Rosen continued fractions and Veech groups

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Thanks ...

In this expository talk, we give an introduction to the Rosen continued fractions, and sketch a geometric application of these and related expansions.

I. Simple Continued Fractions

- Each real x has SCF-expansion

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}}$$

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with *convergents* $\quad [\, a_0; a_1,\, a_2, \ldots,\, a_n] =: p_n/q_n \,.$

Underlying map is shift on continued fractions

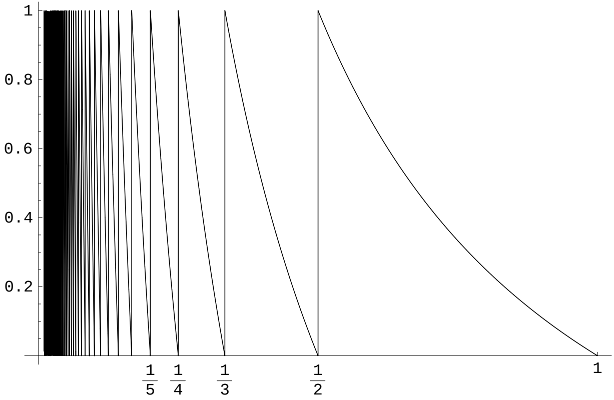
$$T \; : \; [0,1) \to \; [0,1)$$

$$x \mapsto \frac{1}{x} \mod 1$$

$$= \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \; x \neq 0;$$

$(\, T(0) \, = \, 0 \,)$.

Underlying map, Figure



Matrices and Convergents

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- In general,

$$\begin{pmatrix} \epsilon\, p_{n-1} & p_n \\ \epsilon\, q_{n-1} & q_n \end{pmatrix}$$

gives a determinant one matrix, with $\epsilon = \pm 1$.

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- Details of parity for finite expansions. The alternating sign is related to:

  Convergents alternate above and below x .

Matrix group

- With möbius action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x := \frac{ax+b}{cx+d},$$

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- Projective Action

$$\begin{pmatrix} a\mu & b\mu \\ c\mu & d\mu \end{pmatrix} x := \frac{ax+b}{cx+d}.$$

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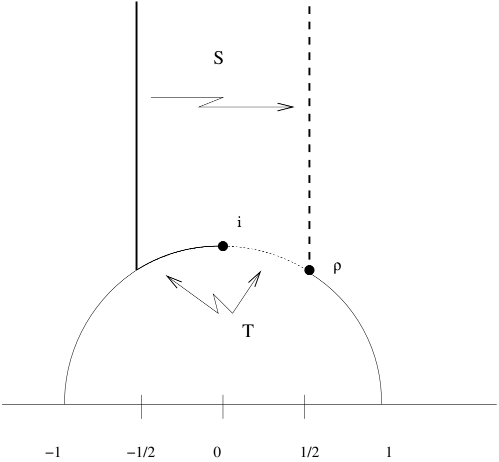
- Extend möbius action to complex z. Circles sent to circles; real line preserved by any real 2×2 matrix.
- Find all of $\mathrm{PSL}(2, \mathbf{R})$ acts on upper half-plane.

Fundamental Domain



Hyperbolic metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$

II. Hecke groups

- The Hecke (triangle Fuchsian) group, G_q, with $q \in \{3, 4, 5, \ldots\}$ is the group generated by

$$S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

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- Example: $G_3 = \mathrm{PSL}(2, \mathbb{Z})$.

- Let $U = ST$, so

$$U = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $U^q = \mathrm{Id}$, and $G_q \cong \mathbb{Z}/2 \star \mathbb{Z}/q$.

Word Problem

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- **Question** Which determinant one matrices with $a, b, c, d \in \mathbb{Z}[\lambda_q]$ are in G_q ?
- **Difficulty** G_q is of infinite index in $\mathrm{PSL}(2, \mathbb{Z}[\lambda_\mathrm{q}])$

Rosen Continued Fractions

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- Determine a_i with nearest integer multiple of λ_q

  Need $\epsilon_i = \pm 1$

$$\alpha = a_0\lambda + \cfrac{\epsilon_1}{a_1\lambda + \cfrac{\epsilon_2}{a_2\lambda + \cfrac{\epsilon_3}{\ddots}}}$$

$$\alpha = [\, a_0\, ;\ \epsilon_1 : a_1\lambda\, ,\ \epsilon_2 : a_2\lambda\, ,\ \ldots\,]$$

Rosen Maps Figure

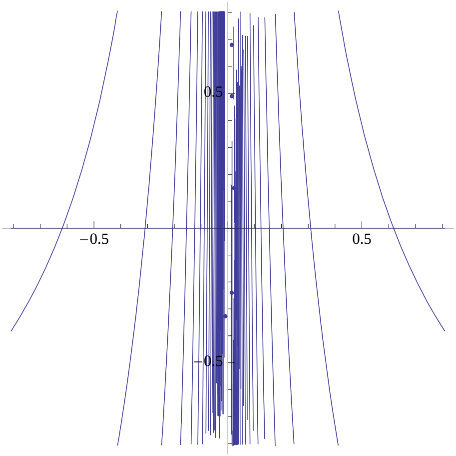

Figure: Approximate graph of $f_q(x)$, q=5

Rosen's Theorem

- For
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
let
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- **Theorem** Let $M \in \mathrm{SL}(2, \mathbb{Z}[\lambda_q])$. Then $M \in G_q$ if and only if, up to sign, the columns of M are made from consecutive convergents of $M \cdot \infty$, or of $M \cdot 0$.

Rosen's Cusp Question

Rosen's Cusp Challenge Determine the cusp set for each G_q:

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- $q = 5$ Rosen '63: Units of $\mathbb{Z}[\lambda_5]$ are cusps

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- McMullen 2003, using techniques related to Veech groups, determined cusp sets of certain 'triangle groups'.

Periodic Expansion

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$$\sqrt{14} = [\,3; \overline{1, 2, 1, 6}\,]$$

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- **Rosen's Periodic Expansions Question** Which real numbers have periodic expansion with respect to the λ continued fractions?

Orbits of elements of $\mathbb{Q}(\lambda)$

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- For odd q, the number of orbits of the field elements must go to infinity with q.

Other Aspects

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- Today's next talk!

III. Veech Groups — flat torus is the touchstone

- The flat torus has *optimal dynamics* — when we follow a line, we either return to starting point or get arbitrarily close to every point.

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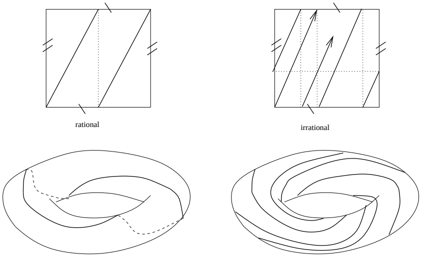
- 

Figure: Indeed, have ergodic invariant measure for this "linear flow" in the second case.

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  - Veech gave examples with this group isomorphic to (index 2 subgroup of) Hecke group, G_q .

Billiards on Square gives Torus

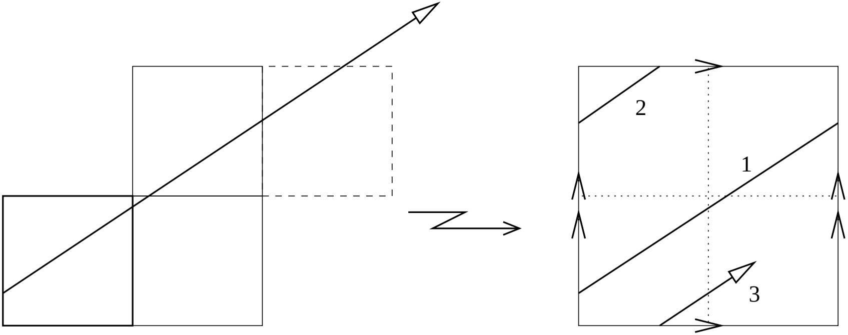

Figure: Unfolding; square table to torus surface.

Billiards to Surfaces: Genus 2

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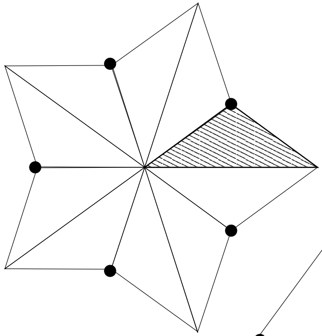

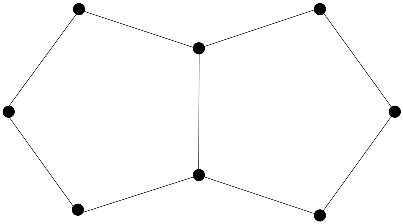

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Translation Surfaces

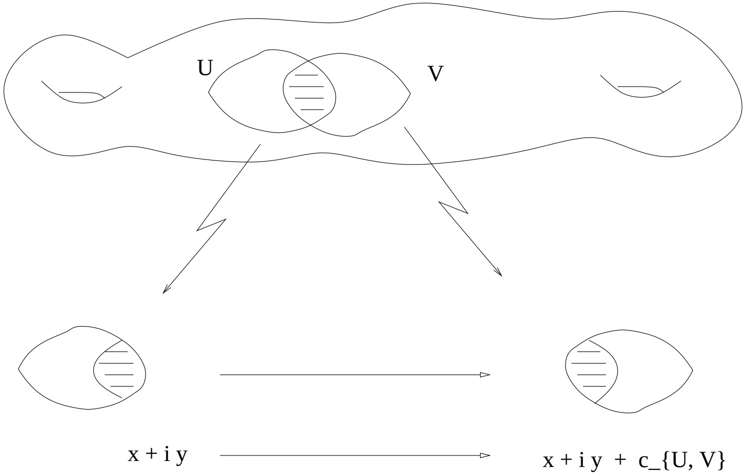
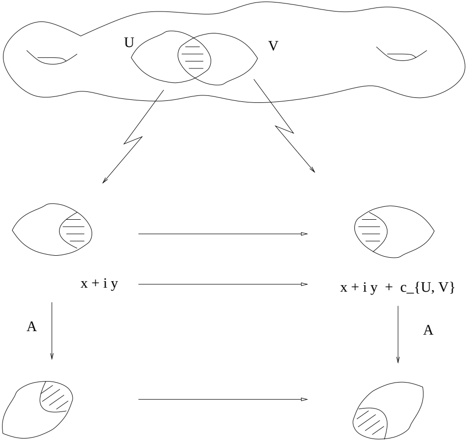


Figure: Idea of translation surface

SL_2R- Action

Post-compose with $A \in SL_2R$.



New translation surface.

Affine Diffeomorphisms

$$(x, y) \dashrightarrow (x , \quad x + y \bmod 1)$$

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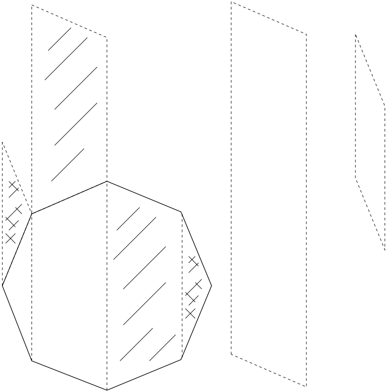
- An *affine diffeomorphism* is some $f : X \to X$ whose *derivative* (off of singularities) is constant $A \in SL_2 R$.
- Group of all these derivatives is the Veech group: $SL(X, \omega)$.

Octagon: vertical direction has two cylinders



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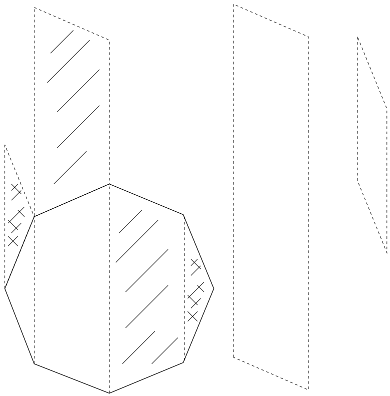

- Gives $\begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$, $\mu = 2(1+\sqrt{2})$.
- With rotation, get triangle group that is isomorphic to index 2 subgroup of G_8 .

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- Rosen's result that 1 has periodic expansion for even q gives $(1+\cos\pi/q)/\sin\pi/q$ is non-periodic direction on the $2q$-gon. In fact, there is a corresponding pseudo-Anosov diffeomorphism.