Introduction to Ergodic Theory of Numbers

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Goal

The aim of these lectures is to show how basic ideas in ergodic theory can be used to understand the structure and global behaviour of different number theoretic expansions.

A sampler of expansions

- (m-adic expansions) $x = \sum_{n=1}^{\infty} \frac{a_n}{m^n}$, $m \in \mathbb{N}$, $m \geq 2$, and $a_n \in \{0, 1 \cdots, m-1\}$.
- (β expansions) $x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}$, $\beta \in \mathbb{R}$, $\beta > 1$ and $a_n \in \{0, 1 \cdots, \lfloor \beta \rfloor\}$.
- (Lüroth series expansion)

$$x = \frac{1}{a_1} + \sum_{n=2}^{\infty} \frac{1}{a_1(a_1-1)\cdots a_{n-1}(a_{n-1}-1)a_n} + \cdots,$$

  here $a_k \in \mathbb{N}$, $a_k \geq 2$ for each $k \geq 1$.
- (Generalized Lüroth series expansion)

$$x = \frac{h_1}{s_1} + \frac{h_2}{s_1 s_2} + \cdots + \frac{h_k}{s_1 s_2 \cdots s_k} + \cdots,$$

  here h_i, s_i belong to a given set of non-negative real numbers.

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(Continued fraction expansion)

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

$a_i \in \mathbb{N}, \ a_i \geq 1.$

What is common?

They are all generated by iterating an appropriate map.

$x = \sum_{n=1}^{\infty} \frac{a_n}{m^n}$

- Generated by iterating $Tx = mx - \lfloor mx \rfloor$, which is defined on $[0, 1)$.

m-adic expansions

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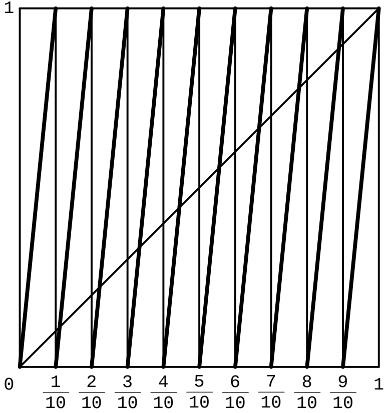
m-adic expansions



- Set $a_1(x) = \lfloor mx \rfloor$, and $a_n(x) = a_1(T^{n-1}x)$, where $T^n = T \circ T \circ \cdots \circ T$.
- $Tx = mx - a_1(x)$.
- Rewriting we get, $x = \dfrac{a_1(x)}{m} + \dfrac{Tx}{m}$
- After k-steps we get

$$x = \frac{a_1(x)}{m} + \frac{a_2(x)}{m^2} + \cdots + \frac{a_k(x)}{m^k} + \frac{T^k x}{m^k}$$

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- All points have unique m-adic expansion except for points of the form $\frac{k}{m^n}$, they have two expansions.
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β-expansions

- Expansions of the form $x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}$, $\beta \in \mathbb{R}$, where $\beta > 1$ and $a_n \in \{0, 1 \cdots, \lfloor \beta \rfloor\}$ are non-unique.
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Greedy expansions

- Introduced by Renyi in the late 50's.
- The greedy map/algorithm generate expansions of the form $x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}$ with the property that for each $n \geq 1$, a_n is the largest element of $\{0, 1, \cdots, \lfloor \beta \rfloor\}$ satisfying

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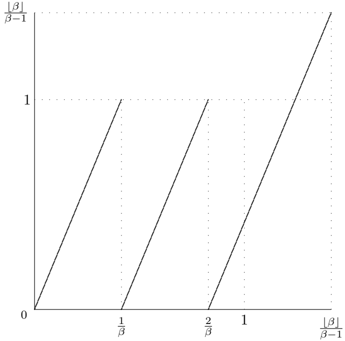
The greedy map

$\beta > 1$ non-integer. Define $T_\beta : [0, \lfloor\beta\rfloor/(\beta-1)) \to: [0, \lfloor\beta\rfloor/(\beta-1))$ by

$$T_\beta(x) = \begin{cases} \beta x \pmod 1, & 0 \le x < 1, \\ \\ \beta x - \lfloor\beta\rfloor, & 1 \le x < \lfloor\beta\rfloor/(\beta-1), \end{cases}$$

The greedy map

1



The *greedy map* T_β (here $\beta = \sqrt{2} + 1$).

Generating greedy expansions

- Define

$$a_1(x) = \begin{cases} \lfloor \beta x \rfloor & 0 \le x < 1, \\ \lfloor \beta \rfloor, & 1 \le x < \lfloor \beta \rfloor / (\beta - 1), \end{cases}$$

- Then $T_\beta x = \beta x - a_1(x)$.
- Rewriting, after k steps we get

$$x = \frac{a_1(x)}{\beta} + \frac{a_2(x)}{\beta^2} + \cdots + \frac{a_k(x)}{\beta^k} + \frac{T_\beta^k x}{\beta^k}.$$

- Taking limits, we get the greedy expansion.

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Lazy expansions

- Introduced by the Hungarian school in the early 90's.
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The lazy map

Consider the map $L_\beta : (0, \lfloor\beta\rfloor/(\beta-1)] \to (0, \lfloor\beta\rfloor/(\beta-1)]$ by

$$L_\beta(x) = \beta x - d, \qquad \text{for } x \in \Delta(d),$$

where

$$\Delta(0) = \left(0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}\right] \tag{1}$$

and

$$\Delta(d) = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d}{\beta}\right], \quad d \in \{1, 2, \ldots, \lfloor\beta\rfloor\}.$$

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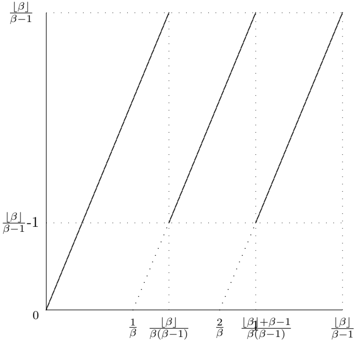

The *lazy map* S_β.

Generating lazy expansions

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Lüroth series expansions

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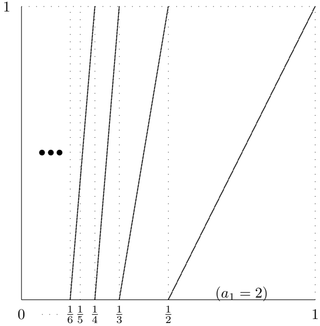
Let $T : [0,1) \to [0,1)$ be defined by

$$Tx = \begin{cases} n(n+1)x - n, & x \in [\frac{1}{n+1}, \frac{1}{n}), \\ 0, & x = 0. \end{cases}$$

The Lüroth map

1



The Lüroth Series map T.

- Let $a_1(x) = n$ if $x \in [\frac{1}{n}, \frac{1}{n-1})$, $n \geq 2$.
- set $a_m(x) = a_1(T^{m-1}x)$.
- $$Tx = \begin{cases} a_1(x)(a_1(x)-1)x - (a_1(x)-1), & x \neq 0, \\ 0, & x = 0. \end{cases}$$
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Generalized Lüroth series expansions(GLS-expansions)

- Let $\mathcal{I} = \{[\ell_n, r_n) : n \in \mathcal{D}\}$ be any finite or countable collection of intervals such that $\mathcal{D} \subset \mathbb{Z}^+$ and $\sum_{n\in\mathcal{D}}(r_n - \ell_n) = 1$.
- Let $I_n = [\ell_n, r_n)$.
- Define T on $[0, 1)$ by

$$Tx = \begin{cases} \dfrac{1}{r_n - \ell_n}x - \dfrac{\ell_n}{r_n - \ell_n}, & x \in I_n,\ n \in \mathcal{D}, \\ 0, & x \in I_\infty = [0,1) \setminus \bigcup_{n\in\mathcal{D}} I_n\,; \end{cases}$$

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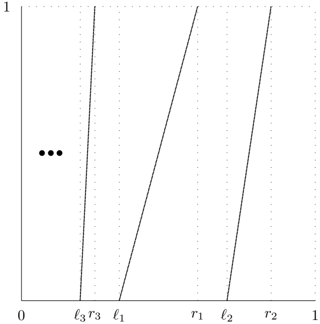

The GLS-map T.

Generalized Lüroth series expansions (GLS-expansions)

- let $s_1(x) = \dfrac{1}{r_n - \ell_n}$, and $h_1(x) = \dfrac{\ell_n}{r_n - \ell_n}$, $x \in I_n$.
- Then, $Tx = xs_1(x) - h_1(x)$.
- set $s_n(x) = s_1(T^{n-1}x)$ and $h_n(x) = h_1(T^{n-1}x)$.
- iterations of T and taking limits, lead to an expansion of the form

$$x = \frac{h_1}{s_1} + \frac{h_2}{s_1 s_2} + \cdots + \frac{h_k}{s_1 s_2 \cdots s_k} + \cdots$$

Generalized Lüroth series expansions (GLS-expansions)

- let $s_1(x) = \dfrac{1}{r_n - \ell_n}$, and $h_1(x) = \dfrac{\ell_n}{r_n - \ell_n}$, $x \in I_n$.
- Then, $Tx = xs_1(x) - h_1(x)$.
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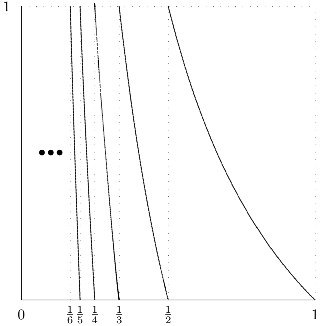
Continued fractions

Generated by the map $T : [0,1) \to [0,1)$ by $T0 = 0$ and for $x \neq 0$

$$Tx = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$

Continued fractions

1



The continued fraction map T.

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- Define $a_1(x) = \left\lfloor \frac{1}{x} \right\rfloor$, and $a_n(x) = a_1(T^{n-1}x)$.
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$$x = \frac{1}{a_1 + Tx} = \cdots = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + T^n x}}}.$$

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- Convergence will be shown later.

We now view our maps in the setup of Ergodic Theory in order to understand their metrical properties.

- Start with a probability space $(X, \mathcal{F}, \mu)$, and a measurable transformation $T : X \to X$.
- Assume T is measure preserving with respect to μ, i.e. $\mu(A) = \mu(T^{-1}A)$ or all $A \in \mathcal{F}$
- We call $(X, \mathcal{F}, \mu, T)$ a measure preserving system.

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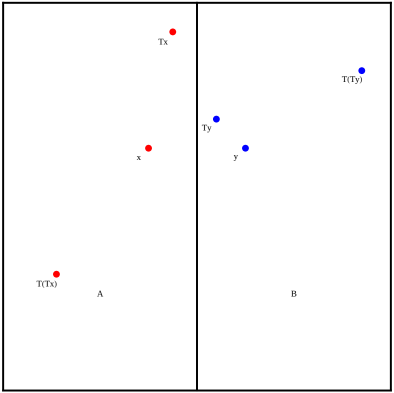
Ergodicity

Roughly speaking we call a map T on a set X ergodic if it is **impossible** to divide X into two pieces A and B (each with positive probability of occuring) such that T acts on each piece separately. Below is a picture of a non-ergodic map T.



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- Let $(X, \mathcal{F}, \mu, T)$ be a measure preserving system.
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Ergodicity

Theorem

Let $(X, \mathcal{F}, \mu)$ be a probability space and $T : X \to X$ measure preserving. The following are equivalent:

(i) *T is ergodic.*

(ii) *If $B \in \mathcal{F}$ with $\mu(T^{-1}B \Delta B) = 0$, then $\mu(B) = 0$ or 1.*

(iii) *If $A \in \mathcal{F}$ with $\mu(A) > 0$, then $\mu\left(\cup_{n=1}^{\infty} T^{-n}A\right) = 1$.*

(iv) *If $A, B \in \mathcal{F}$ with $\mu(A) > 0$ and $\mu(B) > 0$, then there exists $n > 0$ such that $\mu(T^{-n}A \cap B) > 0$.*

(v) *If $f \in L^2$ satisfies $f(x) = f(Tx)$ μ a.e., then f is a constant μ a.e.*

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$(i) \Rightarrow (ii)$:

- Suppose $\mu(T^{-1}B\Delta B) = 0$.
- By induction $\mu(T^{-k}B\Delta B) = 0$ for all $k \geq 1$.
- Let $A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} T^{-m}B$.
- Then, $T^{-1}A = A$, and by (i) $\mu(A) = 0$ or 1.
- $\mu(B\Delta A) \leq \sum_{k=1}^{\infty} \mu(T^{-k}B\Delta B) = 0$.
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- Suppose $A = T^{-1}A$.
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- By (v), $\mathbf{1}_A$ is a constant a.e.
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Theorem

(The Ergodic Theorem) *Let $(X, \mathcal{F}, \mu, T)$ be a measure preserving system. Then, for any f in $L^1(\mu)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = f^*(x)$$

exists a.e., is T-invariant and $\int_X f \, \mathrm{d}\mu = \int_X f^* \, \mathrm{d}\mu$. If moreover T is ergodic, then f^* is a constant a.e. and $f^* = \int_X f \, \mathrm{d}\mu$.

Consequences of the Ergodic Theorem

For example if $\mathbf{1}_A$ is the indicator function of a measurable set, then for μ a.e. x,

$$\lim_{n\to\infty}\frac{1}{n}\#\{0\le i\le n-1: T^i x\in A\}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{1}_A(T^i(x))=\mu(A).$$

Corollary

Let $(X, \mathcal{F}, \mu, T)$ be a measure preserving system. Then, T is ergodic if and only if for all $A, B \in \mathcal{F}$, one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(T^{-i}A \cap B) = \mu(A)\mu(B).$$

Consequences of the Ergodic Theorem

Proof: Suppose T is ergodic, and let $A, B \in \mathcal{F}$.

- By the Ergodic Theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{T^{-i}A\cap B}(x) = \mathbf{1}_B(x) \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_A(T^i x) = \mathbf{1}_B(x)\mu(A)$$

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Conversely, suppose $T^{-1}A = A$. By hypotheses,

- $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(T^{-i}A \cap A) = \mu(A)^2.$$
- By T-invariance of A,
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Consequences of the Ergodic Theorem

Theorem

Suppose μ_1 and μ_2 are probability measures on $(X, \mathcal{F})$, and $T : X \to X$ is measurable and measure preserving with respect to μ_1 and μ_2. Then,

(i) *if T is ergodic with respect to μ_1, and μ_2 is absolutely continuous with respect to μ_1, then $\mu_1 = \mu_2$,*

(ii) *if T is ergodic with respect to μ_1 and μ_2, then either $\mu_1 = \mu_2$ or μ_1 and μ_2 are singular with respect to each other.*

Consequences of the Ergodic Theorem

Proof (ii): Suppose T is ergodic with respect to μ_1 and μ_2, and assume $\mu_1 \neq \mu_2$.

- There exists $A \in \mathcal{F}$ such $\mu_1(A) \neq \mu_2(A)$.
- Define $C_j = \{x : \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbf{1}_A(T^i(x)) = \mu_j(A)\}$, $j = 1, 2$.
- $C_1 \cap C_2 = \emptyset$. By the Ergodic Theorem, $\mu_j(C_j) = 1$.
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Examples Revisited

For each of the examples in the sampler, we will show that there is a measure μ which is absolutely continuous with respect to Lebesgue measure, such that the underlying transformation is measure preserving and ergodic.

Useful tool

Lemma

(Knopp's Lemma) *If B is a Lebesgue set and $\mathcal{C}$ is a class of subintervals of $[0,1)$, satisfying*

(i) *every open subinterval of $[0,1)$ is at most a countable union of disjoint elements from $\mathcal{C}$,*

(ii) $\forall A \in \mathcal{C}\,,\ \lambda(A \cap B) \geq \gamma\lambda(A)$, *where $\gamma > 0$ is independent of A,*

then $\lambda(B) = 1$.

Proof of Knopp's Lemma

- Let $\varepsilon > 0$, there exists a set E_ε which is a finite disjoint union of open intervals such that $\lambda(B^c \triangle E_\varepsilon) < \varepsilon$.
- Then $\lambda(B^c) < \varepsilon + \lambda(B^c \cap E_\varepsilon)$.
- Now by conditions (i) and (ii) (that is, writing E_ε as a countable union of disjoint elements of $\mathcal{C}$) one gets that $\lambda(B \cap E_\varepsilon) \geq \gamma\lambda(E_\varepsilon)$.
- $$\varepsilon > \lambda(B^c \triangle E_\varepsilon) \geq \lambda(B \cap E_\varepsilon) \geq \gamma\lambda(E_\varepsilon) \geq \gamma\lambda(B^c \cap E_\varepsilon) > \gamma(\lambda(B^c) - \varepsilon),$$
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m-adic revisited

- $Tx = mx - \lfloor mx \rfloor$.
- T is measure preserving with respect to Lebesgue measure:

$$T^{-1}[a,b) = \bigcup_{i=0}^{m-1} \left[\frac{a+i}{m}, \frac{b+i}{m} \right),$$

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To prove ergodicity, we use Knopp's Lemma. First few facts:

- $T^n x = m^n x - \lfloor m^n x \rfloor$.
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$$Tx = \begin{cases} n(n+1)x - n, & x \in [\frac{1}{n+1}, \frac{1}{n}), \\ 0, & x = 0. \end{cases}$$

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- $$T_\beta(x) = \begin{cases} \beta x \pmod 1, & 0 \le x < 1, \\ \beta x - \lfloor \beta \rfloor, & 1 \le x < \lfloor \beta \rfloor / (\beta - 1), \end{cases}$$

- Independently, A.O. Gel'fond (in 1959) and W. Parry (in 1960) showed that T_β is measure preserving with respect to the measure $\mu_\beta = \int_A h_\beta d\lambda$ with

$$h_\beta(x) = \begin{cases} \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{1}{\beta^n} 1_{[0, T_\beta^n(1))}(x) & x \in [0,1) \\ 0 & x \in [1, [0, \frac{\lfloor \beta \rfloor}{1-\beta}), \end{cases}$$

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- A fundamental interval of order k:

$$\Delta(i_1, i_2, \ldots, i_k) = \{x : a_1(x) = i_1, a_2(x) = i_2, \ldots, a_k(x) = i_k\}.$$

- $\Delta(i_1, i_2, \ldots, i_k)$ is *full* if $T_\beta^k(\Delta(i_1, i_2, \ldots, i_k)) = [0, 1)$.
- T_β^k restricted to a full interval $\Delta(i_1, i_2, \ldots, i_k)$ is surjective of slope β^k. In this case $\lambda(\Delta(i_1, i_2, \ldots, i_k)) = \frac{1}{\beta^k}$.
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Lazy expansions revisited

$$L_\beta(x) = \beta x - d, \quad \text{for } x \in \Delta(d),$$

where

$$\Delta(0) = \left(0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}\right]$$

and

$$\Delta(d) = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d}{\beta}\right], \quad d \in \{1, 2, \ldots, \lfloor\beta\rfloor\}.$$

Lazy expansions revisited

- The map $\psi : [0, \lfloor\beta\rfloor/(\beta-1)) \to (0, \lfloor\beta\rfloor/(\beta-1)]$ defined by

$$\psi(x) = \frac{\lfloor\beta\rfloor}{\beta-1} - x,$$

then ψ is a continuous bijection.

- $\psi T_\beta = L_\beta \psi$.
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- All fundamental intervals $\Delta(i_1, i_2, \ldots, i_k)$ are full.
- Using properties of continued fractions and the equivalence of the Gauss measure to the Lebesgue measure, one can show

$$\mu(T^{-k}A \cap \Delta(i_1, i_2, \ldots, i_k)) \geq \frac{\log 2}{4}\mu(A)\mu(\Delta(i_1, i_2, \ldots, i_k))$$

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Natural Extension

- The goal is to find an invertible system associated with a given non-invertible system in such a way that all the dynamical properties of the original system are preserved.
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Definition of Natural Extension

An invertible measure-preserving system $(X, \mathcal{F}, \mu, T)$ is called a *natural extension* of the non-invertible measure preserving system $(Y, \mathcal{G}, \nu, S)$ if there exists a measurable surjective (a.e.) map $\psi : X \to Y$ such that

- (i) $\psi \circ T = S \circ \psi$,
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- (iii) $\vee_{m=0}^{\infty} T^m \psi^{-1} \mathcal{G} = \mathcal{F}$, where $\bigvee_{k=0}^{\infty} T^k \psi^{-1} \mathcal{G}$ is the smallest σ-algebra containing the σ-algebras $T^k \psi^{-1} \mathcal{G}$ for all $k \geq 0$.

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Suppose $(X, \mathcal{F}, \mu, T)$ is a natural extension of $(Y, \mathcal{G}, \nu, S)$.

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- Let $Tx = mx - \lfloor mx \rfloor$.
- A natural extension of the measure preserving system $([0,1), \mathcal{B}, \lambda, T)$ is the invertible system $([0,1) \times [0,1), \mathcal{B} \times \mathcal{B}, \lambda \times \lambda, \mathcal{T})$ where

$$\mathcal{T}(x,y) = \left(Tx, \frac{y + a_1(x)}{m}\right),$$

with $a_1(x)$ the *m*-adic digit of x.

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Natural Extension of m-adic expansions

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- So, $\bigvee_{m \geq 0} \mathcal{T}^m \pi^{-1} \mathcal{B} = \bigvee_{m \geq 0} \mathcal{T}^m(\mathcal{B} \times [0, 1)) = \mathcal{B} \times \mathcal{B}$.

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Natural Extension of greedy expansions, a special case

Let β be the positive root of the polynomial $x^m - x^{m-1} - \cdots - x - 1$.

- 1 has a greedy expansion $1 = \frac{1}{\beta} + \frac{1}{\beta^2} + \cdots + \frac{1}{\beta^m}$.
- $T_\beta x = \beta x \pmod 1$, T_β is restricted to $[0,1)$.
- Recall that T_β is measure preserving and ergodic with respect to the Parry measure μ_β with density

$$h_\beta(x) = \frac{1}{F(\beta)} \sum_{i=0}^{m-1} \mathbf{1}_{[0,T_\beta^i 1)},$$

where $F(\beta) = \int \sum_{i=0}^{m-1} \frac{1}{\beta^i} \mathbf{1}_{[0,T_\beta^i 1)} d\lambda$, and $T_\beta^i 1 = \frac{1}{\beta} + \frac{1}{\beta^2} + \cdots + \frac{1}{\beta^{m-i}}$.

Natural Extension of greedy expansions, a special case

Let β be the positive root of the polynomial $x^m - x^{m-1} - \cdots - x - 1$.

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- $X = \bigcup_{k=0}^{m-1} \left[T_\beta^{m-k}1,\ T_\beta^{m-k-1}1\right) \times \left[0,\ T_\beta^k 1\right)$
- Let $\mathcal{L}$ be the restriction of the two dimensional Lebesgue σ-algebra to X, and $\overline{\lambda}$ the normalized two dimensional Lebesgue measure.
- Consider the transformation $\mathcal{T}$ on X defined by

$$\mathcal{T}_\beta(x,y) := \left(T_\beta x,\ \frac{1}{\beta}(\lfloor \beta x \rfloor + y)\right).$$

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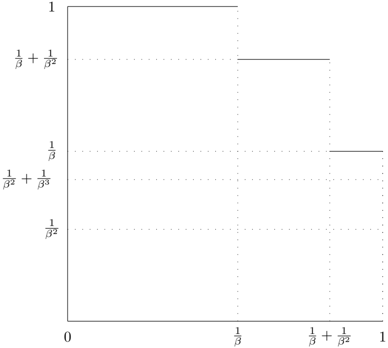

The natural extension of T_β if β is $m$bonacci number with $m = 3$.

Natural Extension of greedy expansions, a special case

- Let $\psi(x, y) = x$. Then $\psi \circ \mathcal{T} = T \circ \psi$ and $\mu_\beta = \overline{\lambda} \circ \psi^{-1}$.
- A similar proof as the one used for the m-adic shows that $\mathcal{T}$ is the natural extension of T_β with one small difference.
- $\mathcal{L}$ is generated by sets of the form

$$\Delta(k_1, \ldots, k_n) \times \Delta(l_1, \ldots, l_m),$$

where $\Delta(k_1, \ldots, k_n)$ and $\Delta(l_1, \ldots, l_m)$, are full fundamental intervals.

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Natural Extension of greedy expansions, the general case

- The map $\mathcal{T}_\beta(x,y) := \left(T_\beta x, \frac{1}{\beta}(\lfloor \beta x \rfloor + y)\right)$ corresponds to a two sided shift.
- Because not all blocks are allowed, it is not always possible to find a nice domain X in $\mathbb{R}^2$ on which $\mathcal{T}$ is bijective.
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- Let $R_0 = [0,1)^2$ and $R_i = [0, T_\beta^i 1) \times [0, \frac{1}{\beta^i})$, $i \geq 1$.
- Underlying space of the natural extension is the set

$$X = R_0 \times \{0\} \cup \bigcup_{n=1}^{\infty} R_i \times \{i\}.$$

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- Let $1 = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \cdots$ be the greedy expansion of 1.
- Define $\mathcal{T}_\beta$ on $R_i \times \{i\} by$

$$\mathcal{T}_\beta(x,y,i) = \begin{cases} (T_\beta x, y^*, 0), & \text{if } a_1(x) < b_{i+1} \\ (T_\beta x, y^*, i+1), & \text{if } a_1(x) = b_{i+1} \end{cases},$$

where, $a_1(x)$ is the greedy digit of x, and

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- Notice that if y has greedy expansion $y = \frac{c_{i+1}}{\beta^{i+1}} + \frac{c_{i+2}}{\beta^{i+2}} + \cdots$ $(y < \frac{1}{\beta^i})$. Then, the greedy digits of y^* are given by
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- Define $\psi(x, y, i) = x$, then $\mu_\beta = \overline{\lambda} \circ \psi^{-1}$, and $\psi \circ \mathcal{T} = T \circ \psi$.
- By working per rectangle R_i, and using a modification of the argument used for the special case (shifting by full blocks) one can show for any full fundamental intervals $\Delta(a_1, \ldots, a_n)$ and $\Delta(b_1, \ldots, b_m)$,

$$(\Delta(a_1, \ldots, a_n) \times \Delta(b_1, \ldots, b_m) \cap R_i) \times \{i\} \in \mathcal{T}^m \psi^{-1} \mathcal{B} \cap (R_i \times \{i\}).$$

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Natural Extension of continued fractions

$Tx = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ is measure preserving and ergodic with respect to the gauss measure $\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} d\lambda$.

Natural Extension of continued fractions

Theorem

(Ito, Nakada, Tanaka, 1977; Nakada, 1981) *Let* $\overline{\Omega} = [0,1) \times [0,1]$, $\overline{\mathcal{B}}$ *be the collection of Borel sets of* $\overline{\Omega}$. *Define the two-dimensional Gauss-measure* $\bar{\mu}$ *on* $(\overline{\Omega}, \bar{\mathcal{B}})$ *by*

$$\bar{\mu}(E) = \frac{1}{\log 2} \iint_E \frac{\mathrm{d}x\,\mathrm{d}y}{(1+xy)^2}, E \in \bar{\mathcal{B}}.$$

and the two-dimensional RCF-*operator* $\mathcal{T} : \overline{\Omega} \to \overline{\Omega}$ *for* $(x,y) \in \overline{\Omega}$ *be defined by*

$$\mathcal{T}(x,y) = \left(T(x), \frac{1}{\lfloor \frac{1}{x} \rfloor + y} \right), \; x \neq 0, \quad \mathcal{T}(0,y) = (0,y). \tag{2}$$

Then $(\overline{\Omega}, \bar{\mathcal{B}}, \bar{\mu}, \mathcal{T})$ *is the natural extension of* $([0,1), \mathcal{B}, \mu, T)$.

Classical Facts

- Let $x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$ be the continued fraction expansion of x.
- $\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}$.
- With some work, one has $x - \frac{p_n}{q_n} = \frac{(-1)^n T^n x}{q_n(q_n + q_{n-1} T^n x)}$.
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- Define $\Theta_n = q_n^2 \left| x - \frac{p_n}{q_n} \right| = \frac{T^n x}{1 + \frac{q_{n-1}}{q_n} T^n x}$.
- Using the recursion relation $q_i = a_i q_{i-1} + q_{i-2}$ repeatedly, one gets

$$\frac{q_{n-1}}{q_n} = \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + \ddots + \cfrac{1}{a_1}}},$$

the past of x.

- Given x, set $T_n = T^n x$ and $V_n = \frac{q_{n-1}}{q_n}$. Then $\mathcal{T}^n(x, 0) = (T_n, V_n)$, and

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Doeblin-Lenstra Conjecture

Theorem

For (Lebesgue) a.e. point x, and for any $0 \leq z \leq 1$,

$$\lim_{n\to\infty} \frac{1}{n}\#\{j\,;\,1 \leq j \leq n,\ \Theta_j(x) \leq z\} = F(z)$$

where

$$F(z) \begin{cases} \dfrac{z}{\log 2} & 0 \leq z \leq \frac{1}{2} \\[2ex] \dfrac{1}{\log 2}(1 - z + \log 2z) & \frac{1}{2} \leq z \leq 1. \end{cases}$$

Doeblin-Lenstra Conjecture

- Let $A_z = \{(x,y) \in \overline{\Omega} : \dfrac{x}{1+xy} \leq z\}$.
- $\overline{\mu}(A_z) = F(z)$.
- $\Theta_j(x) \leq z \Leftrightarrow \mathcal{T}^j(x,0) \in A_z$.
- $\dfrac{1}{n}\#\{j\,;\,1 \leq j \leq n,\, \Theta_j(x) \leq z\} = \dfrac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{A_z}(\mathcal{T}^j(x,0))$.
- Jager (1986) showed that for a.e. x, and for any Borel set C of $\overline{\Omega}$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_C(\mathcal{T}^j(x,0)) = \overline{\mu}(C).$$

- $\lim_{n\to\infty} \dfrac{1}{n}\#\{j\,;\,1 \leq j \leq n,\, \Theta_j(x) \leq z\} = F(z)$.

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- Let $A_z = \{(x,y) \in \overline{\Omega} : \dfrac{x}{1+xy} \leq z\}$.
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