Multidimensional Generalized Automatic Sequences and Shape-Symmetric Morphic Words

Emilie Charlier  Tomi Kärki  Michel Rigo

Department of Mathematics
University of Liège

Numération : Mathématiques et Informatique
Marseille, March 26th 2009
**Background**

A *k*-ary numeration system, $k \geq 2$

$\Sigma_k = \{0, \ldots, k - 1\}$

$$n = \sum_{i=0}^{\ell} d_i k^i, \ d_\ell \neq 0, \ \text{rep}_k(n) = d_\ell \cdots d_0 \in \Sigma_k^*$$

An infinite word $x = (x_n)_{n \geq 0}$ is *k-automatic* if there exists a DFAO $A = (Q, q_0, \Sigma_k, \delta, \Gamma, \tau)$ s.t. for all $n \geq 0$,

$$x_n = \tau(\delta(q_0, \text{rep}_k(n))).$$

**Theorem (Cobham)**

Let $k \geq 2$. An infinite word is *k-automatic* iif it is the image under a coding of an infinite fixed point of a *k-uniform* morphism.
**Abstract Numeration Systems**

**Definition**

An *abstract numeration system* is a triple $S = (L, \Sigma, <)$ where $L$ is a regular language over a totally ordered alphabet $(\Sigma, <)$.

Enumerating the words of $L$ with respect to the genealogical ordering induced by $<$ gives a one-to-one correspondence

$$\text{rep}_S : \mathbb{N} \rightarrow L \quad \text{val}_S = \text{rep}_S^{-1} : L \rightarrow \mathbb{N}.$$ 

**Example**

$L = a^*$, $\Sigma = \{a\}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>⋯</th>
</tr>
</thead>
<tbody>
<tr>
<td>rep$(n)$</td>
<td>$\varepsilon$</td>
<td>$a$</td>
<td>$aa$</td>
<td>$aaa$</td>
<td>$aaaa$</td>
<td>⋯</td>
</tr>
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The **Abstract Numeration Systems** study the structure and properties of numeration systems, which are used to represent numbers in a given basis. A numeration system is defined by a regular language $L$ over an alphabet $\Sigma$ and a linear order $<$ on $\Sigma$. The genealogical ordering induced by $<$ allows for the enumeration of words in $L$, establishing a one-to-one correspondence between the natural numbers and the elements of $L$.

For instance, consider $L = a^*$, the set of all strings consisting of $a$. The alphabet $\Sigma = \{a\}$ consists only of the symbol $a$. The genealogical ordering induced by $<$ defines the order of words in $L$ as $\varepsilon < a < aa < aaa < aaaa < ⋯$. This ordering allows us to map each natural number $n$ to a word in $L$ and vice versa, providing a unique representation of numbers in terms of words over the alphabet $\{a\}$. This mapping is crucial in various applications, including number theory, combinatorics, and computer science.
Abstract Numeration Systems

Example

\[ L = \{a, b\}^*, \Sigma = \{a, b\}, \ a < b \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
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<th>4</th>
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<tbody>
<tr>
<td>( \text{rep}(n) )</td>
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<td>b</td>
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<td>ab</td>
<td>ba</td>
<td>bb</td>
<td>aaa</td>
<td>\ldots</td>
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</table>

Example

\[ L = a^*b^*, \Sigma = \{a, b\}, \ a < b \]

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<thead>
<tr>
<th>( n )</th>
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<th>1</th>
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<td>bb</td>
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<td>\ldots</td>
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</tbody>
</table>
**Definition**

Let $S = (L, \Sigma, <)$ be an abstract numeration system. An infinite word $x = (x_n)_{n \geq 0}$ is $S$-automatic if there exists a DFAO $A = (Q, q_0, \Sigma, \delta, \Gamma, \tau)$ s.t. for all $n \geq 0$,

$$x_n = \tau(\delta(q_0, \text{rep}_S(n))).$$

**Theorem (Maes, Rigo)**

An infinite word is $S$-automatic for some abstract numeration system $S$ iif it is the image under a coding of an infinite fixed point of a morphism, i.e. a morphic word.
Idea of the Proof in Dimension 1

Example (S-Automatic $\rightarrow$ Morphic)

$$S = (L, \{0, 1, 2\}, 0 < 1 < 2) \text{ where } L = \{w \in \Sigma^*: |w|_1 \text{ is odd}\}$$

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<tr>
<th>$n$</th>
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<th>8</th>
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<tr>
<td>rep$_S(n)$</td>
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<td>10</td>
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<td>012</td>
<td>021</td>
<td>...</td>
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<tr>
<td>$x$</td>
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<td>a</td>
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</table>
Example (Continued)

\[ f : \alpha \mapsto \alpha I_a \quad F_a \mapsto F_b I_b F_a \quad g : \alpha, I_a, I_b \mapsto \varepsilon \]
\[ I_a \mapsto I_b F_b I_a \quad F_b \mapsto F_a I_a F_b \quad F_a \mapsto a \]
\[ I_b \mapsto I_a F_a I_b \quad F_b \mapsto b \]

\[ L \subseteq \Sigma^* \]
\[
\begin{array}{ccccccccccc}
\varepsilon & 0 & 1 & 2 & 00 & 01 & 02 & 10 & 11 & 12 \\
\hline
f^\omega(\alpha) & \alpha & I_a & I_b & F_b & I_a & I_a & F_a & I_b & F_a & I_a & F_b \\
\hline
x & b & a & a & b
\end{array}
\]

\[ g(f^\omega(\alpha)) = x \]
Idea of the Proof in Dimension 1

Example (Morphic → $S$-Automatic)

Consider the morphism $\mu$ defined by $a \mapsto abc$; $b \mapsto bc$; $c \mapsto aac$. We have $\mu^\omega(a) = abcbaacbcacabcabcaacabcabc$.

One canonically associates the DFA $A_{\mu,a}$

$$L_{\mu,a} = \{ \varepsilon, 1, 2, 10, 11, 20, 21, 22, 100, 101, 110, 111, 112, 200, \ldots \}$$

If $S = (L_{\mu,a}, \{0,1,2\}, 0 < 1 < 2)$, then

$$(\mu^\omega(a))_n = \delta_{\mu}(a, \text{rep}_S(n)) \text{ for all } n \geq 0.$$
**Multidimensional Case**

A *d-dimensional infinite word* over an alphabet $\Sigma$ is a map $x : \mathbb{N}^d \to \Sigma$. We use notation like $x_{n_1, \ldots, n_d}$ or $x(n_1, \ldots, n_d)$ to denote the value of $x$ at $(n_1, \ldots, n_d)$.

If $w_1, \ldots, w_d$ are finite words over the alphabet $\Sigma$,

$$(w_1, \ldots, w_d)^\#: = (\#^{m-|w_1|}w_1, \ldots, \#^{m-|w_d|}w_d)$$

where $m = \max\{|w_1|, \ldots, |w_d|\}$.

**Example**

$$(ab, bbaa)^\# = (\#\#ab, bbaa)$$
A $d$-dimensional infinite word over an alphabet $\Gamma$ is $k$-automatic if there exists a DFAO

$$\mathcal{A} = (Q, q_0, (\Sigma_k)^d \setminus \{0, \ldots, 0\}, \delta, \Gamma, \tau)$$

s.t. for all $n_1, \ldots, n_d \geq 0$,

$$\tau \left( \delta \left( q_0, (\text{rep}_k(n_1), \ldots, \text{rep}_k(n_d))^0 \right) \right) = x_{n_1,\ldots,n_d}.$$

**Theorem (Salon)**

Let $k \geq 2$ and $d \geq 1$. A $d$-dimensional infinite word is $k$-automatic iff it is the image under a coding of a fixed point of a $k$-uniform $d$-dimensional morphism.
**Background**

A *d-dimensional picture* over the alphabet $\Sigma$ is a map

$$x : [0, s_1 - 1] \times \cdots \times [0, s_d - 1] \to \Sigma.$$ 

The *shape* of $x$ is $|x| = (s_1, \ldots, s_d)$.

If $s_i < \infty$ for all $i \in [1, d]$, then $x$ is said to be *bounded*.

The set of $d$-dimensional bounded pictures over $\Sigma$ is $B_d(\Sigma)$.

A bounded picture $x$ is a *square* of *size* $c$ if $s_i = c$ for all $i \in [1, d]$. 
Example

Consider the two bidimensional pictures

\[ x = \begin{array}{cc} a & b \\ c & d \end{array} \quad \text{and} \quad y = \begin{array}{cccc} a & a & b \\ b & c & d \end{array} \]

of shapes \(|x| = (2, 2)\) and \(|y| = (3, 2)\) respectively.

Since \(|x|_2 = |y|_2 = 2\), we get

\[ x \odot^1 y = \begin{array}{cccc} a & b & a & a & b \\ c & d & b & c & d \end{array}. \]

But notice that \(x \odot^2 y\) is not defined because \(2 = |x|_1 \neq |y|_1 = 3\).
**Example**

Consider the map $\mu$ given by

\[
    a \mapsto \begin{array}{cc}
        a & a \\
        b & d \\
    \end{array},
    b \mapsto \begin{array}{c}
        c
    \end{array},
    c \mapsto \begin{array}{cc}
        a & a \\
    \end{array},
    d \mapsto \begin{array}{c}
        d
    \end{array}.
\]

Let

\[
    x = \begin{array}{cc}
        a & b \\
        c & d
    \end{array}.
\]

Since $|\mu(a)|_2 = |\mu(b)|_2 = 2$, $|\mu(c)|_2 = |\mu(d)|_2 = 1$, $|\mu(a)|_1 = |\mu(c)|_1 = 2$ and $|\mu(b)|_1 = |\mu(d)|_1 = 1$, $\mu(x)$ is well defined and given by

\[
    \mu(x) = \begin{array}{ccc}
        a & a & c \\
        b & d & b \\
        a & a & d
    \end{array}.
\]

Notice that $\mu^2(x)$ is not well defined.
**Background**

**Definition**
Let $\mu : \Sigma \to B_d(\Sigma)$ be a map. If for all $a \in \Sigma$ and all $n \geq 0$, $\mu^n(a)$ is well defined from $\mu^{n-1}(a)$, then $\mu$ is said to be a $d$-dimensional morphism.

**Example**
If for all $a \in \Sigma$, $\mu(a)$ has a fixed shape $(s_1, s_2)$, then the map $\mu$ is a morphism.

**Definition**
If for all $a \in \Sigma$, $\mu(a)$ is a square of size $k$, then $\mu$ is said to be a $k$-uniform morphism.
Let $\mu$ be a $d$-dimensional morphism and $a$ be a letter such that $(\mu(a))_0 = a$. We say that $\mu$ is *prolongable on $a$* and the limit

$$w = \mu^\omega(a) := \lim_{n \to +\infty} \mu^n(a)$$

is well defined and $w = \mu(w)$ is a *fixed point* of $\mu$.

A $d$-dimensional infinite word $x$ over $\Sigma$ is *purely morphic* if it is a fixed point of a $d$-dimensional morphism.

It is *morphic* if there exists a coding $\nu : \Gamma \to \Sigma$ such that $x = \nu(y)$ for some purely morphic word $y$ over $\Gamma$. 
**Definition**

Let $S = (L, \Sigma, <)$ be an abstract numeration system. A *$d$-dimensional infinite word* over the alphabet $\Gamma$ is *$S$-automatic* if there exists a DFAO

$$\mathcal{A} = (Q, q_0, (\Sigma \cup \{\#\})^d, \delta, \Gamma, \tau)$$

s.t. for all $n_1, \ldots, n_d \geq 0$,

$$\tau \left( \delta \left( q_0, (\operatorname{rep}_S(n_1), \ldots, \operatorname{rep}_S(n_d))\# \right) \right) = x_{n_1, \ldots, n_d}.$$
Example

Consider \( S = (\{a, ba\}^* \{\varepsilon, b\}, \{a, b\}, a < b) \) and the DFA

\[
\begin{align*}
\text{State } p: & \quad (\#, a), (\#, b), (a, \#) \\
\text{State } q: & \quad (a, b), (b, \#), (b, a) \\
\text{State } r: & \quad (a, \#), (b, \#), (b, a) \\
\text{State } s: & \quad (a, b), (b, \#), (b, a)
\end{align*}
\]
We produce the following bidimensional infinite $S$-automatic word:

|     | $\omega$ | $a$ | $b$ | $aa$ | $ab$ | $ba$ | $aaa$ | $aab$ | $\ldots$
|-----|----------|-----|-----|------|------|------|-------|-------|-----|
| $\varepsilon$ | $p$ | $q$ | $q$ | $p$ | $q$ | $p$ | $q$ | $q$ | $\ldots$
| $a$ | $p$ | $p$ | $s$ | $s$ | $q$ | $s$ | $p$ | $s$ | $\ldots$
| $b$ | $q$ | $p$ | $s$ | $q$ | $s$ | $q$ | $p$ | $s$ | $\ldots$
| $aa$ | $p$ | $p$ | $s$ | $p$ | $s$ | $q$ | $q$ | $s$ | $\ldots$
| $ab$ | $q$ | $p$ | $s$ | $p$ | $s$ | $s$ | $s$ | $r$ | $\ldots$
| $ba$ | $p$ | $s$ | $q$ | $p$ | $s$ | $q$ | $s$ | $q$ | $\ldots$
| $aaa$ | $p$ | $p$ | $s$ | $p$ | $s$ | $q$ | $p$ | $s$ | $\ldots$
| $aab$ | $q$ | $p$ | $s$ | $p$ | $s$ | $s$ | $p$ | $s$ | $\ldots$
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ldots$
Shape-Symmetry

**Definition (Maes)**

Let $\mu : \Sigma \rightarrow B_d(\Sigma)$ be a $d$-dimensional morphism having the $d$-dimensional infinite word $x$ as a fixed point.

If for any permutation $f$ of $\{1, \ldots, d\}$ and for all $n_1, \ldots, n_d > 0$,

$$|\mu(x_{n_1}, \ldots, n_d)| = (s_1, \ldots, s_d)$$

$$\downarrow$$

$$|\mu(x_{nf(1)}, \ldots, nf(d))| = (sf(1), \ldots, sf(d)),$$

then $x$ is said to be *shape-symmetric with respect to $\mu$.*
Example

$$\mu(a) = \mu(f) = \begin{array}{cc} a & b \\ c & d \end{array}, \quad \mu(b) = \begin{array}{c} e \\ c \end{array}, \quad \mu(c) = \begin{array}{cc} e & b \end{array}, \quad \mu(d) = \begin{array}{c} f \end{array},$$

$$\mu(e) = \begin{array}{cc} e & b \\ g & d \end{array}, \quad \mu(g) = \begin{array}{cc} h & b \end{array}, \quad \mu(h) = \begin{array}{cc} h & b \\ c & d \end{array}.$$
Main Result

**Theorem (C., Kärki, Rigo)**

Let $d \geq 1$. The $d$-dimensional infinite word $x$ is *S-automatic* for some abstract numeration system $S = (L, \Sigma, <)$ where $\varepsilon \in L$ iif $x$ is the image under a coding of a *shape-symmetric* infinite $d$-dimensional word.
Idea of the Proof in Dimension 2

Example ($S$-Automatic $\rightarrow$ Shape-Symmetric)

Consider $S = (\{a, ba\}^*\{\varepsilon, b\}, \{a, b\}, a < b)$ and the DFA $\mathcal{A}$.
One associates a uniform bidimensional morphism $\mu_{A}$ to $A$: If $\sigma_0 = \#$, $\sigma_1 = a$ and $\sigma_2 = b$, then

$$\mu_A(t) = z \text{ with } z_{m,n} = \delta(t, (\sigma_m, \sigma_n)).$$

$$\mu_A(p) = \begin{array}{ccc}
p & q & q \\
p & p & s \\
q & p & s \\
\end{array} ; \quad \mu_A(q) = \begin{array}{ccc}
q & p & q \\
p & s & q \\
p & q & s \\
\end{array} ;$$

$$\mu_A(r) = \begin{array}{ccc}
r & s & s \\
p & r & s \\
p & r & p \\
\end{array} ; \quad \mu_A(s) = \begin{array}{ccc}
s & r & s \\
r & q & s \\
r & s & r \\
\end{array} .$$

Iterating $\mu_A$ from $p$, we obtain ...
The subword in blue is the $S$-automatic word from the beginning.
Idea of the Proof in Dimension 2

Example (Shape-Symmetric $\rightarrow$ $S$-Automatic)

$\mu(a) = \mu(f) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $\mu(b) = \begin{pmatrix} e \\ c \end{pmatrix}$; $\mu(c) = \begin{pmatrix} e & b \end{pmatrix}$; $\mu(d) = \begin{pmatrix} f \end{pmatrix}$

$\mu(e) = \begin{pmatrix} e & b \\ g & d \end{pmatrix}$; $\mu(g) = \begin{pmatrix} h & b \end{pmatrix}$; $\mu(h) = \begin{pmatrix} h & b \\ c & d \end{pmatrix}$.

$\mu^\omega(a) =$

\[
\begin{array}{cccccccc}
  a & b & e & e & b & e & b & e & \cdots \\
  c & d & c & g & d & g & d & c \\
  e & b & f & e & b & h & b & f \\
  e & b & e & a & b & e & b & e \\
  g & d & c & g & d & c & d & c \\
  e & b & e & e & b & a & b & e \\
  g & d & c & g & d & c & d & c \\
  h & b & f & e & b & e & b & f \\
  \vdots & & & & & & & \vdots \\
\end{array}
\]
Example (Continued)

Consider the morphism $\mu_1$ defined by

$$a \mapsto ab ; b \mapsto e ; e \mapsto eb.$$ 

We have $\mu_1^\omega(a) = abeebebeebeebeebeebeebeebeebeebeebeebeebeeb \cdots$. One canonically associates the DFA $A_{\mu_1,a}$

$$L_{\mu_1,a} = \{ \varepsilon, 1, 10, 100, 101, 1000, 1001, 1010, 10000, \ldots \}$$
Consider the morphism $\mu_2$ defined by

$$a \mapsto ac ; c \mapsto e ; e \mapsto eg ; g \mapsto h ; h \mapsto hc.$$ 

We have $\mu_2^\omega(a) = aceegegheghhceghhchceeghhchce \cdots$.

One canonically associates the DFA $A_{\mu_2,a}$

$$L_{\mu_2,a} = \{ \varepsilon, 1, 10, 100, 101, 1000, 1001, 1010, 10000, \ldots \}$$

$$L_{\mu,a} := L_{\mu_1,a} = L_{\mu_2,a}$$
**Definition**

Let $\mu : \Sigma \to \Sigma^*$ be a morphism having the infinite word $x = x_0x_1x_2 \cdots$ as a fixed point. The *shape sequence of $x$ with respect to $\mu$* is

$$\text{Shape}_\mu(x) = (|\mu(x_k)|)_{k \geq 0}.$$ 

**Lemma**

Let $x, y$ be two infinite (unidimensional) words and $\lambda, \mu$ be two morphisms s.t. there exist letters $a, b$ s.t. $x = \lambda^\omega(a)$ and $y = \mu^\omega(b)$.

$$L_{\lambda,a} = L_{\mu,b} \iff \text{Shape}_\lambda(x) = \text{Shape}_\mu(y).$$
Consider $S = (L_{\mu}, a, \{0, 1\}, 0 < 1)$.

<table>
<thead>
<tr>
<th></th>
<th>$\omega$</th>
<th>$\varepsilon$</th>
<th>$1$</th>
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$\left( \mu(\text{val}_S(101), \text{val}_S(10)) \right)_{0, 1} = \text{val}_S(1010), \text{val}_S(101)$
Example (Continued)

\( \delta_\mu(\sigma, (i, j)) = (\mu(\sigma))(i, j) \)

\[
\begin{align*}
(\mu^\omega(a))_{m,n} &= \delta_\mu(a, (\text{rep}_S(m), \text{rep}_S(n))^0) \text{ for all } m, n \geq 0.
\end{align*}
\]