Conjugates of Parry Numbers near the unit circle

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Theorem (Salem, 1945)

Every P.V. number is a limit point of numbers of the class (T) on both sides.

open problem : given a sequence of Salem numbers (β_i) which converges to $x \in (1, +\infty)$, what is x ? (i.e. Pisot or Salem ?)

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then a mysterious cancelling occurs.

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dynamical viewpoint : reformulation, since

- a Pisot number is a Parry number (ex-beta-number) [Theorem A. Bertrand-Mathis, K. Schmidt]
- a Salem number is Parry/or nonParry [probabilistic model of Boyd]
- LEFT/RIGHT : limits may be different [W. Parry, 1960]

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Sequences of Parry numbers : what brings dynamics ?

beta-conjugates [Boyd] (other roots in the Parry polynomial than the Galois conjugates)
+
equidistribution near the unit circle

Observation : another concentration of conjugates...

-> deep link with theorems on the equidistribution of small points in the closure of torsion subgroups [Szpiro, Ullmo, Zhang, on abelian varieties and Bilu, on *n*-dim algebraic tori (1997)] in arithmetic geometry.

-> idea : to bring back “analogs" to numeration. The ingredients : new, and allow to introduce a new notion of convergence for sequences of Parry numbers. (recall that the set of Parry numbers is dense in $(1, +\infty)$ [Parry]).

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$\beta > 1$ Perron number if algebraic integer and all its Galois conjugates $\beta^{(i)}$ satisfy : $|\beta^{(i)}| < \beta$ for all $i = 1, 2, \ldots, d-1$ (degree $d \geq 1$, with $\beta^{(0)} = \beta$).

Let $\beta > 1$. The Rényi β-expansion of 1

$$d_\beta(1) = 0.t_1 t_2 t_3 \ldots \qquad \text{and corresponds to} \qquad 1 = \sum_{i=1}^{+\infty} t_i \beta^{-i} ,$$

$t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta\{\beta\} \rfloor = \lfloor \beta T_\beta(1) \rfloor, t_3 = \lfloor \beta\{\beta\{\beta\}\} \rfloor = \lfloor \beta T_\beta^2(1) \rfloor, \ldots$ The digits t_i belong to $\mathcal{A}_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$.

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Parry number : if $d_\beta(1)$ is finite or ultimately periodic (i.e. eventually periodic) ; in particular, simple if $d_\beta(1)$ is finite.

Lothaire : a Parry number is a Perron number.

Dichotomy : set of Perron numbers

$$\mathbb{P} = \mathbb{P}_P \cup \mathbb{P}_a$$

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Dichotomy : set of Perron numbers

$$\mathbb{P} = \mathbb{P}_P \cup \mathbb{P}_a$$

$$f_\beta(z) := -1 + \sum_{i=1}^{+\infty} t_i z^i \qquad \text{for } \beta \in \mathbb{P}, z \in \mathbb{C},$$

where $d_\beta(1) = 0.t_1 t_2 t_3 \ldots$, for which $f_\beta(z)$ is a rational fraction if and only if $\beta \in \mathbb{P}_P$ (Szegő's Theorem).

Dichotomy of Perron numbers β <—> dichotomy of analytical functions $f_\beta(z)$.

Let $\beta > 1$. The beta-transformation is

$$T_\beta : [0,1] \to [0,1],\ x \to \{\beta x\}.$$

Let $T_\beta^0 =$Id, $T_\beta^j = T_\beta(T_\beta^{j-1}), \quad j \geq 0$.

Define the Artin-Mazur zeta function

$$\zeta_\beta(z) := \exp\Big(\sum_{n\geq 1} \frac{P_n}{n} z^n\Big)$$

with $P_n =$ number of fixed points under T_β^n.

Theorem (Ito - Takahashi ; Flatto, Lagarias, Poonen)

If β is a simple Parry number, then

$$\zeta_\beta(z) = \frac{1 - z^N}{(1 - \beta z)(\sum_{n=0}^{\infty} T_\beta^n(1) z^n)}$$

where N is minimal with $T_\beta^N(1) = 0$.

If β is a (nonsimple) Parry number, then

$$\zeta_\beta(z) = \frac{1}{(1 - \beta z)(\sum_{n=0}^{\infty} T_\beta^n(1) z^n)}$$

Nonsimple Parry numbers :

$$f_\beta(z) = \frac{-1}{\zeta_\beta(z)}$$

Simple Parry numbers :

$$f_\beta(z) = \frac{-1 + z^N}{\zeta_\beta(z)}$$

$$f_\beta(z) = \frac{-n^*_\beta(z)}{1 - z^p}, \qquad n_\beta(z) = \text{multiple of } P_\beta(z) \text{ (minimal polynom}$$

$n_\beta(z)$ = the Parry polynomial (called = characteristic polynomial of the beta-number β, in Parry '60) ;

$m \geq 0$, non-simple Parry number :

$$n_\beta(X) = X^{m+p+1} - t_1 X^{m+p} - t_2 X^{m+p-1} - \ldots - t_{m+p} X - t_{m+p+1}$$

$$- X^m + t_1 X^{m-1} + t_2 X^{m-2} + \ldots + t_{m-1} X + t_m$$

Simple Parry number ($m \geq 1$) :

$$X^m - t_1 X^{m-1} - t_2 X^{m-2} - \ldots - t_{m-1} X - t_m$$

$$t_i := \lfloor \beta T_\beta^{i-1}(1) \rfloor \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}.$$

Key result : the height $\mathrm{H}(n_\beta)$ of the Parry polynomial satisfies

$$\mathrm{H}(n_\beta) \in \{\lfloor\beta\rfloor, \lceil\beta\rceil\}$$

with all coefficients having a modulus $\leq \lfloor\beta\rfloor$ except possibly only one. If β is a simple Parry number :

$$\mathrm{H}(n_\beta) \ = \ \lfloor\beta\rfloor.$$

Key difficulty : factorization of $n_\beta(X)$.

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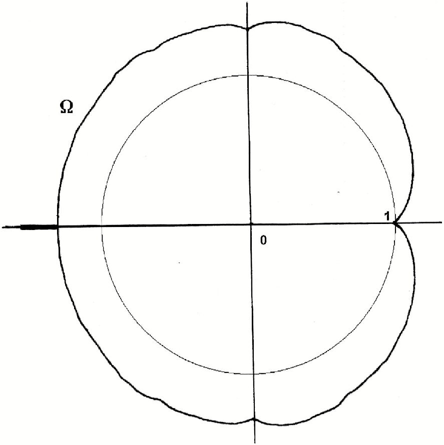

FIG.: Solomyak's fractal set Ω.

$$\mathcal{B} := \{f(z) = 1 + \sum_{j=1}^{\infty} a_j z^j \mid 0 \le a_j \le 1\}$$

functions analytic in the open unit disk $D(0,1)$.

$$\mathcal{G} := \{\xi \in D(0,1) \mid f(\xi) = 0 \text{ for some } f \in \mathcal{B}\}$$

and

$$\mathcal{G}^{-1} := \{\xi^{-1} \mid \xi \in \mathcal{G}\}.$$

External boundary $\partial\mathcal{G}^{-1}$ of $\mathcal{G}^{-1}$: curve with a cusp at $z = 1$, a spike on the negative real axis, $= \left[-\frac{1+\sqrt{5}}{2}, -1\right]$, and is fractal at an infinite number of points.

$$\Omega := \mathcal{G}^{-1} \cup \overline{D(0,1)}.$$

Theorem (Solomyak)

The Galois conjugates ($\neq \beta$) and the beta-conjugates of all Parry numbers β belong to Ω, occupy it densely, and

$$\mathbb{P}_P \cap \Omega = \emptyset.$$

$$f_\beta(z) \;=\; -1+\sum_{i=1}^{\infty} t_i z^i \;=\; (-1+\beta z)\Big(1+\sum_{j=1}^{\infty} T_\beta^j(1) z^j\Big), \qquad |z|<1,$$

-> the zeros $\neq \beta^{-1}$ of $f_\beta(z)$ are those of $1+\sum_{j=1}^{\infty} T_\beta^j(1) z^j$;
and $1+\sum_{j=1}^{\infty} T_\beta^j(1) z^j$ belongs to $\mathcal{B}$.

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$\beta :=$ a Parry number,
$\mathbb{K} :=$ algebraic number field generated by β, its Galois and beta-conjugates (assumed with multiplicity 1), so that $\mathbb{K} \supset \mathbb{Q}(\beta)$.
Weighted sum of Dirac measures :

$$\Delta_\beta := \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{\sigma:\mathbb{K}\to\mathbb{C}} \delta_{\{\sigma(\beta)\}}$$

where (images are Galois- or beta-conjugates) :

$$\sigma(\neq Id) : \quad \beta \to \beta^{(i)} \qquad \text{or} \qquad \sigma : \quad \beta \to \xi_j .$$

Definition :
a sequence of Parry numbers (β_i) is convergent if the sequence

$$(\Delta_{\beta_i})_i$$

has a unique accumulation point.

Topology : a sequence of probability measures $\{\mu_k\}$ on a metric space S wealky converges to μ if for any bounded continuous function $f : S \to \mathbb{R}$ we have $(f, \mu_k) \to (f, \mu)$ as $k \to \infty$.
-> Need : subspace of standard continuous functions with compact support in Solomyak's fractal set Ω.

Absolute logarithmic height of a Parry number β :

$$h(\beta) := \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{v} [\mathbb{K}_v : \mathbb{Q}_v] \max(0, \mathrm{Log}|\beta|_v)$$

Theorem

Let $(\beta_i)_{i\geq 1}$ be a strict sequence of Parry numbers, for which the beta-conjugates have multiplicity one, which satisfies

$$\lim_{i\to\infty} h(\beta_i) \to 0.$$

Then

$$\lim_{i\to\infty} \Delta_{\beta_i} = \nu_{\{|z|=1\}} \qquad \textit{Haar measure.}$$

<u>Strict</u> : A sequence $\{\alpha_k\}$ of points in $\overline{\mathbb{Q}}^*$ is strict if any proper algebraic subgroup of $\overline{\mathbb{Q}}^*$ contains α_k for only finitely many values of k.

Proof : ingredients : Erdős - Turán's Theory, improved by Amoroso and Mignotte. Ex. : if $(\beta_i)_i$ tends to θ, the family of Parry polynomials $(n_{\beta_i})_i$ has CONSTANT height (up to ± 1).

Possible generalizations : to general convergent sequences of Parry numbers with

$$\lim_{i \to +\infty} d_{P,i} = +\infty \quad \text{and} \quad \lim_{i \to +\infty} \frac{\operatorname{Log} \beta_i}{d_{P,i}} = 0,$$

Need : p-adic control of the beta-conjugates to have convergence property for the measure : given by the forms of irreducible factors in the factorization of the Parry polynomials.

Rumely : reformulation in terms of Potential Theory, equilibrium measures, -> A. Granville Theorem. Like in electrostatics, repulsive effects between conjugates...

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Let $k \geq 2$. β_k is the dominant root of the minimal polynomial

$$P_{\beta_k}(X) = X^3 - (k+2)X^2 + 2kX - k.$$

We have : $k < \beta_k < k+1$ and $\lim_{k\to+\infty}(\beta_k - k) = 0$. The length of $d_{\beta_k}(1)$ is $2k+2 = d_P$;

$$f_{\beta_k}(z) = -1 + kz + \sum_{i=2}^{k-1}\left((i-1)z^i + (k-i+1)z^{k+i+1}\right) + kz^k + z^{k+1} + kz^{2k+2}$$

is minus the reciprocal polynomial of the Parry polynomial $n_\beta(X)$. Convergence condition : $(\log \beta_k)/(2k+2) \to 0$.
$k = 30$: the beta-conjugates are the roots of
$\phi_2(X)\phi_3(X)\phi_5(X)\phi_6(X)\phi_{10}(X)\phi_{15}(X)\phi_{30}(X)\phi_{31}(X)$.

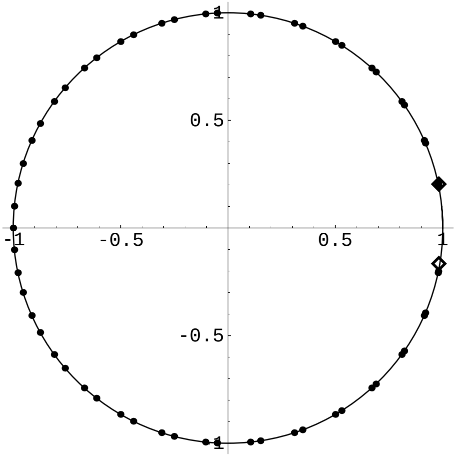


FIG.: Galois conjugates (⋄) and beta-conjugates (•) of the cubic Pisot number $\beta = 30.0356\ldots$, dominant root of $X^3 - 32X^2 + 60X - 30$.

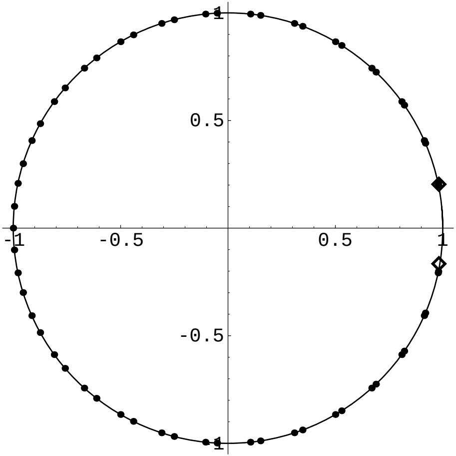


FIG.: $k = 30$

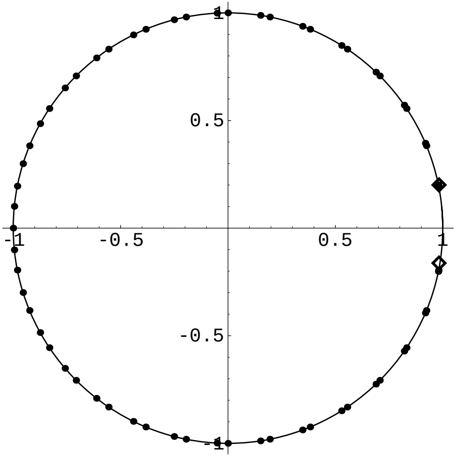


FIG.: $k = 31$

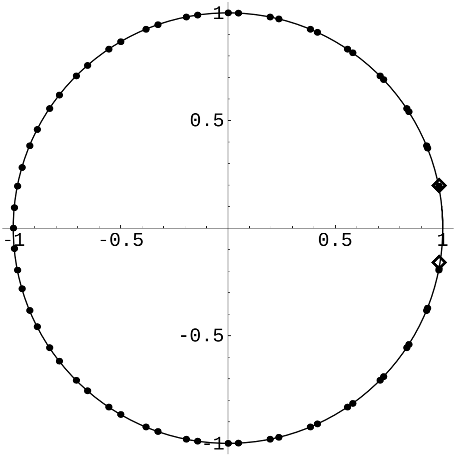


FIG.: $k = 32$

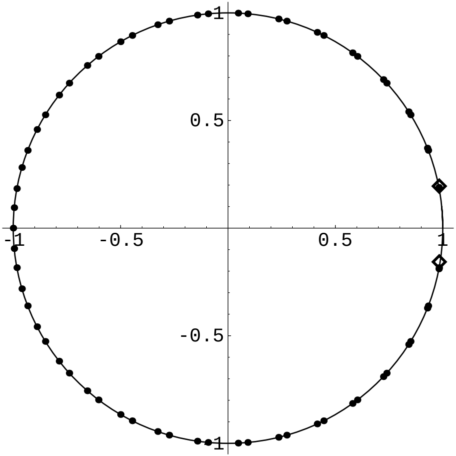


FIG.: $k = 33$

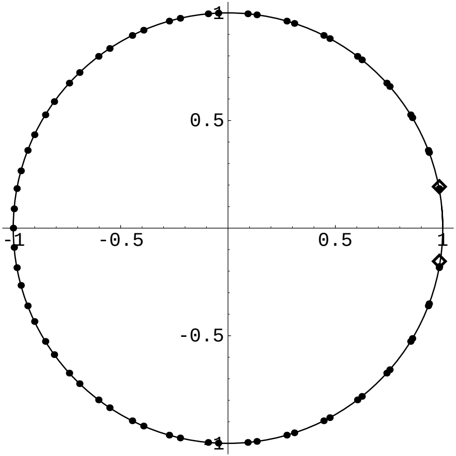


FIG.: $k = 34$

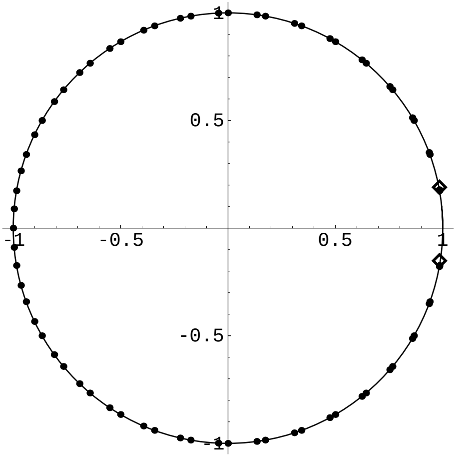


FIG.: $k = 35$

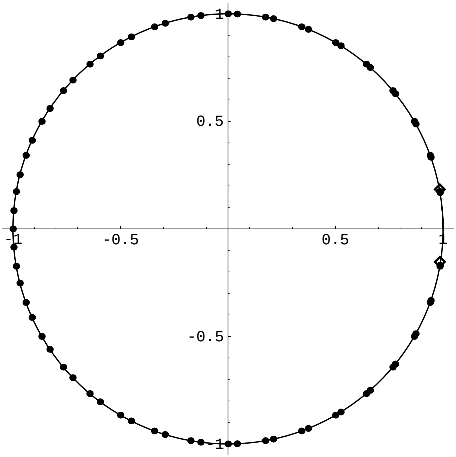


FIG.: $k = 36$

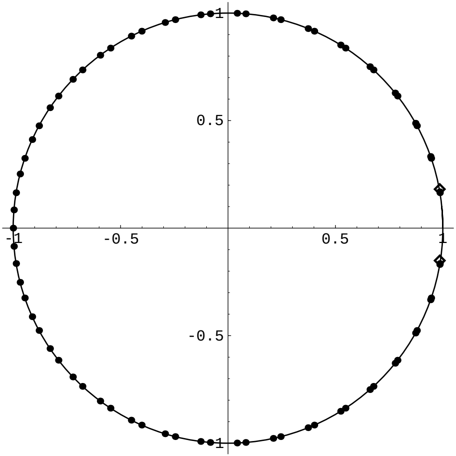
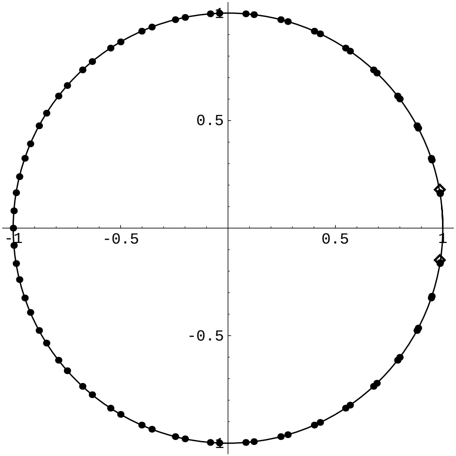


FIG.: $k = 37$



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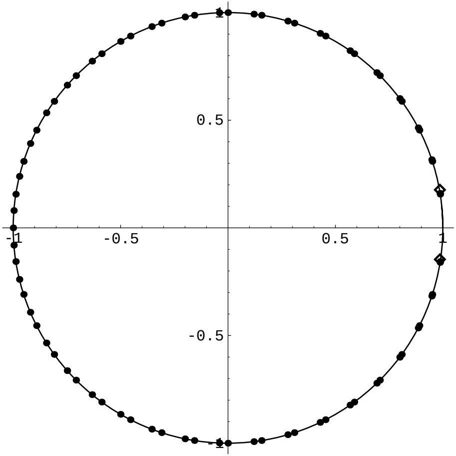

FIG.: $k = 39$

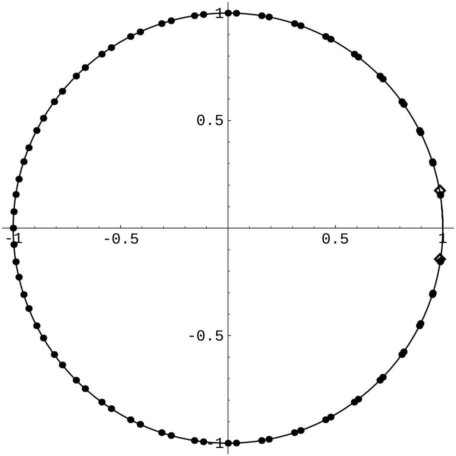

FIG.: $k = 40$

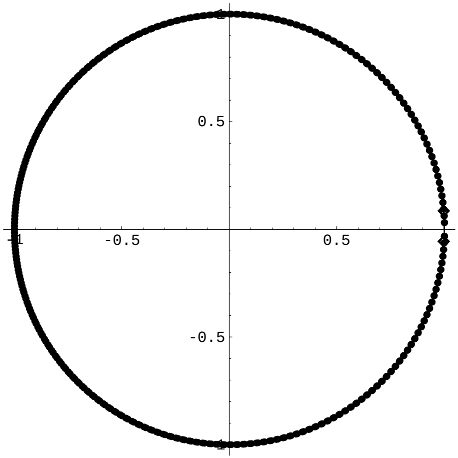

FIG.: $k = 200$

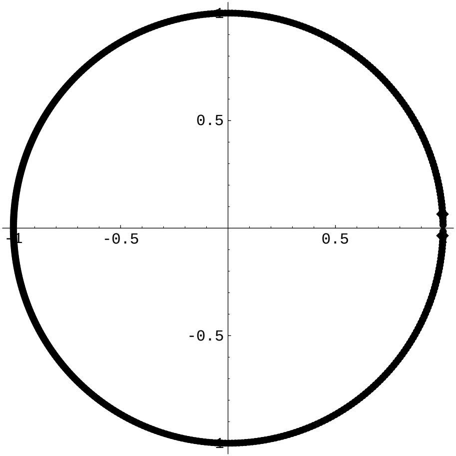

FIG.: $k = 400$

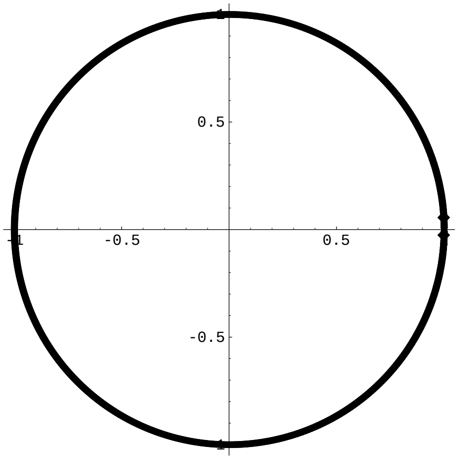

FIG.: $k = 600$