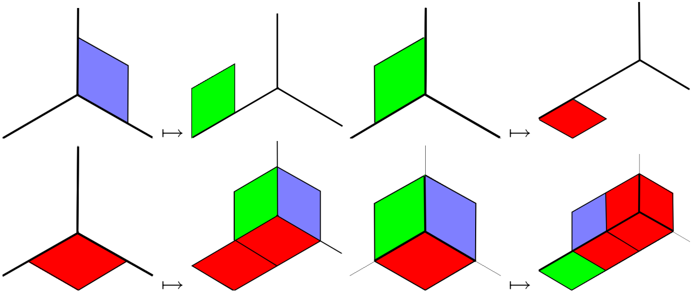


Jacobi-Perron substitutions

The linear Jacobi-Perron algorithm is defined by

$$F(a,b,c) = (a_1, b_1, c_1) = (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a).$$

We have

$$\sigma_{B,C}(1) = 3, \qquad \sigma_{B,C}(2) = 13^B, \qquad \sigma_{B,C}(3) = 23^C, \ \mathbf{M} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & B & C \end{pmatrix}$$

We thus have

$$(a,b,c) = {}^t\mathbf{M}\,(a_1, b_1, c_1)$$

with $B = \lfloor b/a \rfloor$ and $C = \lfloor c/a \rfloor$.

Hence

$$E_1^*(\sigma)(\mathcal{P}_{(a_1,b_1,c_1)}) = \mathcal{P}_{(a,b,c)}.$$

Jacobi-Perron substitutions

We have

$$E_1^*(\sigma)(\mathcal{P}_{(a_1,b_1,c_1)}) = \mathcal{P}_{(a,b,c)}.$$

If we obtain the JP digits $(B_1, C_1)\cdots(B_n, C_n)$, then

$$E_1^*(\sigma_{(B_1,C_1)}) \circ E_1^*(\sigma_{(B_2,C_2)}) \cdots \circ E_1^*(\sigma_{(B_n,C_n)})\mathcal{P}_{(a_n,b_n,c_n)} = \mathcal{P}_{(a,b,c)}.$$

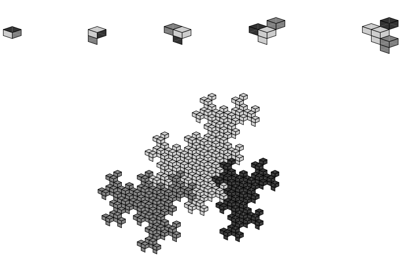
Since the unit cube $\mathcal{U}$ belongs to every discrete plane, we conclude

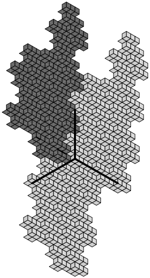
$$(a_n, b_n, c_n) = F^n(a, b, c) \implies E_1^*(\sigma_{(B_1,C_1)}) \ldots E_1^*(\sigma_{(B_n,C_n)})(\mathcal{U}) \subset \mathcal{P}_{(a,b,c)}.$$

Question

Does $E_1^*(\sigma_{(B_1,C_1)}) \ldots E_1^*(\sigma_{(B_n,C_n)})(\mathcal{U}) \subset \mathcal{P}_{(a,b,c)}$ generate the whole plane $\mathcal{P}_{(a,b,c)}$?

Generation of a discrete plane





Question

Consider a discrete plane $\mathfrak{P}_{(a,b,c)}$. Let $\mathcal{U}$ be the unit cube at the origin. By considering the iterates of $\mathcal{U}$ under the action of a generalized substitution, are we able to generate generate the whole discrete plane $\mathfrak{P}_{(a,b,c)}$?

Geometric Finiteness Property

Definition

Let σ be a unimodular Pisot substitution. The geometric (F)-property is satisfied if and only if

$$\mathcal{P}_\sigma = \bigcup_{k\in\mathbb{N}} (E_1^*(\sigma))^k(\mathcal{U}).$$

(F) $d_\beta(x)$ finite for all $x \in \mathbb{Z}[1/\beta] \cap [0,1)$

Jacobi Perron expansions

Theorem [Ito-Ohtsuki]

There exists a finite set of faces $\mathcal{V}$ with $\mathcal{U} \subset \mathcal{V}$ s.t. if there exists n such that for all k

1. $B_{n+3k} = C_{n+3k}$
2. $C_{n+3k} - B_{n+3k} \geq 1$
3. $B_{n+3k+2} = 0$,

then the sequence of patterns

$$E_1^*(\sigma_{(B_1,C_1)}) \ldots E_1^*(\sigma_{(B_n,C_n)})(\mathcal{V})$$

generates the whole plane $\mathcal{P}_{(a,b,c)}$.
Otherwise, the sequence of patterns

$$E_1^*(\sigma_{(B_1,C_1)}) \ldots E_1^*(\sigma_{(B_n,C_n)})(\mathcal{U})$$

generates the whole plane $\mathcal{P}_{(a,b,c)}$.

Boundary of fundamental domains

The sequence of patterns

$$E_1^*(\sigma_{(B_1,C_1)})\dots E_1^*(\sigma_{(B_n,C_n)})(\mathcal{U})$$

generates the whole plane $\mathcal{P}_{(a,b,c)}$

What is the shape of these patterns?

Theorem [Ei] Let σ be an invertible three-letter substitution. The boundary of $E_1^*(\sigma)(\mathcal{U})$ is given by $\widetilde{\sigma^{-1}}$, the mirror image of the inverse of σ

Theorem [B.,Lacasse,Paquin,Provençal] Take any admissible JP expansion. The boundaries of the patterns

$$E_1^*(\sigma_{(B_1,C_1)})\dots E_1^*(\sigma_{(B_n,C_n)})(\mathcal{U})$$

are selfavoiding curves.
Question Does the renormalization provide a Rauzy fractal with disjoint subpieces?

Multidimensional continued fractions and discrete geometry

- Approximate a direction par nested cones
- Lattice reduction
- Multidimensional Euclid algorithm
- A sequence of best approximations

We would like to get

- Reasonable convergence speed
- Reasonable computation time of the rational approximations with respect to the precision
- Detect rational dependencies
- Characterization of cubic numbers

Convergence

Theorem

There exists $\delta > 0$ s.t. for a.e. (α, β) , there exists $n_0 = n_0(\alpha, \beta)$ s.t. for all $n \geq n_0$

$$|\alpha - p_n/q_n| < \frac{1}{q_n^{1+\delta}}$$

$$|\beta - r_n/q_n| < \frac{1}{q_n^{1+\delta}},$$

where p_n, q_n, r_n are given by Brun/JP.

Brun [Ito-Fujita-Keane-Ohtsuki '93+'96]; Jacobi-Perron [Broise-Guivarc'h '99]

Brun's algorithm

Brun's transformation is defined on $[0,1]^d \backslash \{0\}$ by

$$T(\alpha_1, \cdots, \alpha_d) = (\frac{\alpha_1}{\alpha_d}, \cdots, \frac{\alpha_{i-1}}{\alpha_i}, \{\frac{1}{\alpha_i}\}, \frac{\alpha_{i+1}}{\alpha_i}, \cdots, \frac{\alpha_d}{\alpha_i}),$$

where

$$i = \min\{j | \ \alpha_j = ||\alpha||_\infty\}.$$

The linear version is obtained by subtracting the second largest entry to the largest one

Arithmetics	Geometry
d-uple $\alpha \in [0,1]^d$	stepped plane $\mathcal{P}_{(1,\alpha)}$
$(1,\alpha_n) \propto B_n(1,\alpha_{n+1})$	$\mathcal{P}_{(1,\alpha_n)} = \Theta^*_{\sigma_n}(\mathcal{P}_{(1,\alpha_{n+1})})$ with ${}^tB_n$ incidence matrice of σ_n

Arithmetics	Geometry
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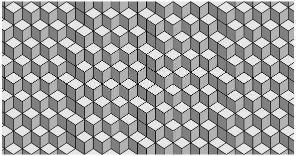
$(1, \alpha_0, \beta_0) = (1, \frac{11}{14}, \frac{19}{21})$	

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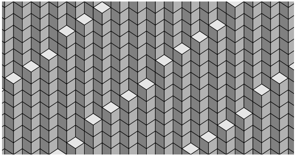
<tr><td>$(1, \frac{11}{14}, \frac{19}{21}) \propto B_{1,2}(1, \frac{33}{38}, \frac{2}{19})$</td><td></td></tr>
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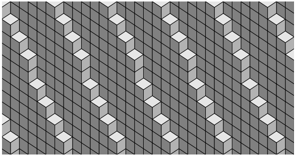
$(1, \frac{33}{38}, \frac{2}{19}) \propto B_{1,1}(1, \frac{5}{33}, \frac{4}{33})$	

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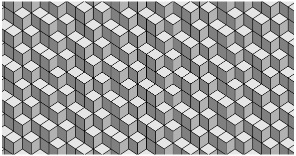
<tr><td>$(1, \frac{5}{33}, \frac{4}{33}) \propto B_{6,1}(1, \frac{3}{4}, \frac{4}{5})$</td><td></td></tr>
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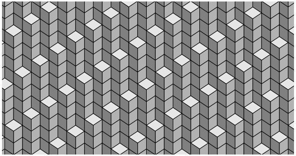
<tr><td>$(1, \frac{3}{4}, \frac{4}{5}) \propto B_{1,2}(1, \frac{3}{4}, \frac{1}{4})$</td><td></td></tr>
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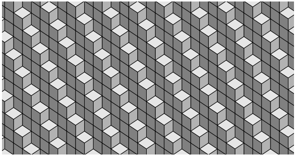
$(1, \frac{3}{4}, \frac{1}{4}) \propto B_{1,1}(1, \frac{1}{3}, \frac{1}{3})$	

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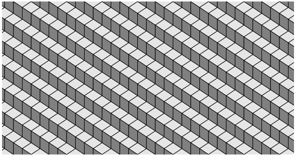
$(1, \frac{1}{3}, \frac{1}{3}) \propto B_{3,1}(1,0,1)$	

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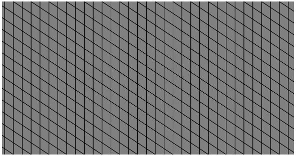
$(1, 0, 1) \propto B_{1,2}(1, 0, 0)$	

S-adic expansions

We want to consider not only a substitution but a sequence of substitutions.

Definition

A sequence u is said S-adic if there exist

- a finite set of substitutions $\mathcal{S}$ over an alphabet $\mathcal{D} = \{0, \ldots, d-1\}$
- a morphism φ from $\mathcal{D}^\star$ to $\mathcal{A}^\star$
- an infinite sequence of substitutions $(\sigma_n)_{n\geq 1}$ with values in $\mathcal{S}$

such that

$$u = \lim_{n\to+\infty} \sigma_1\sigma_2\ldots\sigma_n(0).$$