On negative bases

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β-expansions

β-expansions are a particular class of representations in a non integer base $\beta > 1$ and alphabet $\{0, 1, \ldots, \lfloor\beta\rfloor\}$.

The β-expansion of a real number x, $\mathsf{d}_\beta(x)$, is computed by the greedy algorithm, based upon the iteration of the map $T_\beta(x) := \beta x - \lfloor\beta x\rfloor$.

The closure of the set of β-expansions is called β-shift.

Theorem (Parry)

The sequence $x_1 x_2 \cdots$ belongs to the β-shift if and only if for each $n \geqslant 1$

$$x_n x_{n+1} \cdots \leqslant_{lex} \mathsf{d}^*{}_\beta(1)$$

where

$$\mathsf{d}^*{}_\beta(1) := \begin{cases} (d_1 d_2 \cdots d_{n-1}(d_n - 1))^\omega & \text{if } \mathsf{d}_\beta(1) = d_1 d_2 \cdots d_{n-1} d_n \text{ is finite} \\ \mathsf{d}_\beta(1) & \text{otherwise} \end{cases}$$

Representation in negative base

We consider a negative value $-\beta$ with $\beta > 1$ and an alphabet with integer digits A.
A $(-\beta)$-representation of a real number x with alphabet A is a sequence (x_i) in $A^{\mathbb{N}}$ satisfying the equality

$$x = x_{-n}(-\beta)^n + x_{-n+1}(-\beta)^{n-1} + \cdots + x_1(-\beta) + x_0 + \frac{x_1}{-\beta} + \frac{x_2}{(-\beta)^2} + \cdots .$$

We also write

$$x = (x_{-n}x_{-n+1}\cdots x_{-1}x_0 \bullet x_1 x_2 \cdots)_{-\beta}.$$

Representability in base $-b$, $b > 1$ integer

Every real number admits a representation with base $-b$ and digits in $\{0, 1, \ldots, b-1\}$.
The representation of a real number is not necessarily unique.

Example. The greatest value representable in the form $\bullet x_1 x_2 \cdots$ admits two representations: $(\bullet(b-1)^\omega)_b = (1\bullet)_b$. This is also true in the case of $-b$ representations:

$$\frac{1}{(b+1)} = (\bullet(0(b-1))^\omega)_{-b} = (1\bullet((b-1)0)^\omega)_{-b}.$$

If x is an integer (positive or negative), then the representation is unique.

Properties of representations in base $-b$, $b > 1$ integer

Example. Representation of the integers base -2

1.	1	11.	-1
110.	2	10.	-2
111.	3	1101.	-3
100.	4	1100.	-4
101.	5	1111.	-5
11010.	6	1110.	-6
11011.	7	1001.	-7
11000.	8	1000.	-8

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Grünwald (1885) showed that:

- every number in $\mathbb{N}$ (resp. $-\mathbb{N}$) is representable with an odd (resp. even) number of digits;
- if $x = (w)_{-b} = (v)_b$ then $|w| \geqslant |v|$.

and he introduced the first algorithms for addition, multiplication, square root operation.

Ordering the $-b$-representations...

Definition. Two finite words of the same length satisfy:
$w_{-n} \cdots w_0 \prec v_{-n} \cdots v_0$ if and only if exists k such that

$$w_i = v_i \quad \text{for every } \; -n \leqslant i < k \quad \text{and} \quad (-1)^k (w_k - v_k) < 0.$$

Example. $(3)_{-2} = 111\bullet \prec 100\bullet = (4)_{-2}$: in fact the first digits in which the sequences differ, 1 and 0, are in an odd position.

In general, if $w, v \in \{0, \ldots, b-1\}^*$ and $|w| = |v|$:

$$w \prec v \Leftrightarrow (w\bullet)_{-b} < (v\bullet)_{-b}$$

Real negative bases

Ito and Sadahiro (2008) introduced an algorithm to represent any real number with real base $-\beta$, $\beta > 1$ and with digits in $A = \{0, 1, \ldots, \lfloor\beta\rfloor\}$.

- $-\beta$-transformation on $I_{-\beta} := \left[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}\right)$

$$T_{-\beta}(x) := -\beta x - \lfloor -\beta x + \frac{\beta}{\beta+1} \rfloor.$$

- $-\beta$-expansion of $x \in I_{-\beta}$: $\mathsf{d}_{-\beta}(x) = x_1 x_2 \cdots$ with

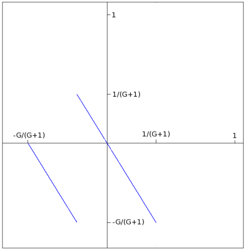
$$x_k := \lfloor -\beta T_{-\beta}^{k-1}(x) + \frac{\beta}{\beta+1} \rfloor.$$

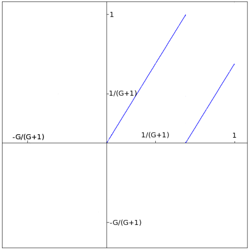
By shifting every real number has a $-\beta$-expansion.

Example: golden mean case

If $\beta = G := \frac{1+\sqrt{5}}{2}$ then

$$I_{-\beta} = \left[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}\right) = \left[-\frac{1}{\beta}, \frac{1}{\beta+1}\right)$$





$-\beta$-transformation on $I_{-\beta}$
$T_{-\beta}(x) = -\beta x - \lfloor -\beta x + \frac{\beta}{\beta+1} \rfloor$

Classical β-transformation on $[0, 1)$
$T_{\beta}(x) = \beta x - \lfloor \beta x \rfloor$

The $-\beta$-shift

The $-\beta$-shift is the closure of the $-\beta$-expansions.
Definition. $x_1x_2\cdots\prec y_1y_2\cdots$ if and only if there exists $k\geqslant 1$ such that:

$$x_i = y_i \text{ for } 1\leqslant i<k \quad\text{and}\quad (-1)^k(x_k-y_k)<0.$$

Property. Set $x,y\in I_\beta$.

$$\mathsf{d}_{-\beta}(x)\prec \mathsf{d}_{-\beta}(y) \quad\Leftrightarrow\quad x<y$$

Characterization of the $-\beta$-shift

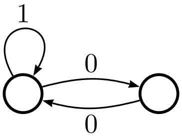
$$\mathsf{d}^*_{-\beta}\left(\frac{1}{\beta+1}\right) := \begin{cases} (0d_1\cdots d_{2n}(d_{2n+1}-1))^\omega & \text{if } \mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1}) = (d_1\cdots d_{2n+1})^\omega; \\ \mathsf{d}_{-\beta}(\frac{1}{\beta+1}) = 0d_1d_2\cdots; & \text{otherwise.} \end{cases}$$

Theorem (Ito-Sadahiro)

The sequence $x_1x_2\cdots$ belongs to the $(-\beta)$-shift if and only if for each $n \geqslant 1$

$$\mathsf{d}_{-\beta}\left(\frac{-\beta}{\beta+1}\right) \preceq x_n x_{n+1}\cdots \preceq \mathsf{d}^*_{-\beta}\left(\frac{1}{\beta+1}\right)$$

Example. $\mathsf{d}_{-G}(-\frac{G}{G+1}) = 10^\omega$ and $\mathsf{d}^*{}_{-G}(\frac{1}{G+1}) = 010^\omega$:



Some recalls on symbolic dynamical systems

$S \subseteq A^{\mathbb{N}}$ is a symbolic dynamical system if and only if

- S is shift-invariant;
- S is closed.

S is a sofic dynamical system if and only if the set of finite factors $F(S)$ is recognizable by a finite automaton.

S is of finite type if and only if

- S can be defined by the interdiction of a finite set of words.

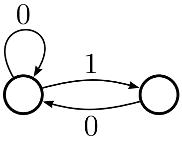
$$\Leftrightarrow$$

- S is recognized by a local finite automaton.

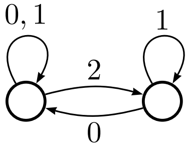
Example: classical β-shifts

The β-shift, i.e. the closure of the set of β-expansions, is a symbolic dynamical system.

The G-shift is of finite type: 11 is forbidden.



The G^2-shift is sofic but not of finite type: the finite automaton recognizing it is not local.

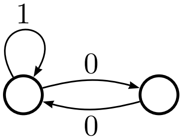


Characterization of sofic $-\beta$-shifts

Theorem (Ito and Sadahiro)

The $-\beta$-shift is sofic if and only if $\mathsf{d}_{-\beta}\left(-\frac{\beta}{\beta+1}\right)$ *is eventually periodic.*

Example. The $-G$-shift is sofic but not of finite type: the finite automaton recognizing it is not local.



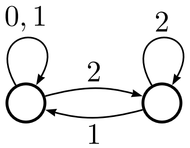
$$\mathsf{d}_{-G}\left(-\frac{G}{G+1}\right) = 10^{\omega}$$

Characterization of $-\beta$-shifts of finite type

Theorem

The $-\beta$-shift is of finite type if and only if $\mathsf{d}_{-\beta}\left(-\frac{\beta}{\beta+1}\right)$ is purely periodic

Example. The $-G^2$-shift is of finite type.



$\mathsf{d}_{-G^2}\left(-\frac{G^2}{G^2+1}\right) = (21)^\omega$ and the set of minimal forbidden words is $\{20\}$.

Entropy

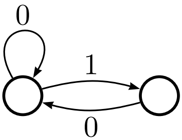
The entropy of a symbolic dynamical system S is

$$h(S) := \lim_{n\to\infty} \frac{1}{n} \log F_n(S) \tag{1}$$

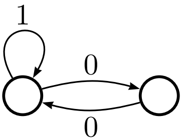
with $F_n(S) = \sharp$ factors of S of length n.

If S is sofic, $h(S)$ is equal to the logarithm of the greatest eigenvalue of the adjacency matrix of the automaton recognizing S.

Example. The G-shift and the $-G$-shift have the same entropy:



Automaton recognizing the G-shift



Automaton recognizing the $-G$-shift

Entropy of the $-\beta$-shift

The entropy of the classical β-shift is known to be $\log \beta$.

In our case, it follows from Fotiades and Boudourides (2001)

Proposition

The entropy of the $-\beta$-shift is $\log \beta$.

A particular class of bases: Pisot numbers

A Pisot number is a positive algebraic integer greater than 1 all of whose conjugate elements have absolute value less than 1.

Examples

- all integers are Pisot numbers;
- G is a Pisot number;
- all the positive zeros of the polynomial:

$$X^2 - aX - b \in \mathbb{Z}[X]$$

with $0 < b \leqslant a$ or $-a + 1 < b < 0$ are Pisot numbers.

The Pisot case

Theorem

If β is a Pisot number, then for every x in $\mathbb{Q}(\beta) \cap I_{-\beta}$ the sequence $\mathsf{d}_{-\beta}(x)$ is eventually periodic.

Corollary If β is Pisot the $-\beta$-shift is sofic.

Normalization with Pisot bases

The normalization on an alphabet $C \supset A$ is the partial function

$$\begin{array}{rrl} \nu_{-\beta,C}: & C^{\mathbb{N}} & \mapsto A^{\mathbb{N}} \\ & (c_1 c_2 \cdots) & \mapsto \mathsf{d}_{-\beta}(\sum_{i \geqslant 1} c_i(-\beta^{-i})), \end{array}$$

if $\sum_{i \geqslant 1} c_i(-\beta^{-i}) \in I_{-\beta}$

Proposition

If β is a Pisot number, for every $C \supset A$ the normalization is realizable by a finite transducer.

Addition with Pisot bases

Corollary

If x, y and $x + y \in I_{-\beta}$ the addition is realizable by a finite transducer.

In fact if $\mathsf{d}_{-\beta}(x) = x_1 x_2 \cdots$ and $\mathsf{d}_{-\beta}(y) = y_1 y_2 \cdots$ then:

$$z_i := x_i + y_i \in C := \{0, 1, \ldots, 2\lfloor\beta\rfloor\},$$

the normalization on the alphabet C yields:

$$\mathsf{d}_{-\beta}(x + y) = \nu_{-\beta,C}(z_1 z_2 \cdots)$$

Integer

- Unique representation a.e.
- Monotonicity w.r.t. $<_{lex}$

- Unique representation a.e.
- Monotonicity w.r.t $\prec$

General

- $I_\beta = [0,1)$ and $T_\beta = \beta x - \lfloor \beta x \rfloor$
- Monotonicity w.r.t. $<_{lex}$
- Characterization lays on $\mathbf{d}_\beta(1)$
- Entropy $= \log \beta$

- $I_{-\beta} = [-\frac{\beta}{\beta+1}, \frac{1}{\beta+1})$ and $T_{-\beta} = -\beta x - \lfloor -\beta x + \frac{\beta}{\beta+1} \rfloor$
- Monotonicity w.r.t. $\prec$
- Characterization lays on $\mathbf{d}_{-\beta}(\frac{-\beta}{\beta+1})$
- Entropy $= \log \beta$

Pisot

- β-shift is sofic
- Normalization and addition are rational

- $-\beta$-shift is sofic
- Normalization and addition are rational