Combinatorial interpretation of Rosen continued fractions and generalizations

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CIRM - Numeration: Mathematics and Computer Science
March 23$^{\text{th}}$, 2009

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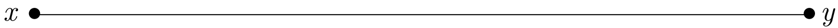

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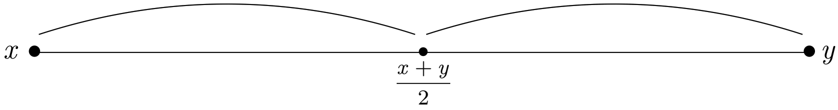

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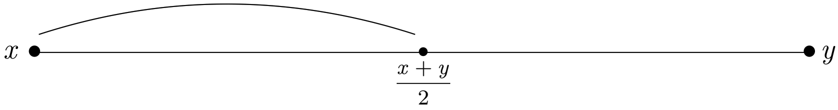

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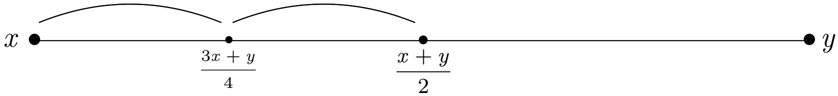

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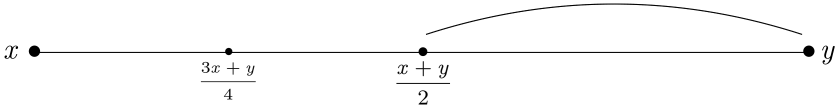

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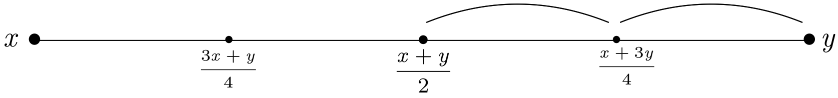

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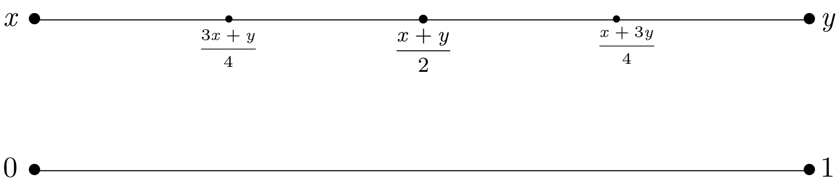

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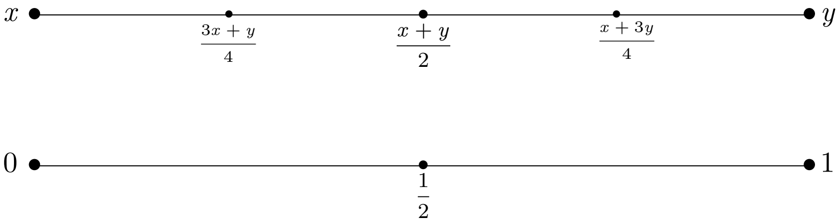

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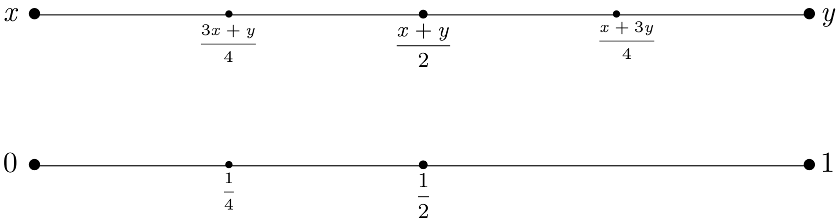

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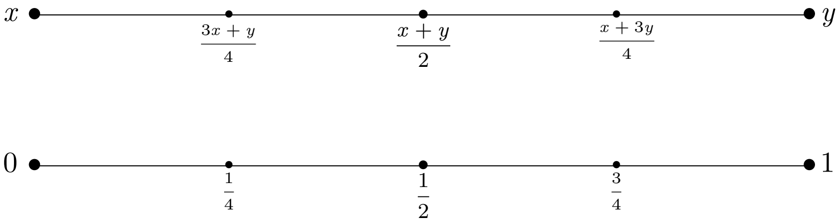

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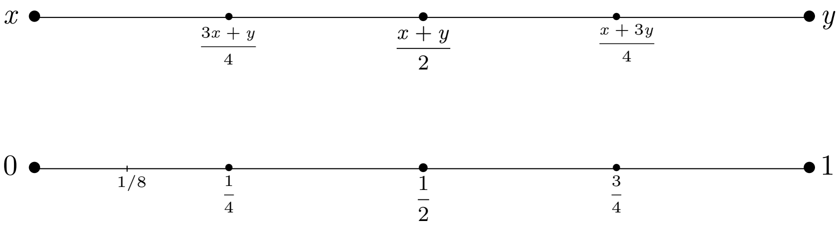

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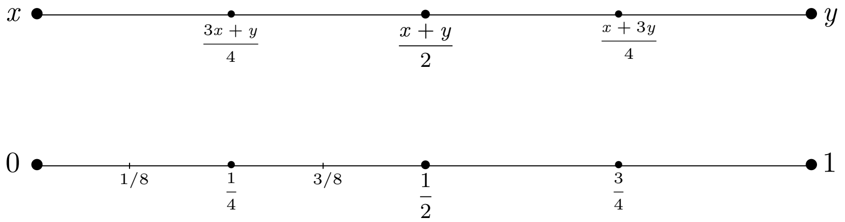

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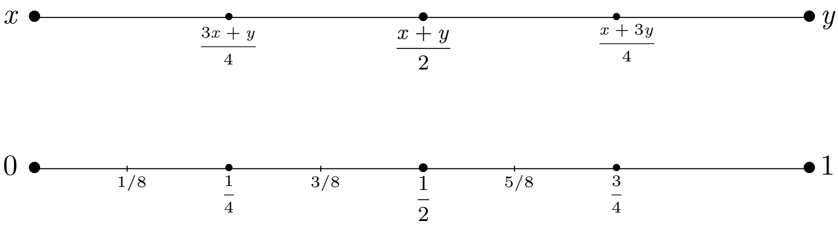

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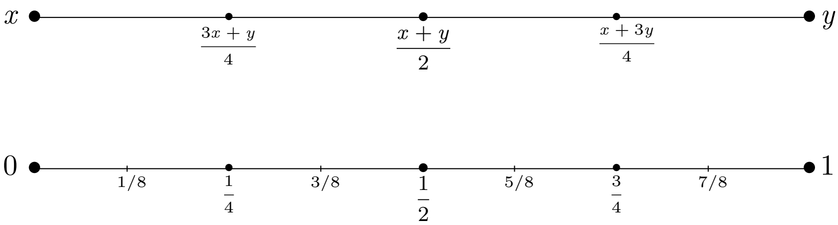

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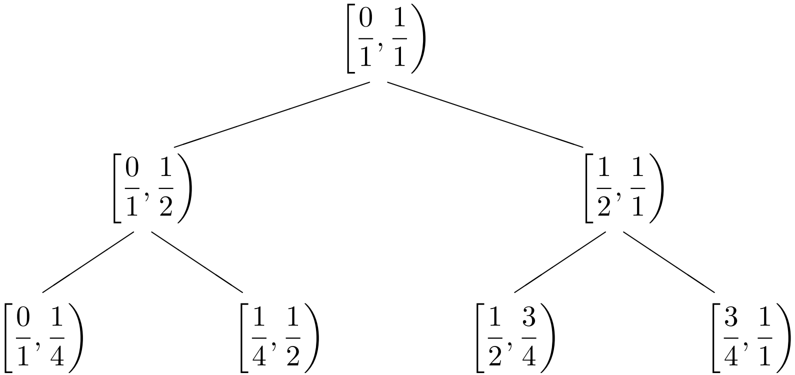

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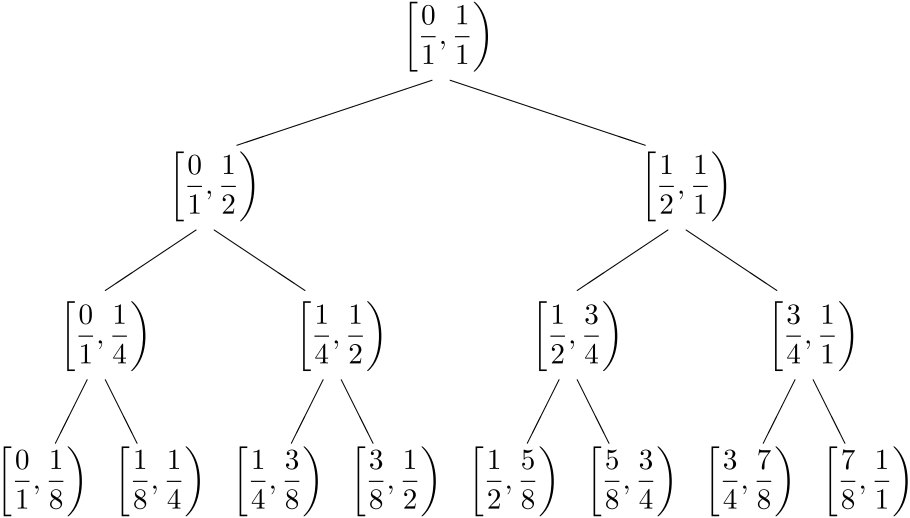

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Theorem

The codage of $x \in [0, 1)$ given by the dichotomy algorithm produces an analytical expression of x, with the following rule: writing $x_n = 0$ (resp. $x_n = 1$) if the left (resp. right) interval is chosen at the n-th step, we have

$$x = \sum_{n \geq 1} \frac{x_n}{2^n}.$$

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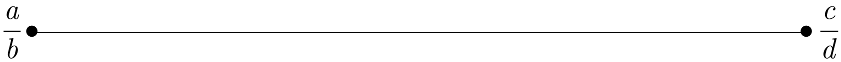

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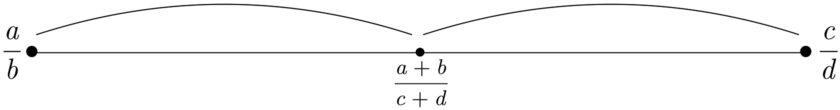

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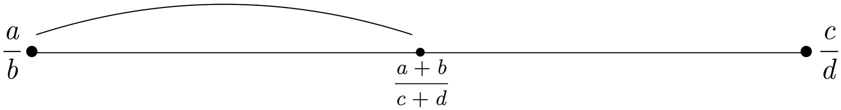

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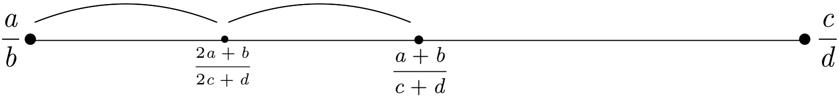

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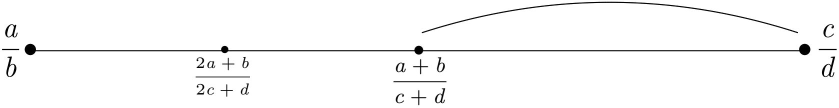

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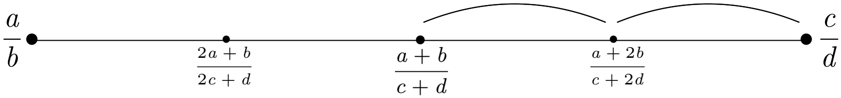

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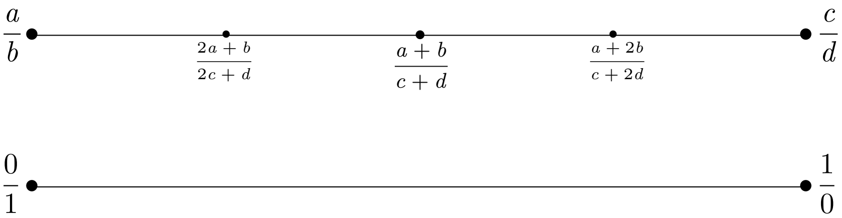

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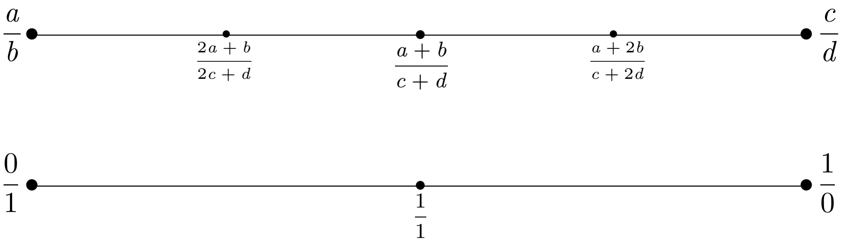

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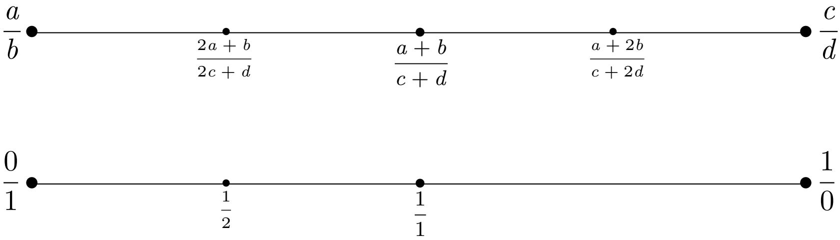

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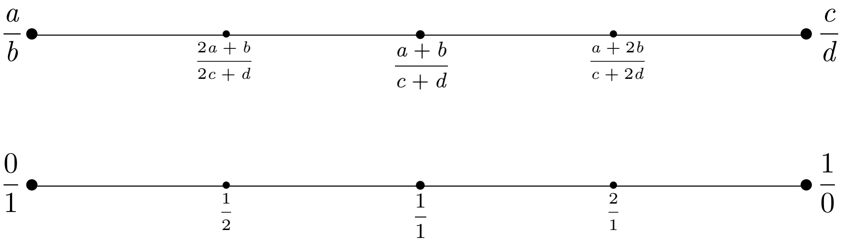

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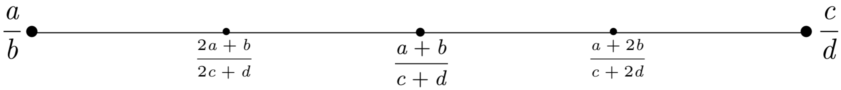

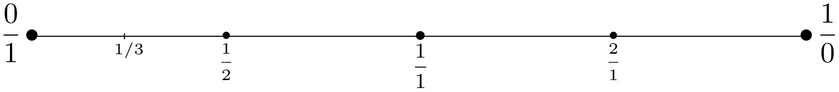

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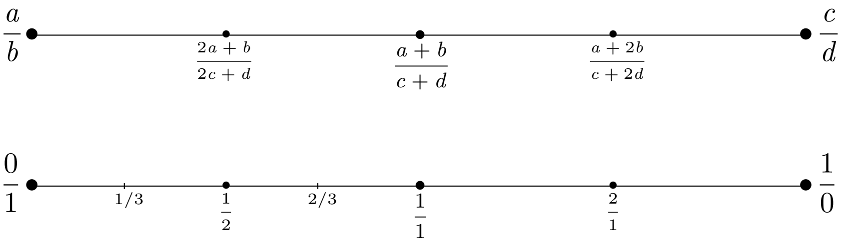

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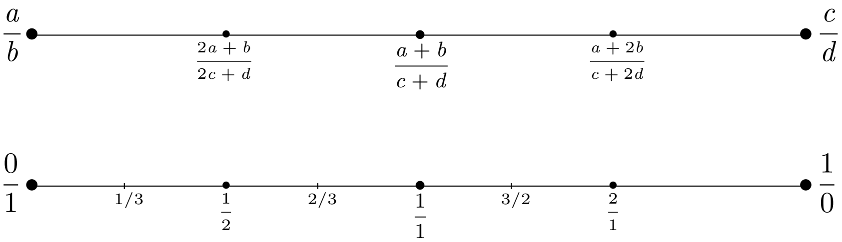

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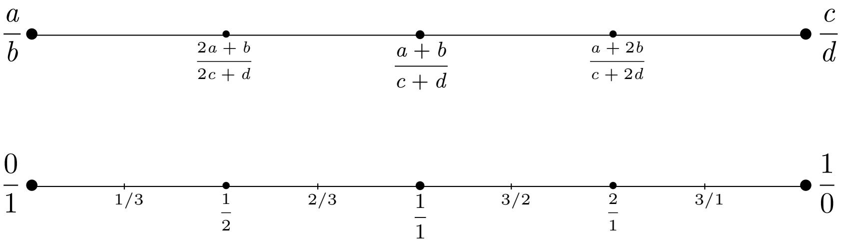

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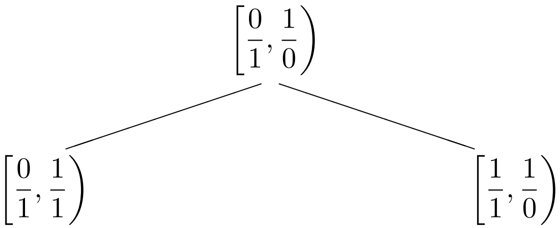

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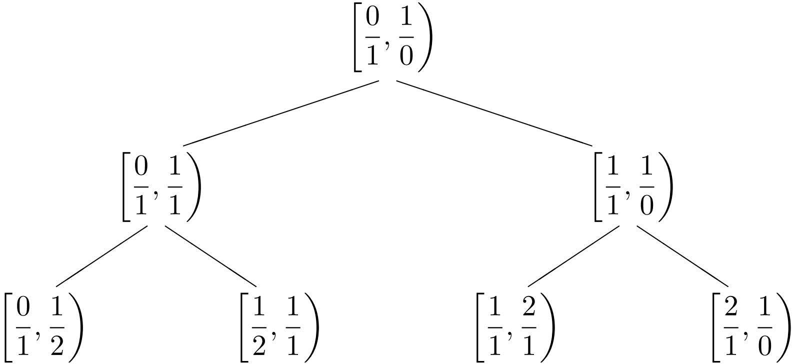

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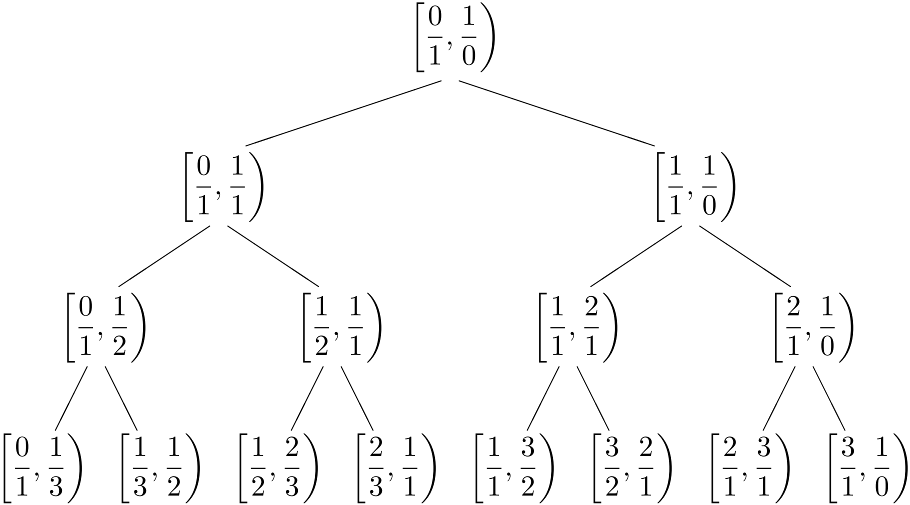

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Theorem

The codage of $x \in [0, 1)$ given by the Stern-Brocot algorithm produces an analytical expression of x with the following rule: denoting by $D^{a_0} G^{a_1} D^{a_2} G^{a_3} \ldots$ the codage of x, where $a_0 \geq 0$ and $a_n > 0$ for any $n > 0$, we have

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

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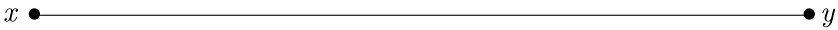

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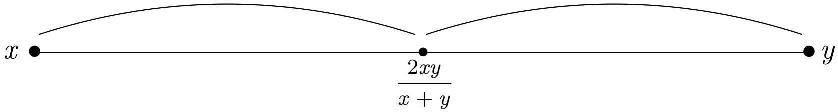

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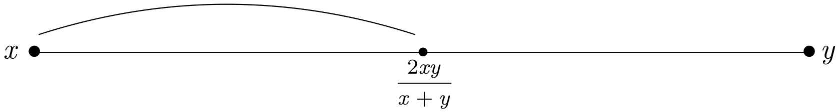

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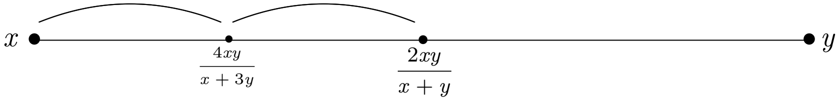

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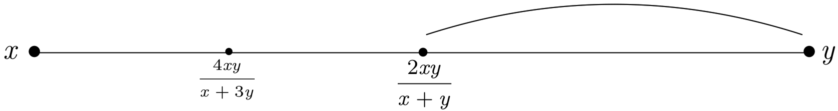

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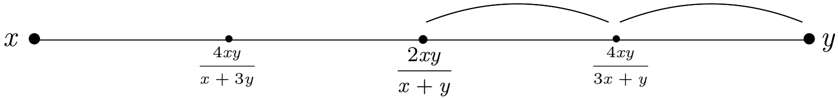

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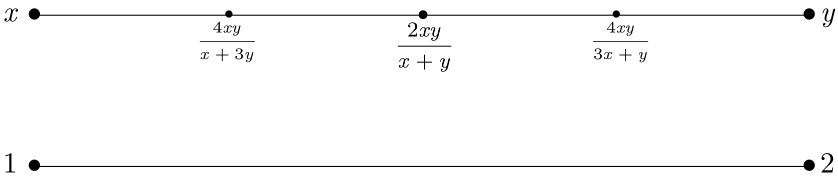

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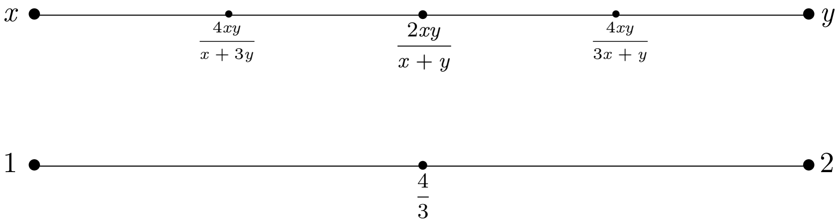

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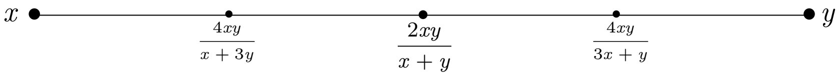

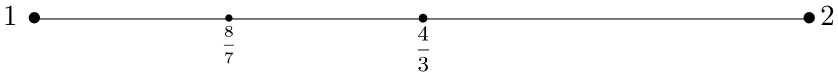

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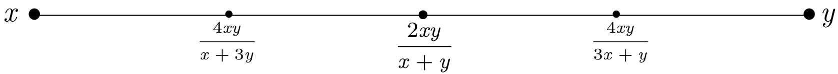

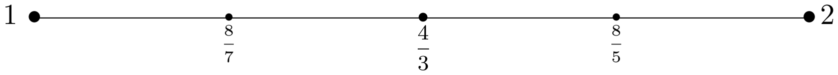

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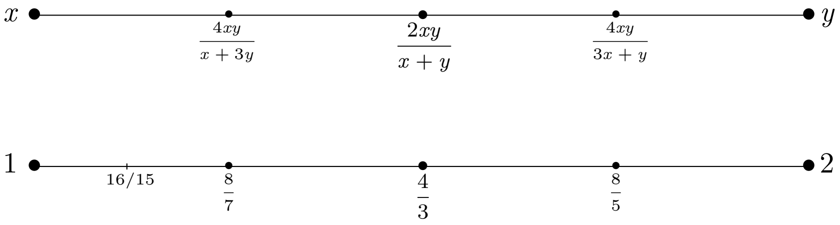

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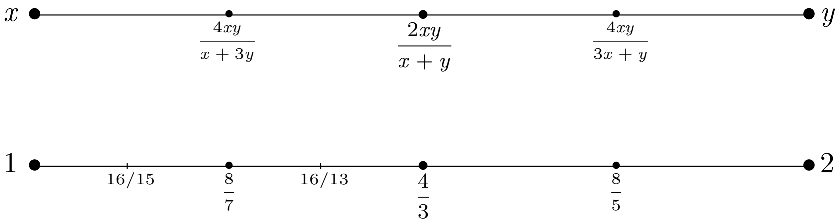

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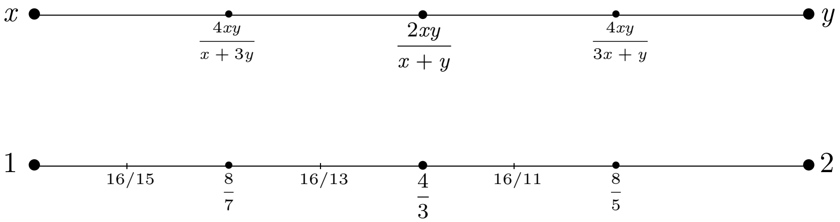

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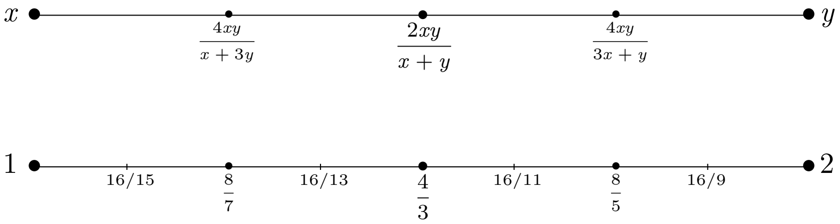

Numeration from the harmonic mean

Theorem

The square root of 2 is the only real number in $[1, 2)$ whose arithmetic and harmonic codages, $(a_n)_n$ and $(h_n)_n$, satisfy $a_n = 1 - h_n$.

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Numeration in base 3

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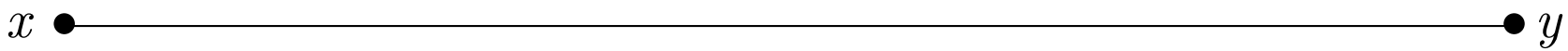

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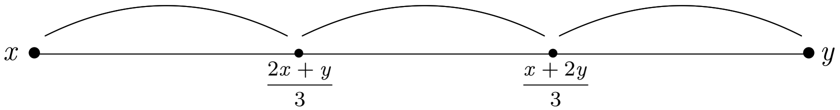

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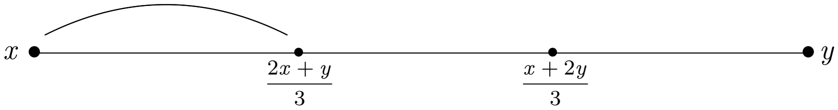

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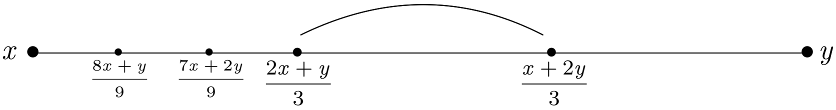

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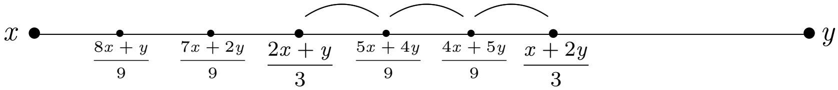

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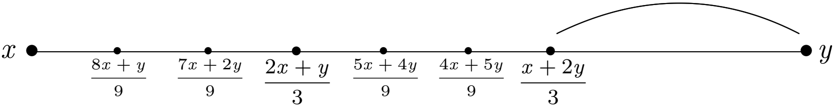

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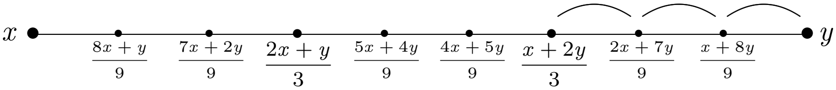

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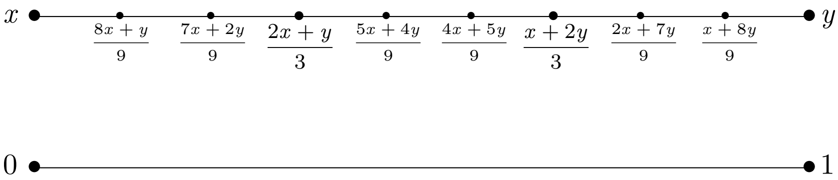

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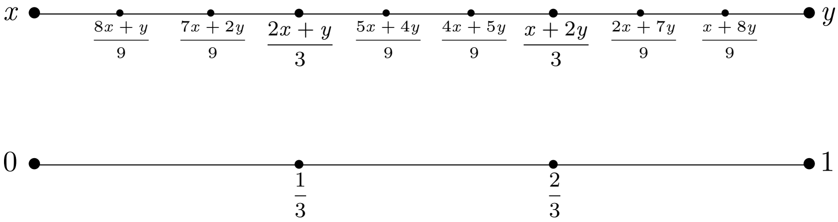

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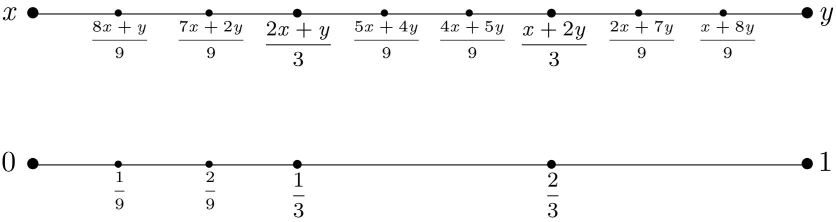

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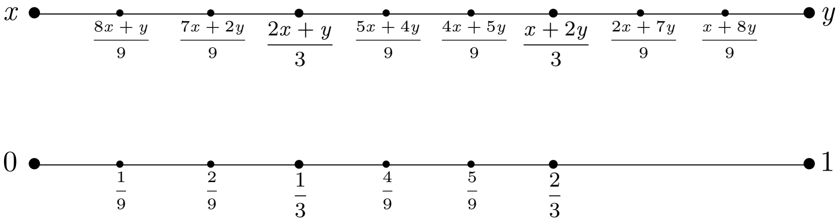

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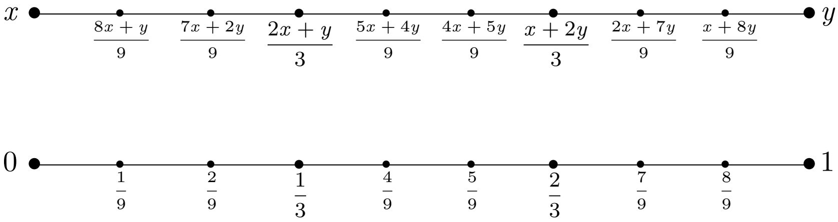

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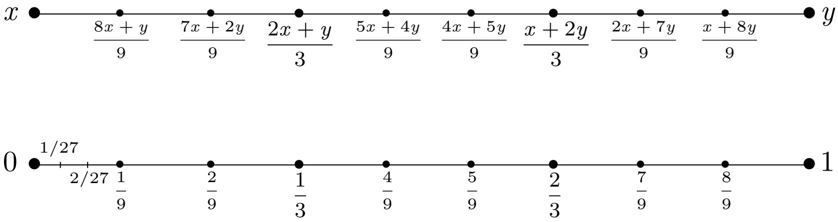

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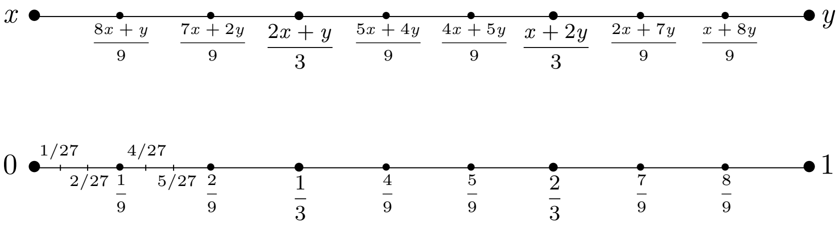

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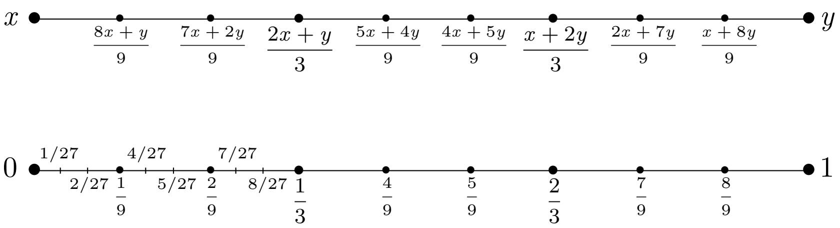

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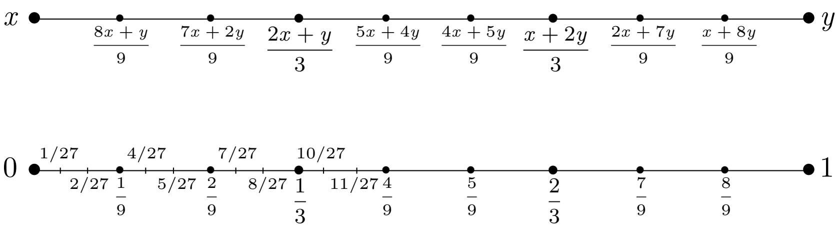

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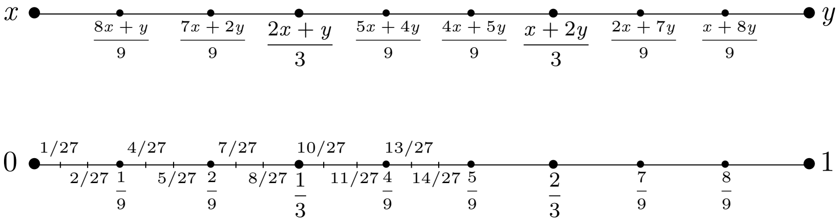

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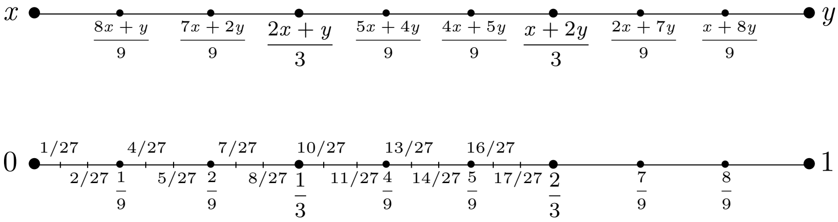

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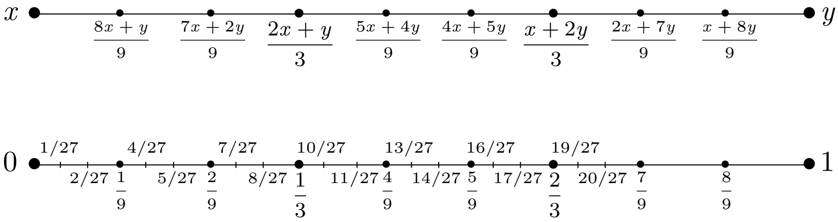

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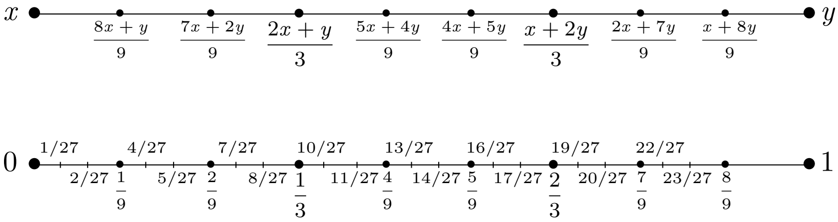

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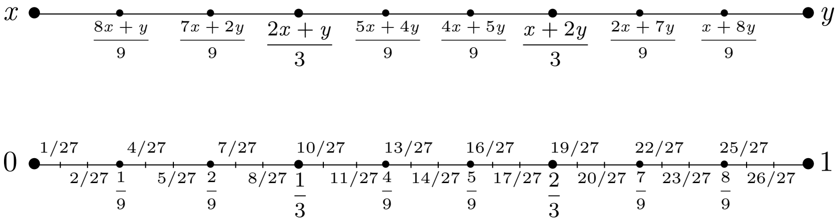

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$$\left[\frac{0}{1}, \frac{1}{1}\right)$$

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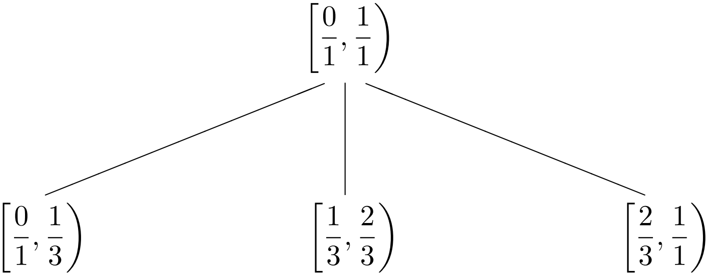

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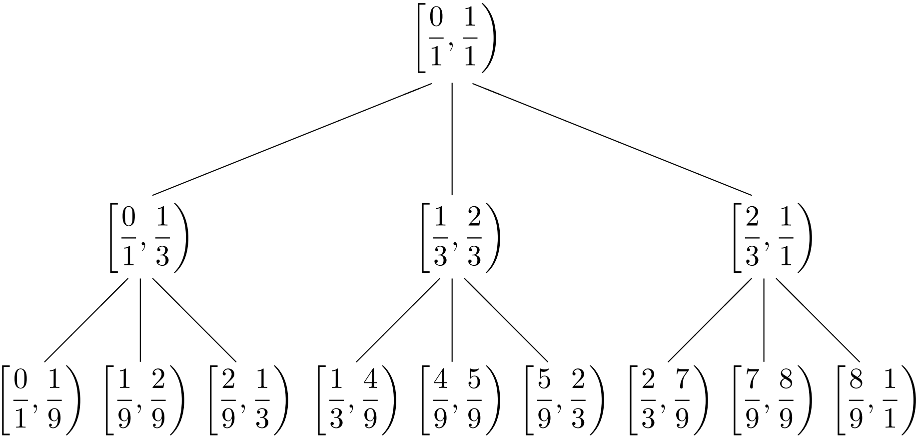

Numeration in base 3

Theorem

The codage of $x \in [0, 1)$ given by the base 3-algorithm produces an analytical expression of x, with the following rule: writing $x_n = 0$ (resp. $x_n = 1$, $x_n = 2$) if the left (resp. center, right) interval is chosen at the n-th step, we have

$$x = \sum_{n \geq 1} \frac{x_n}{3^n}.$$

“Continued fractions with 3 letters”

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Cutting rule: $\dfrac{a}{b}, \dfrac{c}{d} \longrightarrow \dfrac{s}{t}$ $\underline{\textit{and}}$ $\dfrac{u}{v}$

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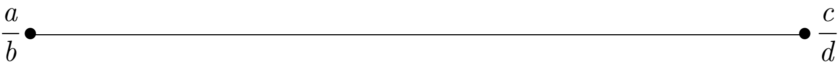

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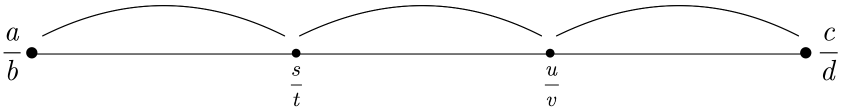

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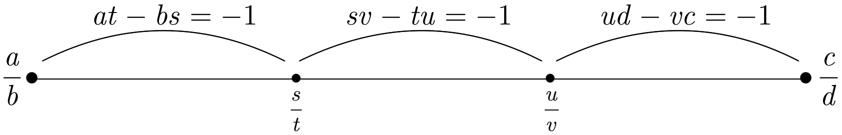

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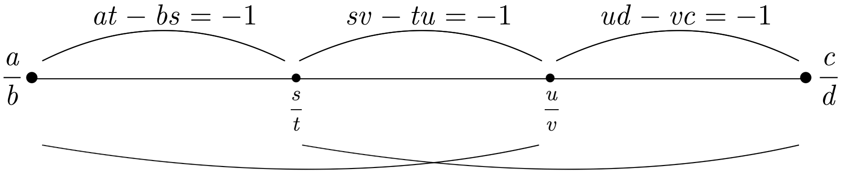

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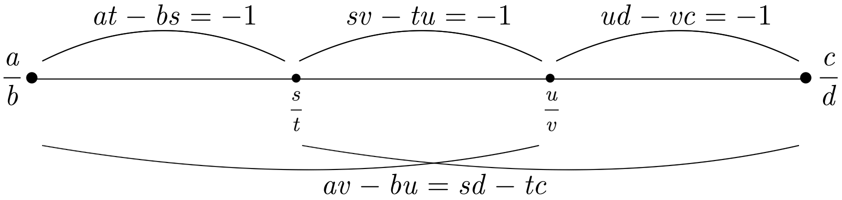

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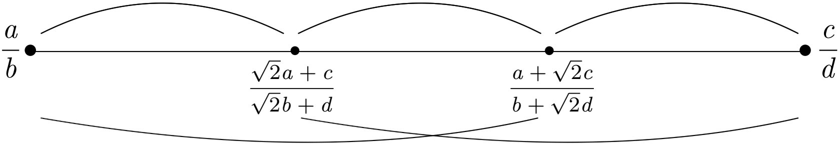
Cutting rule: $$\frac{a}{b}, \frac{c}{d} \longrightarrow \frac{\sqrt{2}a + c}{\sqrt{2}b + d} \quad \underline{\textit{and}} \quad \frac{a + \sqrt{2}c}{b + \sqrt{2}d}$$



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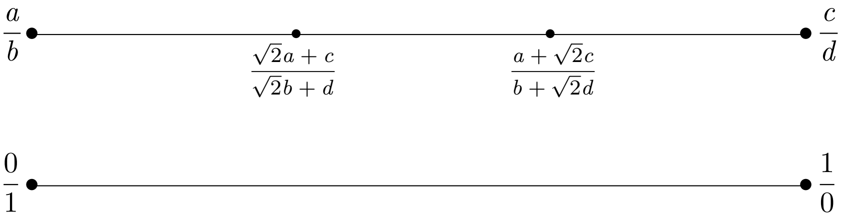

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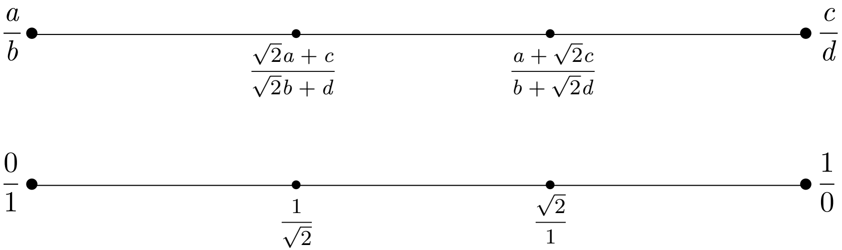

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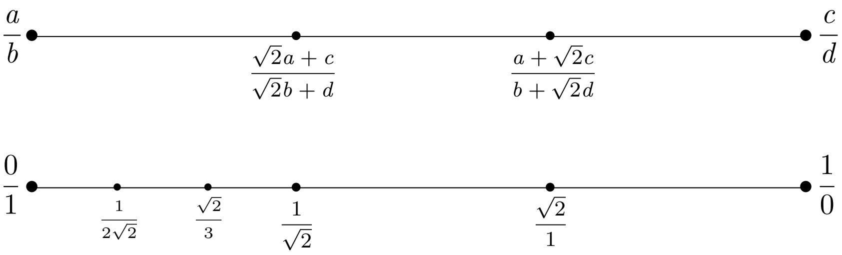

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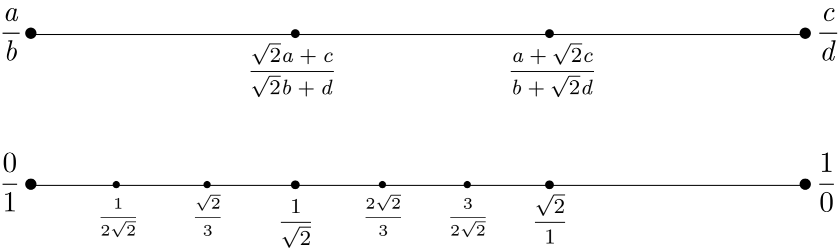

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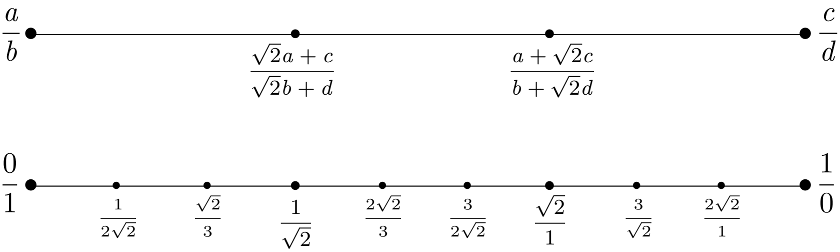

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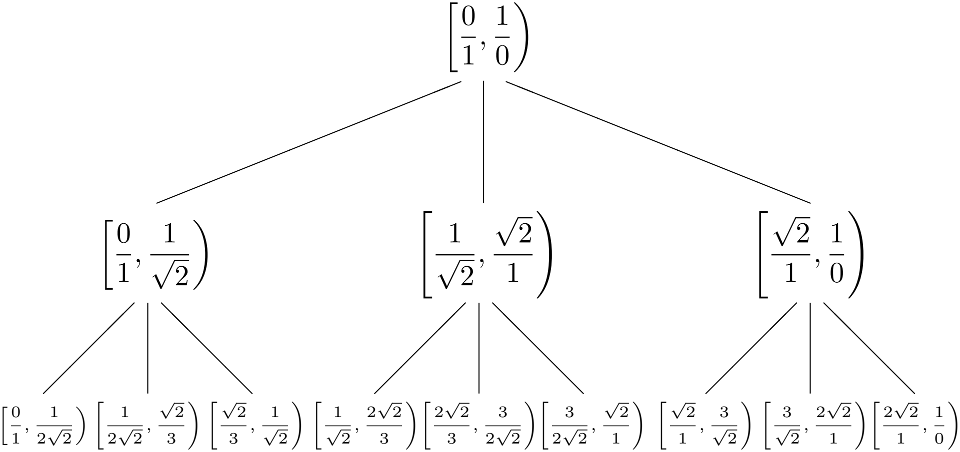

“Continued fractions with 3 letters”

Theorem

The codage of $x \in [0,1)$ given by the “3-Stern-Brocot” algorithm produces an analytical expression of x in $\sqrt{2}$-continued fraction:

$$x = a_0\sqrt{2} + \cfrac{1}{a_1\sqrt{2} + \cfrac{1}{a_2\sqrt{2} + \cfrac{1}{a_3\sqrt{2} + \cdots}}}.$$

Rosen continued fractions

Rosen continued fractions

Definition

Let $k \geq 3$ be an integer and $\lambda := 2\cos(\pi/k)$. A λ-*continued fraction* of $x \in \mathbb{R}$ is an expression of the form

$$x = a_0\lambda + \cfrac{1}{a_1\lambda + \cfrac{1}{a_2\lambda + \cdots}} =: [a_0, a_1, a_2, \ldots]_\lambda,$$

where the a_ns are integers (positive or not), all different from 0 except, possibly, a_0.

Rosen continued fractions

Theorem

In the interval $[0, 2)$, values of the form $\lambda = 2\cos(\pi/k)$ are the only ones for which the subgroup of homographies of $\mathbb{H}^2$ generated by

$$z \longmapsto z + \lambda \quad \text{and} \quad z \longmapsto -\frac{1}{z}$$

is discrete.

Symbolic version of Rosen continued fractions

Definition

Let $k \geq 3$, $\lambda := 2\cos(\pi/k)$, and

$$\begin{aligned}
\lambda_0 &:= \lambda = [1]_\lambda, \\
\lambda_1 &:= [1, -1]_\lambda, \\
\lambda_2 &:= [1, -1, 1]_\lambda, \\
&\vdots \\
\lambda_{k-2} &:= [1, -1, 1, \ldots, (-1)^{k-3}]_\lambda.
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The *mediants* of $\dfrac{a}{b}$ and $\dfrac{c}{d}$ (with $ad - bc = -1$) are

$$\frac{\alpha_0(a+\lambda_0 c)}{\alpha_0(b+\lambda_0 d)}, \quad \frac{\alpha_1(a+\lambda_1 c)}{\alpha_1(b+\lambda_1 d)}, \quad \frac{\alpha_2(a+\lambda_2 c)}{\alpha_2(b+\lambda_2 d)}, \quad \dots, \quad \frac{\alpha_{k-2}(a+\lambda_{k-2} c)}{\alpha_{k-2}(b+\lambda_{k-2} d)}.$$

Symbolic version of Rosen continued fractions

Example

For $k = 5$ ($\lambda = \varphi$).

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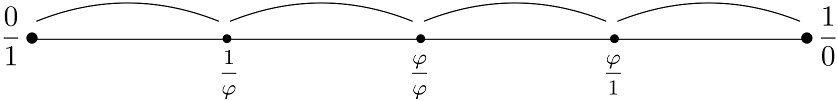

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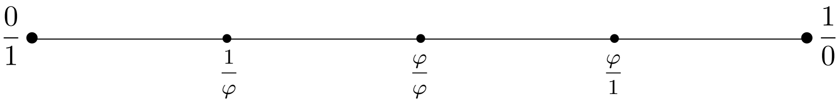

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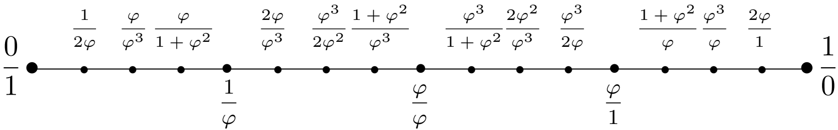

Symbolic version of Rosen continued fractions

Theorem

There exists a correspondence between the codage with $(k-1)$ letters given by the “Rosen-Stern-Brocot” algorithm and the $2\cos(\pi/k)$-continued fraction expansion.

Beyond Rosen: $\lambda > 2$

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The set C_λ of λ-expandable x is of null measure.

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Let p_n/p_{n-1} be the convergents to λ_k. We have

Beyond Rosen: $\lambda > 2$

The set C_λ of λ-expandable x is of null measure.

Remarkable values for λ: $\lambda = \lambda_k := [\overline{k}]$.

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Some examples:

$$\begin{aligned} C_{\varphi_2} \cap \mathbb{N} &= \{0, 2, 10, 12, 14, 24, 34, 36, 46, 58, 60, 70, \dots\}, \\ C_{\varphi_3} \cap \mathbb{N} &= \{3, 30, 33, 36, 327, 330, 360, 363, 393, 882, \dots\}, \\ C_{\varphi_4} \cap \mathbb{N} &= \{4, 68, 72, 76, 144, 216, 432, 644, 648, 860, \dots\}. \end{aligned}$$

Beyond Rosen: $\lambda < 2$

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Euclidean base	Hyperbolic base
2	$1 = 2\cos(\pi/3)$
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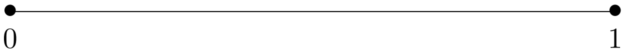
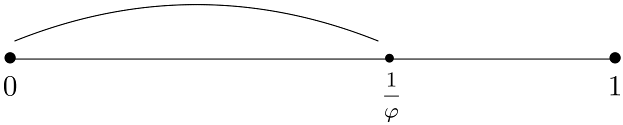
4	$\varphi = 2\cos(\pi/5)$
5	$\sqrt{3} = 2\cos(\pi/6)$
$\vdots$	$\vdots$

In base golden ratio



In base golden ratio



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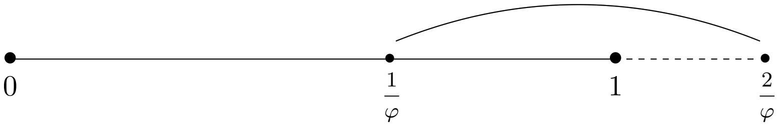

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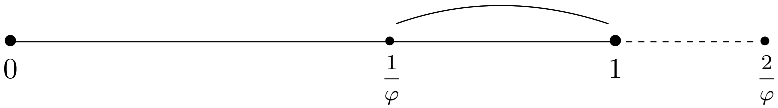

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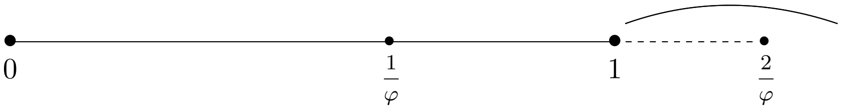

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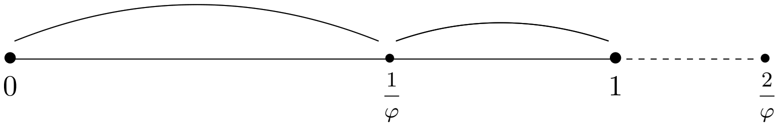

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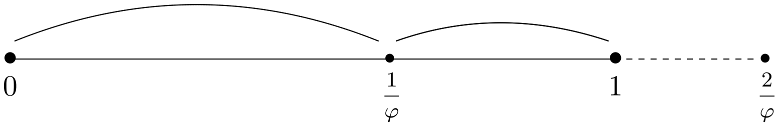
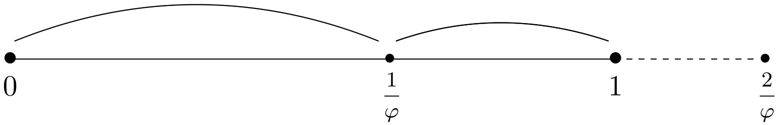


The codage of a real number x by a sequence $(x_n)_n$ of 0s (when we go to the left) and 1s (when we go to the right) satisfies

In base golden ratio



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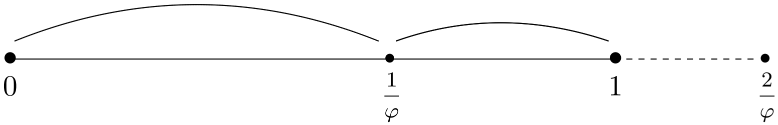
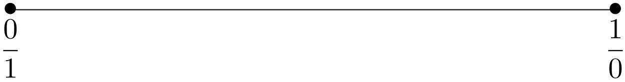


The codage of a real number x by a sequence $(x_n)_n$ of 0s (when we go to the left) and 1s (when we go to the right) satisfies

- we never get the sequence 11 ;
- $x = \sum_{n \geq 1} \frac{x_n}{\varphi^n}$.

“Continued fractions without 11”



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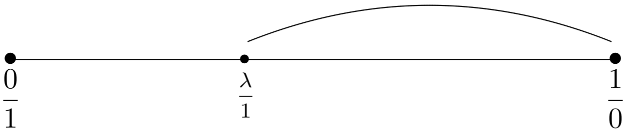

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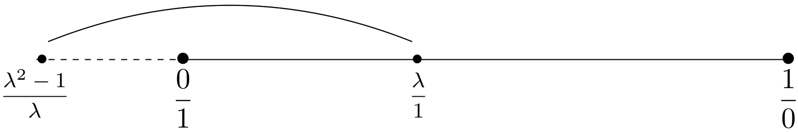

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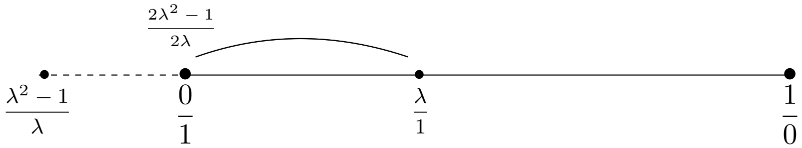

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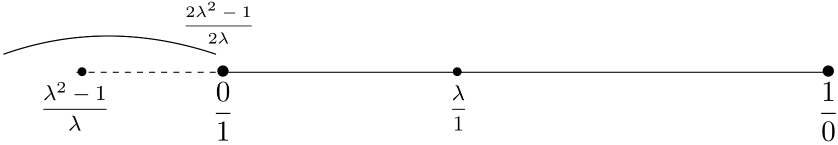

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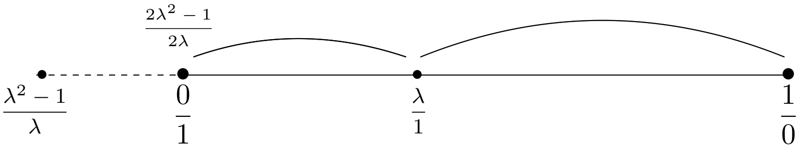

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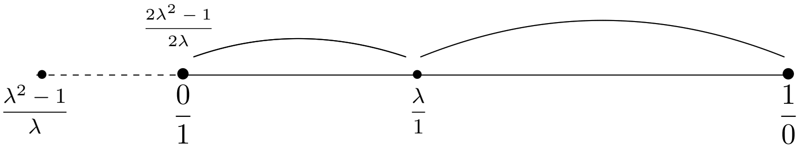

To get a good fit, we must have

$$\frac{2\lambda^2 - 1}{2\lambda} = \frac{0}{1}$$

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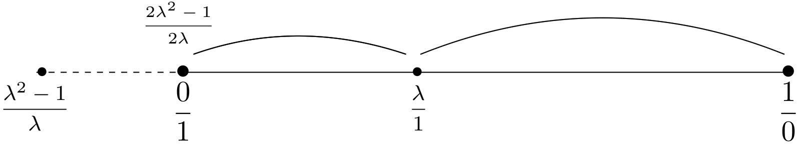

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i.e. $\lambda = 1/\sqrt{2}$.

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