The tilings of Kari and Culik

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Outline

Wang tiles

KC tiles

Proof of aperiodicity

Proof of existence

How do these tilings work?

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Proof of existence

How do these tilings work?

Wang tiles

- Set $\mathcal{W}$ of 2-dimensional square dominos.
  - with “colored" (or numbered) edges.
- In a *valid* tiling, colors of adjacent edges must match.
- Essentialy a 2-dimensional SFT,
  - (any 2-d SFT can be coded in terms of Wang tiles by using higher block code).

Example: 2-d Fibonacci set $\mathcal{W}$

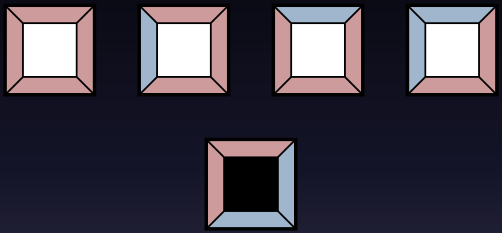


Figure: A Wang tile set $\mathcal{W}$ with two edge color (pink and blue) that enforce a rule on center colors (black and white): in a valid tiling two black tiles cannot be adjacent.

Fibonacci Wang tiling

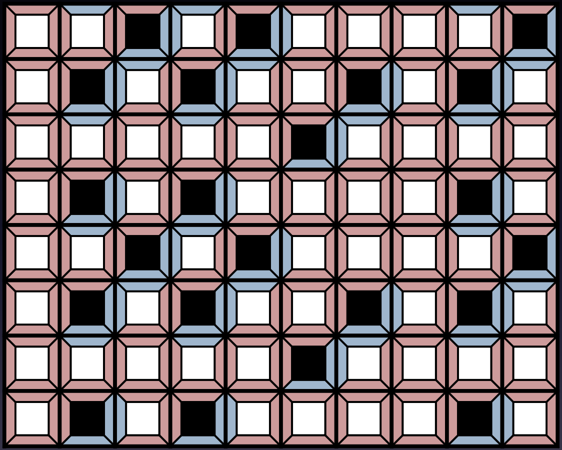
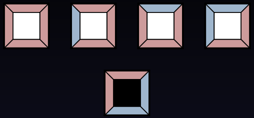
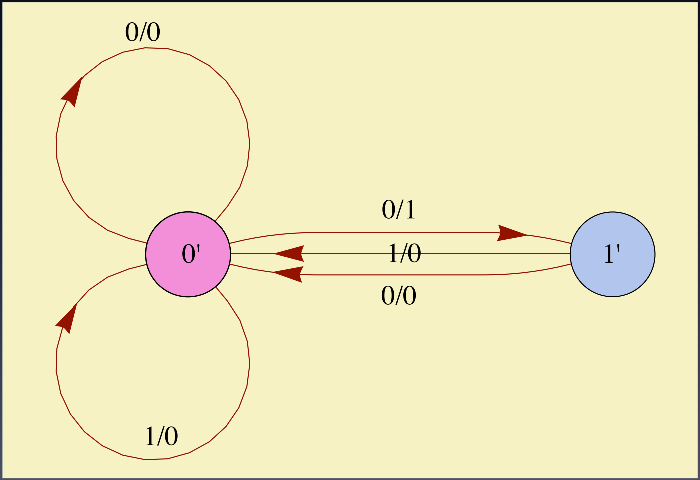


Figure: Patch of 2-d Fibonacci tiling. If edge colors are erased then tiling by black and white tiles is 2-d Fibonacci SFT.

Finite state machine





Hao Wang, 1961

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Theorem (Wang's Theorem)

If for every $r, s \in \mathbb{N}$, a Wang tile set $\mathcal{W}$ admits a valid tiling of an $r \times s$ rectangle, then $\mathcal{W}$ admits a valid tiling of the plane.

- Essentially a compactness theorem. Equivalent to König's lemma.

Wang's Conjecture

Conjecture (Wang's Conjecture)

Every valid Wang tile set $\mathcal{W}$ admits a valid periodic *tiling of the plane.*

- Equivalently: every nonempty 2-dimensional SFT has a periodic orbit.
  - (Wang *did not* use the language of "SFT".)
- This conjecture *is* true for 1-dimensional SFT,
  - (and easy).

Tiling Theorem

"Theorem"
Assuming Wang's conjecture is true, given a set $\mathcal{W}$ *of Wang tiles, there is an* algorithm *to determine whether or not* $\mathcal{W}$ *is valid.*

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- i.e., Wang conjectured that the question of whether or not $\mathcal{W}$ is valid is *decidable*.

Proof

- **Proof:**
- For each $r, s = 1, 2, 3, \ldots$, construct at all valid tilings of an $r \times s$ rectangle.
- If some rectangle cannot be tiled, then $\mathcal{W}$ is not valid.
- **Output:** "No".
  - (Given $\mathcal{W}$, the question of whether an $r \times s$ block can be validly tiled by $\mathcal{W}$ is known to be NP-complete.)
- Then check each valid tiling of an $r \times s$ block for periodic boundary conditions. If a periodic tiling is found:
- **Output:** "Yes".

Proof (continued)

- **Proof (continued).**
- The algorithm must stop in finite time.
  - If $\mathcal{W}$ is not valid, then by Wang's Theorem, some $r \times s$ block cannot be validly tiled.
  - If $\mathcal{W}$ is valid, then by Wang's Conjecture, some $r \times s$ block can be validly tiled periodically.
- $\square$

Aperiodic Tilings

- **But!**

Aperiodic Tilings

- **But!** Wang's Conjecture is FALSE!
- Robert Berger, 1966:
  - In general, it is *undecidable* whether $\mathcal{W}$ is valid.
  - There exist sets $\mathcal{W}$ that admit tilings, but only aperiodic ones.
  - Call such a $\mathcal{W}$ *aperiodic*.
- In Berger's aperiodic example, $\#(\mathcal{W}) \sim 20,000$.
- …there is a big difference between $d = 1$ and $d = 2$.

Types of $\mathcal{W}$:

Possibilities for SFT, $d = 1$:

- Empty.
- Periodic points only.
- Periodic and aperiodic points both.

Non-emptiness problem is *decidable*.

Possibilities for $\mathcal{W}$, $d = 2$:

- No valid tilings.
- All valid tilings periodic.
- Periodic and aperiodic valid tilings both.
- *All valid tilings are aperiodic**.

* Call such $\mathcal{W}$ *aperiodic*.

Non-emptiness problem is *undecidable*.

Aperiodic $\mathcal{W}$ milestones

- Breger (1966): $\#(\mathcal{W}) = 20,426$.
- Breger (1966): $\#(\mathcal{W}) = 104$.
- D. E. Knuth (1966): $\#(\mathcal{W}) = 92$.
- R. Pensose (1976): $\#(\mathcal{W}) = 20$ (only 2 if counted differently, but tiles not squares).
- R. M. Robinson (1977): $\#(\mathcal{W}) = 18$. (6 if counted differently).
- R. Ammann (1978): $\#(\mathcal{W}) = 16$ (2 if counted differently, but not squares).

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- R. Ammann (1978): $\#(\mathcal{W}) = 16$ (2 if counted differently, but not squares).
- Kari (1996): $\#(\mathcal{W}) = 14$.
- Culick (1996): $\#(\mathcal{W}) = 13$.
    - (Based on idea of Kari; Holds current record.)

Hierarchy

- All the known examples of aperiodic $\mathcal{W}$, before 1996, are based on *hierarchy*.
  - Penrose tilings are a *substitution*.
  - Essentially all substitution tilings give rise to aperiodic Wang tiles (S. Mozes, 1989).
  - R. M. Robinson tilings are essentially 2-dimensional Töeplitz sequences.

Penrose tilings & Beatty sequences

- However, Penrose tilings are also based on 2-dimensional Beatty (or Sturmian) sequences (deBruijn, 1981, R).

    - Also known as *model sets* (see Meyer, 1972) or *cut and project* tilings.
    - Some model set tilings come from aperiodic $\mathcal{W}$*,
    - …but others do not (see e.g., T. Le. 1995)

- ***Conjecture:*** All these *are* hierarchical.

Question: Are the KC tilings hierarchical?

Outline

- Wang tiles
- KC tiles
- Proof of aperiodicity
- Proof of existence
- How do these tilings work?

The 13 KC tiles

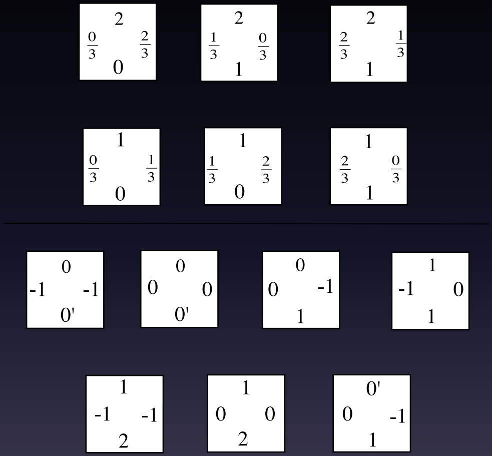


Figure: Note the two types of tiles: Top are called type $\lambda = \frac{1}{3}$; bottom are called type $\lambda = 2$. This version of KC tiles due to Eigen, Navarro & Prasad.

The 13 KC tiles

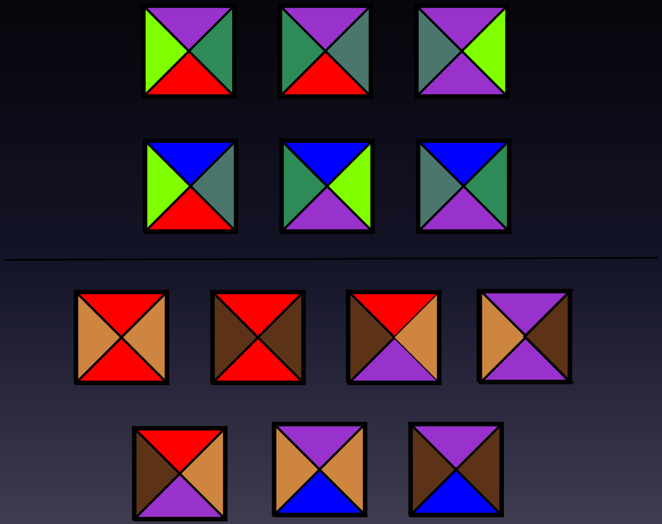
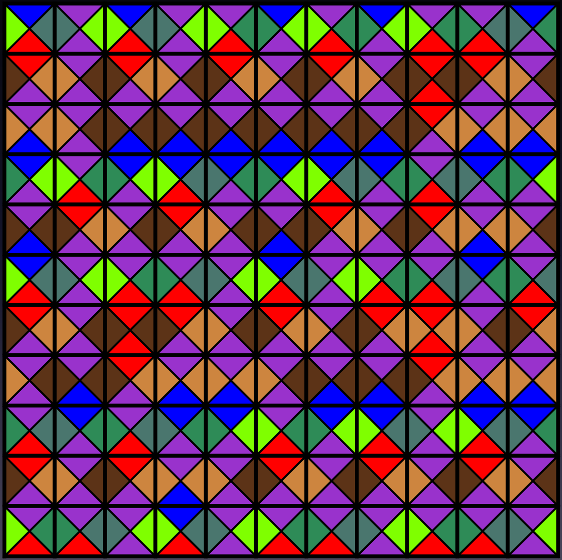


Figure: KC tiles as color tiles

KC tiling patch



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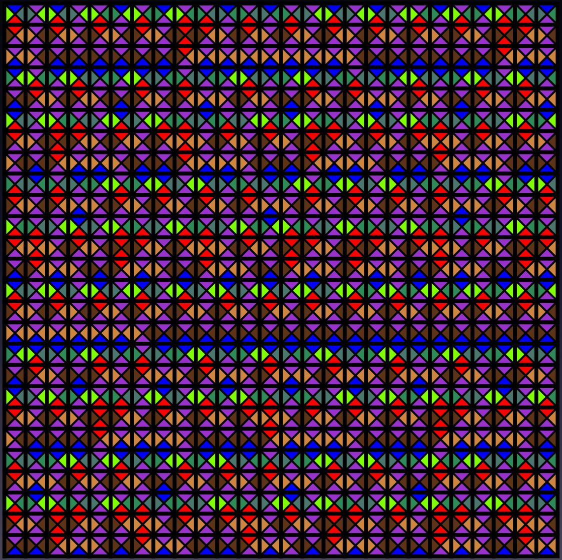

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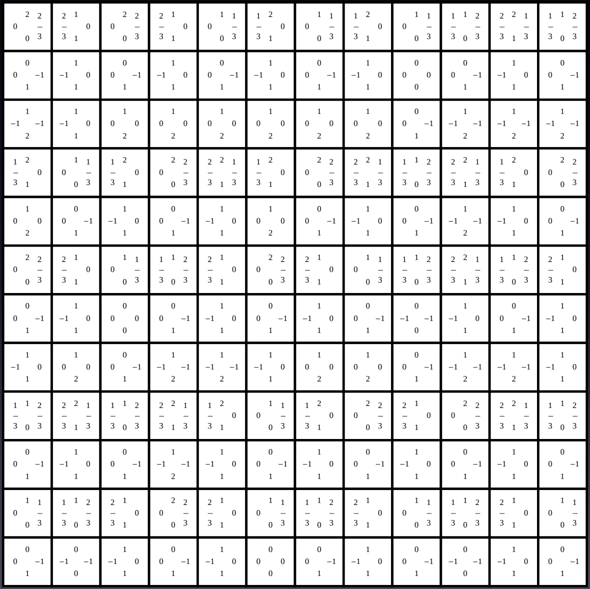

KC tiles $\mathcal{W}$ are aperiodic set

Theorem (Aperiodicity)

Any tiling of the plane by the 13 *Kari-Culik Wang tiles* $\mathcal{W}$ *must be aperiodic.*

Theorem (Valid tilings exist)

There exists a valid tiling by the Kari-Culik Wang tiles $\mathcal{W}$.

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Wang tiles

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Alternating rows

Lemma

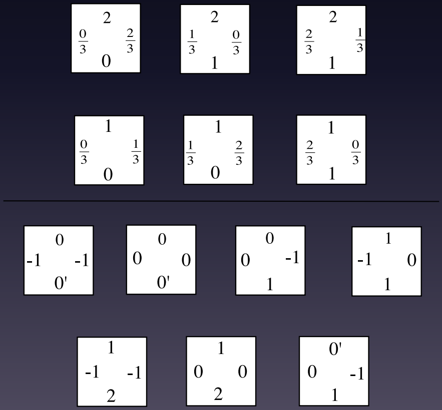
Any rows in a valid tiling by $\mathcal{W}$ *must all be either all type* $\lambda = \frac{1}{3}$ *or all type* $\lambda = 2$*. Moreover, any valid tiling contains rows of both types.*

Alternating rows

Lemma

Any rows in a valid tiling by $\mathcal{W}$ must all be either all type $\lambda = \frac{1}{3}$ or all type $\lambda = 2$. Moreover, any valid tiling contains rows of both types.

Proof:



Multiplication tiles

Definition

A tile with numbered edges n, s, e, w is called a λ multiplication tile if

$$\lambda n + w = s + e.$$

Lemma

All KC tiles are multiplication tiles of their type λ.

Clumping

- Now consider a valid $u \times v$ block. Let $\lambda_1, \ldots, \lambda_v$ be the multipliers of the rows.
- By induction, the block is a multiplier tile:
  - $\lambda = \lambda_1 \lambda_2 \ldots \lambda_u$,
  - $n = n_{1,1} + n_{1,2} + \cdots + n_{1,v}$,
  - $s = s_{u,1} + s_{u,2} + \cdots + s_{u,v}$,
  - $e = e_{1,1} + \lambda_1(e_{2,1} + \lambda_2(e_{3,1} + \ldots \lambda_{u-1} e_{u,1}) \ldots)$
  - $w = w_{1,v} + \lambda_1(e_{2,v} + \lambda_3(e_{3,v} + \ldots \lambda_{u-1} e_{u,v}) \ldots)$

Proof of aperiodicity

- If the tiling is *periodic*, then there exists a periodic $u \times v$ block.
- This block has a *periodic boundary*: $n_{1,j} = s_{u,j}$ and $e_{i,1} = w_{i,v}$.
- Thus we have $n = s$ and $e = w$.
- Then $\lambda n + w = e + s \implies \lambda = 1$.
- But $\lambda = \lambda_1 \lambda_2 \ldots \lambda_u = 2^k \left(\frac{1}{3}\right)^{u-k}$.
- Thus $2^k = 3^{u-k}$ where $k, u - k \geq 1$.

Contradiction! $\square$

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How do these tilings work?

Horizontal h and $\vec{n}$

- Let $\alpha \in [\frac{1}{3}, 2]$, $\beta \in \mathbb{R}$, $\lambda \in \{\frac{1}{3}, 2\}$.
- Define $h(\alpha, \beta) = \lfloor \alpha + \beta \rfloor - \lfloor \beta \rfloor$.
- Define $\vec{n} = \vec{n}(\alpha, \beta) = (\dots n_{-1}, n_0, n_1 \dots)$ by

$$n_k = h(\alpha, \beta + k\alpha) = \lfloor (k+1)\alpha + \beta \rfloor - \lfloor k\alpha + \beta \rfloor.$$

- This is a *Beatty difference sequence*.
- $n_k \in \{a, a+1\}$ where $a = \lfloor \alpha \rfloor$, so
  - $n_k \in \{0, 1\}$ for $\alpha \in [\frac{1}{3}, 1]$.
  - $n_k \in \{1, 2\}$ for $\alpha \in [1, 2]$.

Properties

- $\vec{n}(\alpha, \beta) = \vec{n}(\alpha, \{\beta\})$.
  (it only depends on $\beta \mod 1$).
- If $\alpha < 1$ then $\vec{n}(\alpha, \beta) \in \{0, 1\}^{\mathbb{Z}}$ is a *Sturmian sequence*.

- If $\alpha > 1$ then $\vec{n}(\alpha, \beta) - \lfloor \alpha \rfloor = \vec{n}(\alpha - \lfloor \alpha \rfloor, \beta)$.
  (i.e., $\vec{n}$ is Sturmian, but in the wrong alphabet.)
- There exists $A : \{a, a+1\} \to \mathbb{R}$, so that:
  - $A(\vec{n}(\alpha, \beta)) = \alpha$,

  (essentially the ergodic theorem).

Numeration

Now let us think of $\vec{n}(\alpha, \beta)$ as a "numeration" of β in "base" α.

- First suppose $\alpha \in [\frac{1}{3}, 1)$
- Consider the piecewise linear mapping $g : [0, 1) \to \mathbb{R}_{\geq 0}$ defined $g(x) = x + \alpha$.
  - Initially define $n_0 = \lfloor g(\beta) \rfloor$,
  - and $b_0 = \{g(\beta)\}$.
  - Then for $k > 0$, define by induction, $n_k = \lfloor g(b_{k-1}) \rfloor$,
  - and $b_k = \{g(b_{k-1})\}$,
  - (with a similar definition for $k < 0$).
- Note that $\{g(\beta)\} = R_\alpha(\beta)$, i.e., the *irrational rotation map*.

Call this the *Sturmian numeration system*.

Numeration (continued)

Sturmian numeration is not quite so nice as other numeration systems. However, there exists $B : \{0,1\} \to \mathbb{R}$ so that:

- $B(\vec{n}(\alpha,\beta)) = \{\beta\}$ if $\alpha \notin \mathbb{Q}$,
- $B(\vec{n}(\alpha,\beta)) = \frac{\lfloor q\{\beta\}\rfloor}{q}$ if $\alpha = \frac{p}{q} \in \mathbb{Q}$.

Numeration (continued)

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In the case $\alpha \in [1, 2]$ use $\alpha' = \alpha - 1$, and replace n_k with $n_k + 1$. Call this a *Beatty numeration system*.

Vertical v

- Fix $\lambda \in \mathbb{Q}$, $\lambda > 0$.
- Define $v(\beta, \lambda) = \lambda\lfloor\beta\rfloor - \lfloor\lambda\beta\rfloor$.
- Think of $v : \mathbb{R} \to \mathbb{R}$.

Lemma
For $\lambda = \frac{p}{q}$, v has period q, and

$$v(\beta, \lambda) \in \left\{-\frac{1-p}{q}, -\frac{2-p}{q}, \ldots, \frac{q-1}{q}\right\}.$$

- If $\lambda = p \in \mathbb{N}$ then $v(\beta, \lambda) \in \{1-p, \ldots, 0\}$.
- If $\lambda = \frac{1}{q}$ then $v(\beta, \lambda) \in \left\{\frac{0}{q}, \ldots, \frac{q-1}{q}\right\}$.

The case $\lambda = 2$

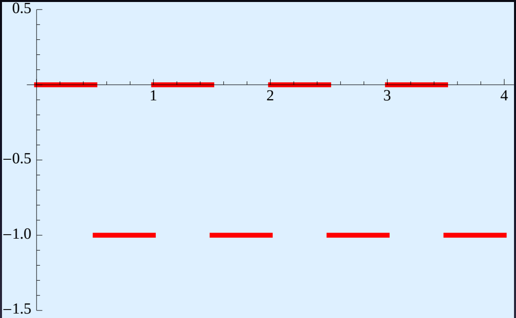

Figure: $v(\beta, 2)$ has period 1, and values in $\{-1, 0\}$.

The case $\lambda = \frac{1}{3}$

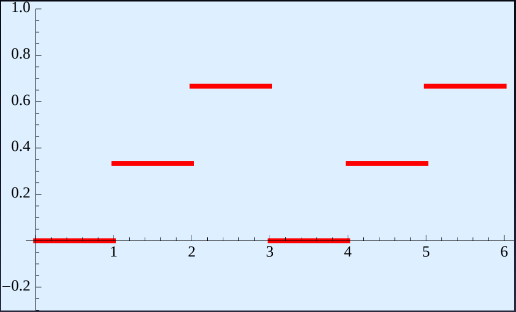

Figure: $v\left(\beta, \frac{1}{3}\right)$ has period 3, and values in $\left\{\frac{0}{3}, \frac{1}{3}, \frac{2}{3}\right\}$.

A typical general case: $\lambda = \frac{4}{9}$

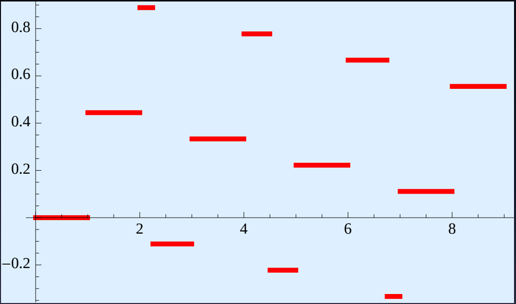

Figure: In general, $v(\beta, \lambda)$ can be quite complicated.

A map of the interval

- Define $f : [\frac{1}{3}, 2] \to [\frac{1}{3}, 2]$ by

$$f(x) = \begin{cases} 2x & \text{if } x \in [\frac{1}{3}, 1), \\ \frac{1}{3}x & \text{if } x \in [1, 2]. \end{cases}$$

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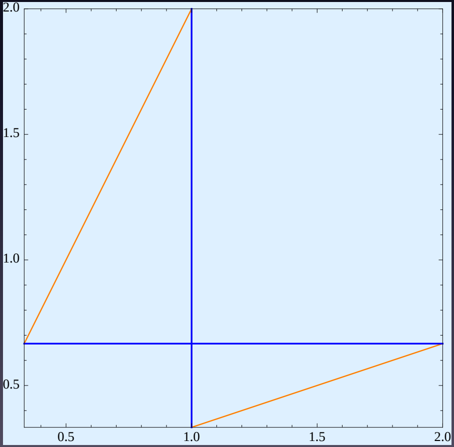

Aperiodicity of f

Proposition

The mapping f has no periodic points.

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Proof.

Suppose $f^u(x) = x$. Then $2^k = 3^{u-k}$ for some $0 < k < u$. Contradiction. $\square$

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- Let $\lambda = f'(\alpha) \in \{\frac{1}{3}, 2\}$.

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- Let $\lambda = f'(\alpha) \in \{\frac{1}{3}, 2\}$.
- Define a Wang tile $T(\alpha, \beta)$ to have side colors:
  - $n = h(\alpha, \beta)$,
  - $s = h(\lambda\alpha, \lambda\beta)$,
  - $e = v(\beta + \alpha, \lambda)$,
  - $w = v(\beta, \lambda)$.

Multiplication property

Proposition

The tile $T(\alpha, \beta)$ is a λ-multiplication tile.

Proof.

$$
\begin{aligned}
\lambda n + w - e - s &= \lambda h(\alpha, \beta) + v(\beta, \lambda) - v(\beta + \alpha, \lambda) - h(\lambda\alpha, \lambda\beta) \\
&= \lambda(\lfloor \alpha + \beta \rfloor - \lfloor \beta \rfloor) + (\lambda \lfloor \beta \rfloor - \lfloor \lambda\beta \rfloor) \\
&\quad -(\lambda \lfloor \beta + \alpha \rfloor - \lfloor \lambda\beta + \lambda\alpha \rfloor) \\
&\quad -(\lfloor \lambda\alpha + \lambda\beta \rfloor - \lfloor \lambda\beta \rfloor) \\
&= 0.
\end{aligned}
$$

□

The "modified" tile

- The *modified Wang* tile $\widetilde{T}(\alpha, \beta)$ is the same as the basic tile $T(\alpha, \beta)$, except:
  - If $\alpha \in [\frac{1}{3}, \frac{1}{2})$, then $n = 0$ is replaced with $n' = 0'$, and
  - If $\lambda\alpha \in [\frac{1}{3}, \frac{1}{2})$, then $s = 0$ is replaced with $s' = 0'$.
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Theorem (Kari, Culik, Eigen/Navarro/Prasad,R)
The set $\mathcal{W} = \{\widetilde{T}(\alpha,\beta) : \alpha \in [\frac{1}{3},2], \beta \in \mathbb{R}\}$ *is precisely the set of 13 KC Wang tiles.*

The 13 KC tiles

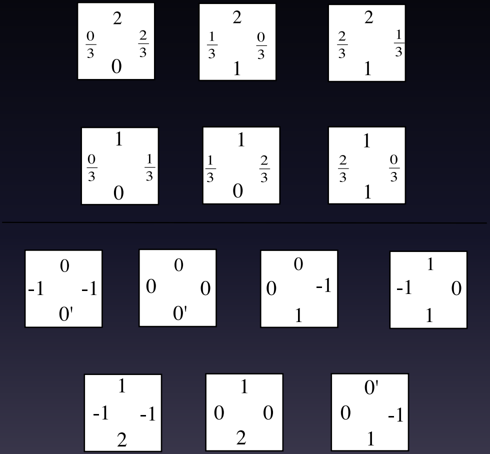

Figure: $\mathcal{W} = \{\tilde{T}(\alpha, \beta) : \alpha \in [\frac{1}{3}, 2], \beta \in \mathbb{R}\}$.

Some definitions

- Let $\lambda_k = f'(f^k(\alpha))$.
- Define μ_k as follows:
  - $\mu_0 = 1$,
  - $\mu_k = \lambda_k \lambda_{k-1} \ldots \lambda_1$, for $k > 0$,
  - $\mu_k = \lambda_{-1} \lambda_{-2} \ldots \lambda_k$, for $k < 0$.
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  - (Note that $\mu_k = (f^k)'(\alpha)$.
- Define $\vec{\alpha}$ by $\alpha_k = \mu_k \alpha = f^k(\alpha)$.
- Define $\vec{\beta}$ by $\beta_k = \mu_k \beta$.

Constructing a valid tiling

- Index $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ like a matrix:
- …by (k, ℓ), where k is the *row* (↓), and ℓ is the *column* (→).
- Fix $\alpha \in [\frac{1}{3}, 2]$ and $\beta \in \mathbb{R}$.
- Define a tiling $\mathbf{T} = \mathbf{T}(\alpha, \beta)$ by placing the the tile

$$\widetilde{T}(k, \ell) = \widetilde{T}(\alpha_k, \beta_k + \ell\alpha_k)$$

at (k, ℓ) in $\mathbb{Z}^2$.

Constructing a valid tiling

- Index $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ like a matrix:
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Theorem (Kari, Culik, Eigen/Navarro/Prasad,R)

The tiling $\mathbf{T}$ *is a valid tiling by* $\mathcal{W}$*. (The existence theorem is true).*

- **Proof:**
- $n(\ell + 1, k) = s(\ell, k)$,
- $e(\ell, k) = w(\ell, k + 1)$. $\square$

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Wang tiles

KC tiles

Proof of aperiodicity

Proof of existence

How do these tilings work?

Tile tops

- Row k is type $\lambda_k \in \{\frac{1}{3}, 2\}$ (read from alphabet).
- The tops of row k are Beatty difference sequence $\vec{n}(\alpha_k, \beta_k)$.

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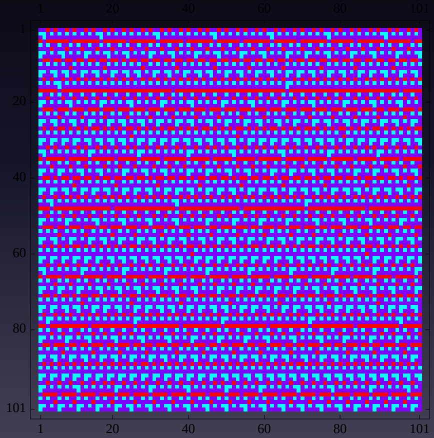
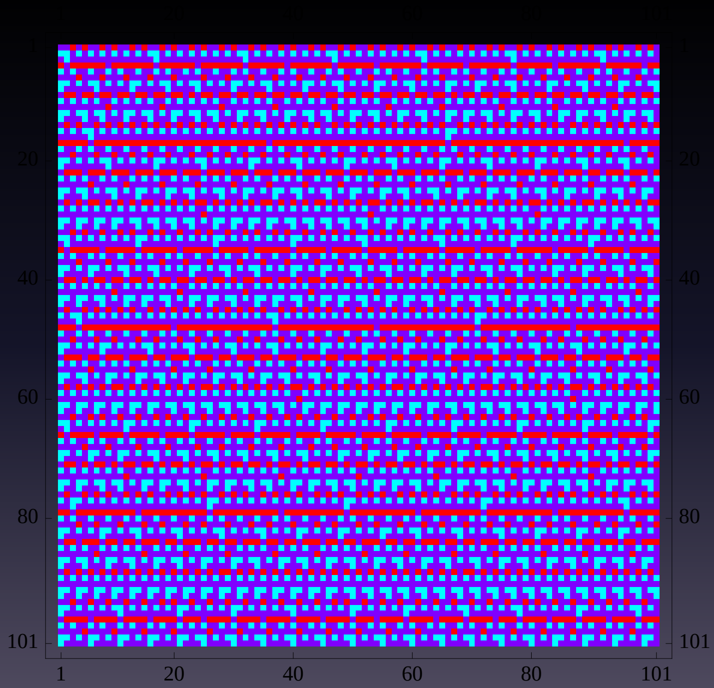


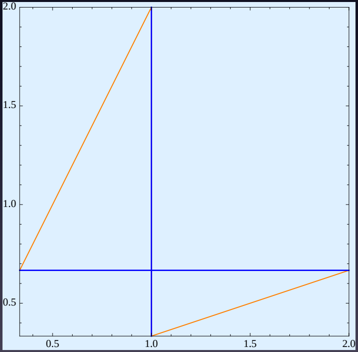
Figure: Blue$= 0 = 0'$, Purple=1, Red=2.


1
20
40
60
80
101

The map f ...

- Recall that

$$f(x) = \begin{cases} 2x & \text{if } x \in [\tfrac{1}{3}, 1), \\ \tfrac{1}{3}x & \text{if } x \in [1, 2]. \end{cases}$$



…is conjugate to a rotation

…is conjugate to a rotation

Theorem (Liousse, 2004)

A 2-piece, piecewise-linear homeomorphism of the circle (like f), with (left & right) slopes $\lambda > 1 > \lambda'$, is topologically conjugate to the rotation on the circle with rotation number

$$\alpha' = \frac{\log \lambda}{\log \lambda - \log \lambda'}.$$

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$$\alpha' = \frac{\log \lambda}{\log \lambda - \log \lambda'}.$$

- In the case of f as above,
  $\varphi \circ f = R_{\alpha'} \circ \varphi$, where $\varphi : [\frac{1}{3}, 2] \to [0, 1]$ is given by

$$\varphi(x) = \frac{\log(x) + \log 3}{\log 2 + \log 3}.$$

Nearly Sturmian

- As before, define $\lambda_k \in \{\frac{1}{3}, 2\}$ by $\lambda_k = f'(f^k(\alpha))$,
- and let $\vec{s}(\alpha)$ by

$$s_k = \begin{cases} 0 & \text{if } \lambda_k = \frac{1}{3}, \\ 1 & \text{if } \lambda_k = 2. \end{cases}$$

Corollary

The sequence $\vec{s}$ is Sturmian. In particular,

$$\vec{s}(\alpha) = \vec{n}\left(\frac{\log 2}{\log 2 + \log 3}, \frac{3}{5}\alpha - \frac{1}{5}\right).$$

Sturmian row alteration

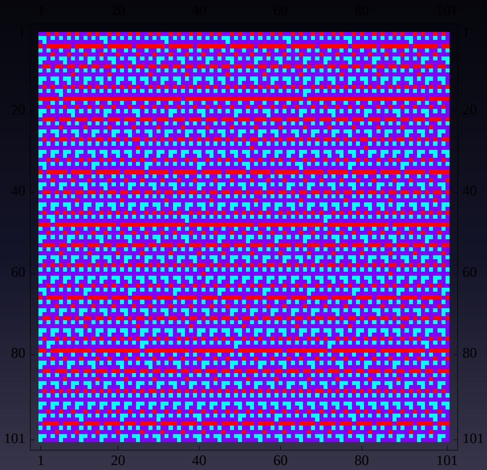


Figure: The row alteration pattern is Sturmian with $\alpha' = \frac{\log 2}{\log 3 + \log 2} \approx 0.38685280723$.

Some basic tiles

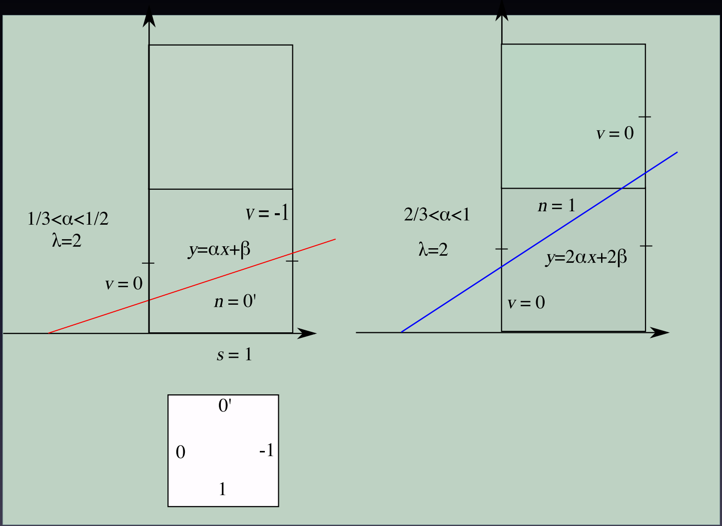
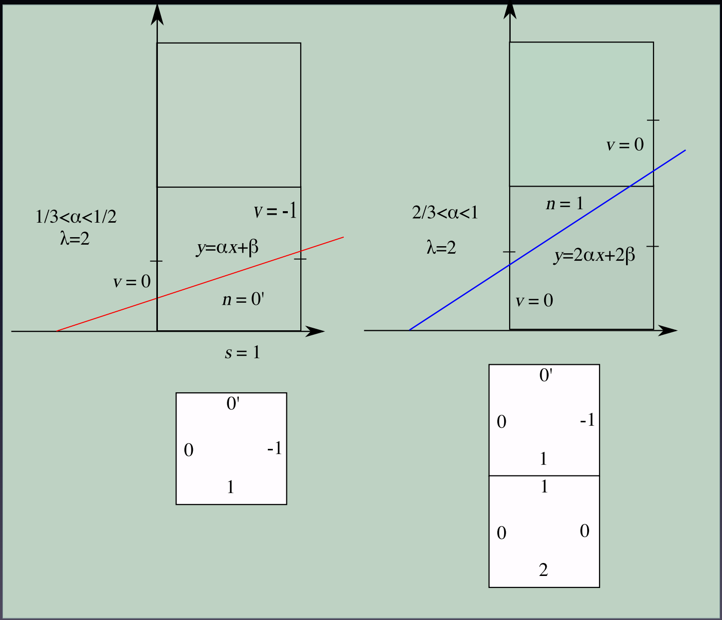
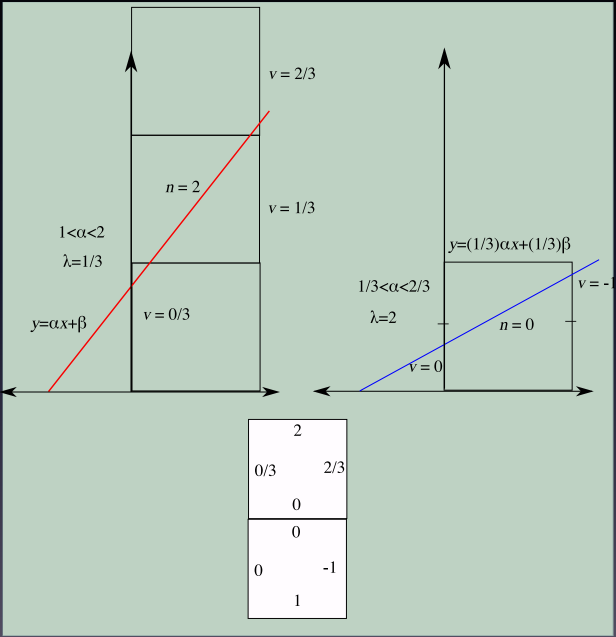


Figure: Line for typical $\lambda = 2$ tile, $\alpha < \frac{1}{2}$. Sides *n*, *e* and *w* read directly off. Side *s* is *n* for line to right (next tile down in **T**).

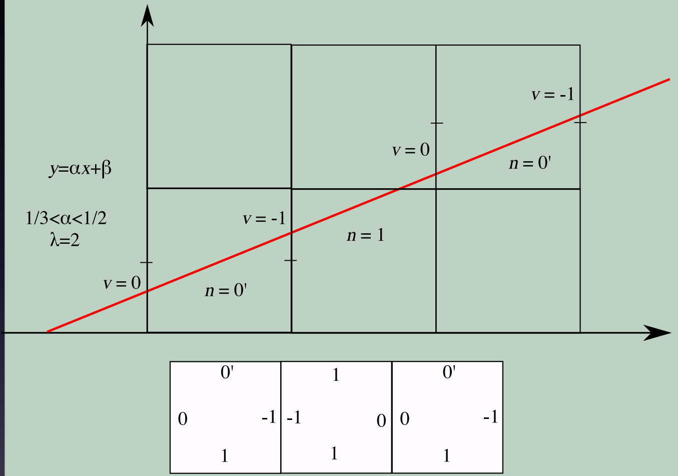
Part of a column



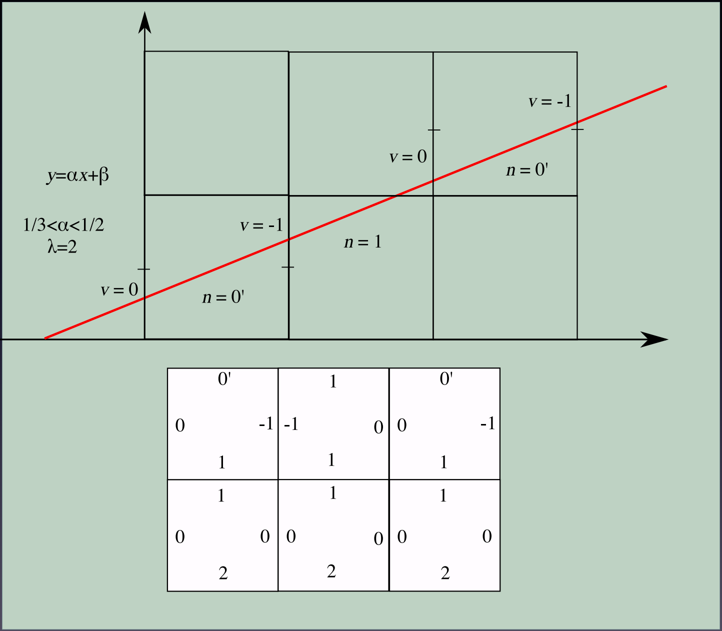
A different column



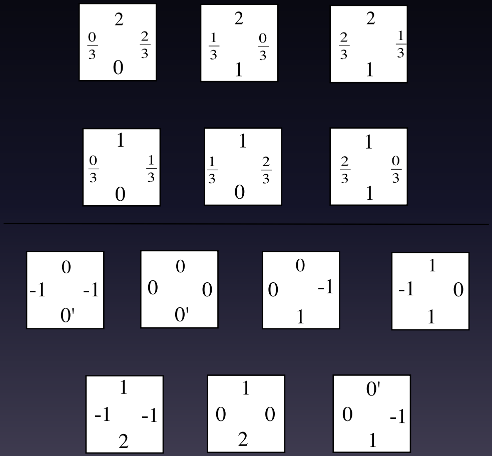
Part of a row



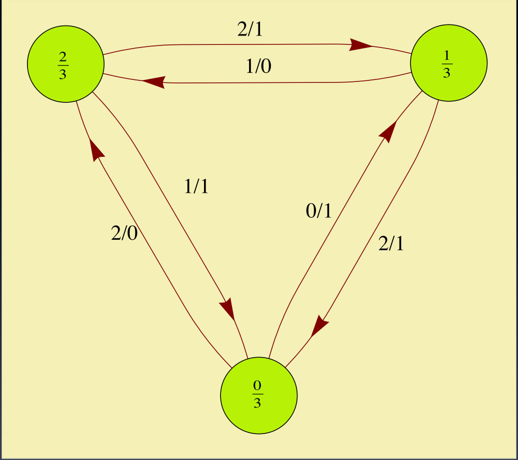
The next row



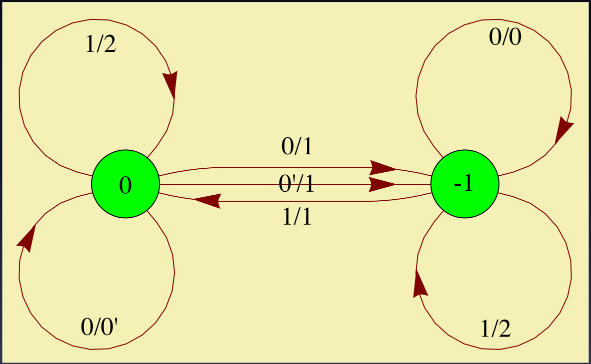
The KC tiles again



Automaton: type $\lambda = 1/3$



Automaton: type $\lambda = 2$



The case $\lambda = 2$

- Consider the irrational rotation maps $R_\alpha : [0,1) \to [0,1)$ and $R_{2\alpha} : [0,1) \to [0,1)$.
- These maps are connected:

$$\begin{array}{ccc} [0,1) & \xrightarrow{R_\alpha} & [0,1) \\ {\scriptstyle 2x \mod 1}\downarrow & & \downarrow{\scriptstyle 2x \mod 1} \\ [0,1) & \xrightarrow{R_{2\alpha}} & [0,1). \end{array}$$

The case $\lambda = \frac{1}{3}$

- Consider the irrational rotation maps $R_\alpha : [0, 1) \to [0, 1)$ and $R_{\frac{1}{3}\alpha} : [0, 1) \to [0, 1)$.
- These maps are connected:

$$\begin{array}{ccc} [0,1) & \xrightarrow{R_\alpha} & [0,1) \\ {\scriptstyle 3x \mod 1}\uparrow & & \uparrow{\scriptstyle 3x \mod 1} \\ [0,1) & \xrightarrow{R_{\frac{1}{3}\alpha}} & [0,1). \end{array}$$

$\lambda = \frac{1}{3}$ (continued)

Note

$$\begin{array}{ccc} [0,1) & \xrightarrow{R_\alpha} & [0,1) \\ {\scriptstyle y=\frac{1}{3}x+v(y)}\big\downarrow & & \big\downarrow{\scriptstyle y=\frac{1}{3}x+v(y)} \\ [0,1) & \xrightarrow{R_{\frac{1}{3}\alpha}} & [0,1), \end{array}$$

where $v \in \{\frac{0}{3}, \frac{1}{3}, \frac{2}{3}\}$.

$\lambda = \frac{1}{3}$ (continued)

Note

$$\begin{array}{ccc} [0,1) & \xrightarrow{R_\alpha} & [0,1) \\ {\scriptstyle y=\frac{1}{3}x+v(y)}\Big\downarrow & & \Big\downarrow{\scriptstyle y=\frac{1}{3}x+v(y)} \\ [0,1) & \xrightarrow{R_{\frac{1}{3}\alpha}} & [0,1), \end{array}$$

where $v \in \{\frac{0}{3}, \frac{1}{3}, \frac{2}{3}\}$.

Here there are *three* different maps, depending on the choice of v.

$\lambda = 2$ (again)

Note

$$\begin{array}{ccc} [0,1) & \xrightarrow{R_\alpha} & [0,1) \\ {\scriptstyle 2x+v(x)}\downarrow & & \downarrow{\scriptstyle 2x+v(x)} \\ [0,1) & \xrightarrow{R_{2\alpha}} & [0,1), \end{array}$$

where $v(x) \in \{0, -1\}$ is defined

$$v(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}), \\ -1 & \text{if } x \in [\frac{1}{2}, 1). \end{cases}$$

Here, both values of v are used in defining a single map, namely $h(x) = 2x \mod 1$.

A "solenoid"

- Consider the group $\mathbb{T}^{\mathbb{Z}} = \oplus_{k\in\mathbb{Z}}\mathbb{T}_k = \oplus_{k\in\mathbb{Z}}[0,1)_k$.
- Fix $\alpha \in [\frac{1}{3}, 2]$ and $\beta \in \mathbb{R}$.
- Let μ_k be as above (i.e. $\mu_k = \lambda_1\lambda_2\ldots\lambda_k$, $k \geq 0$).
- Define $\varphi : \mathbb{R} \to \mathbb{T}^{\mathbb{Z}}$ by $\varphi(\beta) = \vec{x}$ where $x_k = \{\mu_k\beta\}$.

- Define a subgroup $\mathbb{S}_\alpha \subseteq \mathbb{Z}^{\mathbb{T}}$ to be the set of all $\vec{x} \in \mathbb{T}^{\mathbb{Z}}$ so that
  - $x_{k+1} = 2x_k$ if $\lambda_k = 2$, and
  - $3x_{k+1} = x_k$ is $\lambda_k = \frac{1}{3}$.

Results

Lemma
For all $\beta \in \mathbb{R}$, $\varphi(\beta) \in \mathbb{S}_\alpha$. *Moreover,* φ *is an embedding of* $\mathbb{R}$ *into* $\mathbb{S}_\alpha$.

Theorem
Each $\vec{x} \in \mathbb{S}_\alpha$ *determines a valid tiling* $\mathbf{S}(\alpha, \vec{x})$ *of* $\mathbb{R}^2$ *by* $\mathcal{W}$. *If* $\vec{x} = \varphi(\beta)$ *then* $\mathbf{S}(\alpha, \vec{x}) = \mathbf{T}(\alpha, \beta)$.

Comments.

- Essentially these are all tilings in the closure of $\{\mathbf{T}(\alpha, \beta) : \alpha \in [\frac{1}{3}, 2], \beta \in \mathbb{R}\}$.
- From the tile $\mathbf{T}(\alpha, \beta)$ it is possible to read the exact value of α and β (even if $\alpha \in \mathbb{Q}$).

Idea of proof of theorem