Sets of integers recognized by countable automata

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Starting point

Fixed points of uniform morphisms over a countable alphabet A

Let σ be a substitution over A.

$$\forall a \in A, \sigma(a) = \sigma_0(a)\sigma_1(a)\ldots\sigma_{k-1}(a)$$

Condition to ensure the existence of an infinite fixed point :

$$\text{Some letter} a_0 \text{ satisfies } \sigma_0(a_0) = a_0.$$

The only restriction we make is : $\#\left(\sigma_i^{-1}(\{a\})\right) < \infty$ for all i and a.

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What about the supports of letters in this kind of words ?

Let w be the infinite fixed point of some subsitution σ and consider a finite set $F \subset A$.

- $I_F = \{n,\ w_n \in F\}$ can be generated by some countable automata with finite final set.
- If we define the projection $\Pi_F : A \to F \cup \{0\}$ and assume that the graph of the substitution have bounded degree, then $p_{\Pi_F(w)}$ is at most polinomial.

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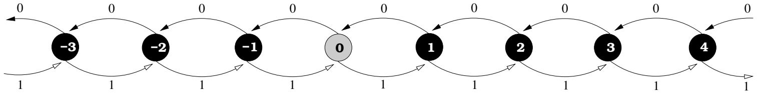

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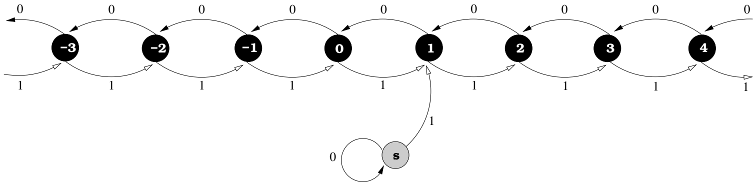

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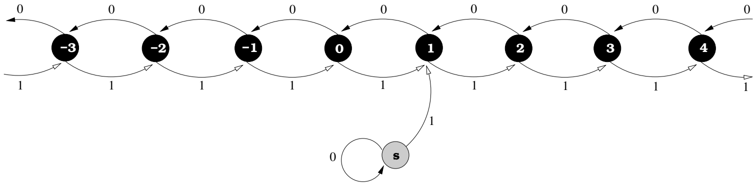

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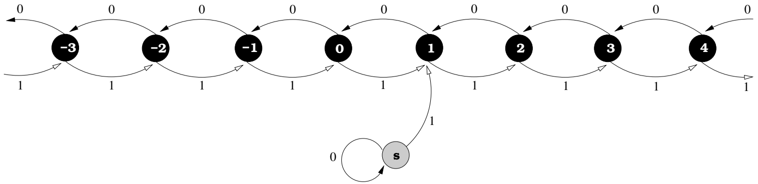

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$$m_n = a \iff n \in \{n \in \mathbb{N},\ |[n]_2|_0 = |[n]_2|_1\}$$

from finite to countable case...

A set of integers S is k^{∞}-recognizable if :

There exists a finite automata (E, Φ, e_0, F) :

- E is a finite or countable set,
- $\Phi : E \times [0, k-1] \to E$ extended to $E \times [0, k-1]^*$ satisfying, for all $e \in E$, $\left|\Phi^{-1}(\{e\})\right| < \infty$,
- $e_0 \in E$,
- $F \subset E$ is finite.

such that :

$$S = \{n \in \mathbb{N},\ \Phi([n]_k, e_0) \in F\}.$$

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- Finite case
  It does not really matter.
- Countable or finite case
  The two directions seem to capture really different properties...
  A priori : $\mathrm{Rec}^{\infty}_{\mathrm{M}\ell}(k) \neq \mathrm{Rec}^{\infty}_{\ell\mathrm{M}}(k)$.

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  Existence of prefix sequences,
- $\ell\mathrm{M}$ direction
  Existence of suffix sequences.

On the set of primes

- $\mathrm{M}\ell$ direction

$$\{\{\log_k p\},\ p \in \mathbb{P}\} \text{ is dense in } [0, 1],$$

- $\ell\mathrm{M}$ direction
  For any fixed prime $p > k$, there exists infinitely many $p' \in \mathbb{P}$ such that :

$$p' = p \mod k^{\lfloor \log_k p \rfloor + 1}.$$

Some general results

<table>
<tr><th>FA</th><th>FA or CA reading $M\ell$</th><th>FA or CA reading ℓM</th></tr>
<tr><td>$\mathrm{Rec}_{M\ell}(k) = \mathrm{Rec}_{\ell M}(k)$</td><td>$\mathrm{Rec}^{\infty}_{M\ell}(k) \setminus \mathrm{Rec}^{\infty}_{\ell M}(k) \neq \emptyset$</td><td>$\mathrm{Rec}^{\infty}_{\ell M}(k) \setminus \mathrm{Rec}^{\infty}_{M\ell}(k) \neq \emptyset$</td></tr>
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<tr><td colspan="3">Stability by intersection</td></tr>
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<tr><td>Stability by union</td><td colspan="2">Unstability by union</td></tr>
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What about Cobham's theorem ?

- finite case
  A set S is k- and ℓ-automatic, for some multiplicatively independent integers k and ℓ , if and only if

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  - Mℓ direction
    Open question...

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- Big problem : Non-symmetrical notion.

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- Cobham extension,
- Properties of numbers generated this wayS...
- Characterization of sets recognizable in both directions.

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  related to ets recognized by deterministic pushdown automata.