Conjugates of Parry Numbers near the unit circle

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Conjugates of Parry numbers near the unit circle

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1. $d_\beta(1)$ and Artin-Mazur zeta function

2. Solomyak’s fractal set $\Omega$

3. An analog of Bilu’s Equidistribution Limit Theorem

4. Ex. : Bassino’s convergent family of cubic Pisot numbers
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Theorem (Salem, 1945)

*Every P.V. number is a limit point of numbers of the class $(T)$ on both sides.*

open problem : given a sequence of Salem numbers $(\beta_i)$ which converges to $x \in (1, +\infty)$, what is $x$? (i.e. Pisot or Salem?)

degrees $\to +\infty$, huge collection of Galois conjugates,

then a mysterious cancelling occurs.

2 phenomena : 1) limit - concentration, 2) removal.
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dynamical viewpoint: reformulation, since

- a Pisot number is a Parry number (ex-beta-number) [Theorem A. Bertrand-Mathis, K. Schmidt]
- a Salem number is Parry/or nonParry [probabilistic model of Boyd]
- LEFT/RIGHT: limits may be different [W. Parry, 1960]

-> expected: 12 modes of convergence

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Sequences of Parry numbers: what brings dynamics?

beta-conjugates [Boyd] (other roots in the Parry polynomial than the Galois conjugates)
+
equidistribution near the unit circle
Observation: another concentration of conjugates...

-> deep link with theorems on the equidistribution of small points in the closure of torsion subgroups [Szpiro, Ullmo, Zhang, on abelian varieties and Bilu, on $n$-dim algebraic tori (1997)] in arithmetic geometry.

-> idea: to bring back “analogs” to numeration. The ingredients: new, and allow to introduce a new notion of convergence for sequences of Parry numbers. (recall that the set of Parry numbers is dense in $(1, +\infty)$ [Parry]).
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$\beta > 1$ Perron number if algebraic integer and all its Galois conjugates $\beta^{(i)}$ satisfy: $|\beta^{(i)}| < \beta$ for all $i = 1, 2, \ldots, d - 1$ (degree $d \geq 1$, with $\beta^{(0)} = \beta$).

Let $\beta > 1$. The Rényi $\beta$-expansion of 1

$$d_\beta(1) = 0.t_1t_2t_3\ldots$$

and corresponds to

$$1 = \sum_{i=1}^{+\infty} t_i \beta^{-i},$$

$$t_1 = \lfloor \beta \rfloor, \quad t_2 = \lfloor \beta \{\beta \} \rfloor = \lfloor \beta T_\beta(1) \rfloor, \quad t_3 = \lfloor \beta \{\beta \{\beta \} \} \rfloor = \lfloor \beta T^2_\beta(1) \rfloor, \ldots$$

The digits $t_i$ belong to $A_\beta := \{0, 1, 2, \ldots, \lfloor \beta - 1 \rfloor\}$. 
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The digits $t_i$ belong to $A_\beta := \{0, 1, 2, \ldots, \lfloor \beta - 1 \rfloor\}$. 
Parry number: if $d_\beta(1)$ is finite or ultimately periodic (i.e. eventually periodic); in particular, simple if $d_\beta(1)$ is finite.

Lothaire: a Parry number is a Perron number.

Dichotomy: set of Perron numbers

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\[ d_\beta(1) \] and Artin-Mazur zeta function

\[ f_\beta(z) := -1 + \sum_{i=1}^{+\infty} t_i z^i \]

for \( \beta \in \mathbb{P}, z \in \mathbb{C} \),

where \( d_\beta(1) = 0.t_1 t_2 t_3 \ldots \), for which \( f_\beta(z) \) is a rational fraction if and only if \( \beta \in \mathbb{P}_P \) (Szegő’s Theorem).

Dichotomy of Perron numbers \( \beta \leftrightarrow \) dichotomy of analytical functions \( f_\beta(z) \).
Let $\beta > 1$. The beta-transformation is

$$T_\beta : [0, 1] \rightarrow [0, 1], \ x \rightarrow \{ \beta x \}.$$ 

Let $T_\beta^0 = \text{Id}, \ T_\beta^j = T_\beta(T_\beta^{j-1}), \ j \geq 0$.

Define the Artin-Mazur zeta function

$$\zeta_\beta(z) := \exp \left( \sum_{n \geq 1} \frac{P_n}{n} z^n \right)$$

with $P_n = \text{number of fixed points under } T_\beta^n$. 
Theorem (Ito - Takahashi; Flatto, Lagarias, Poonen)

If $\beta$ is a simple Parry number, then

$$\zeta_\beta(z) = \frac{1 - z^N}{(1 - \beta z)(\sum_{n=0}^{\infty} T^*_n(1)z^n)}$$

where $N$ is minimal with $T^*_N(1) = 0$.

If $\beta$ is a (nonsimple) Parry number, then

$$\zeta_\beta(z) = \frac{1}{(1 - \beta z)(\sum_{n=0}^{\infty} T^*_n(1)z^n)}$$
Nonsimple Parry numbers:

\[ f_\beta(z) = \frac{-1}{\zeta_\beta(z)} \]

Simple Parry numbers:

\[ f_\beta(z) = \frac{-1 + z^N}{\zeta_\beta(z)} \]

\[ f_\beta(z) = \frac{-n^*_\beta(z)}{1 - z^p}, \quad n_\beta(z) = \text{multiple of } P_\beta(z) \text{ (minimal polynomial)} \]
$n_\beta(z) =$ the Parry polynomial (called = characteristic polynomial of the beta-number $\beta$, in Parry '60);

$m \geq 0$, non-simple Parry number:

$$n_\beta(X) = X^{m+p+1} - t_1 X^{m+p} - t_2 X^{m+p-1} - \ldots - t_{m+p} X - t_{m+p+1}$$

$$- X^m + t_1 X^{m-1} + t_2 X^{m-2} + \ldots + t_{m-1} X + t_m$$

Simple Parry number ($m \geq 1$):

$$X^m - t_1 X^{m-1} - t_2 X^{m-2} - \ldots - t_{m-1} X - t_m$$

$$t_i := \lfloor \beta T_\beta^{i-1}(1) \rfloor \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}.$$
Key result: the height $H(n_\beta)$ of the Parry polynomial satisfies

$$H(n_\beta) \in \{\lfloor \beta \rfloor, \lceil \beta \rceil \}$$

with all coefficients having a modulus $\leq \lfloor \beta \rfloor$ except possibly only one. If $\beta$ is a simple Parry number:

$$H(n_\beta) = \lfloor \beta \rfloor.$$

Key difficulty: factorization of $n_\beta(X)$. 

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$d_\beta(1)$ and Artin-Mazur zeta function
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Solomyak’s fractal set $\Omega$

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Fig.: Solomyak’s fractal set $\Omega$. 

Solomyak’s fractal set $\Omega$
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Solomyak’s fractal set \( \Omega \)

\[ \mathcal{B} := \{ f(z) = 1 + \sum_{j=1}^{\infty} a_j z^j \mid 0 \leq a_j \leq 1 \} \]

functions analytic in the open unit disk \( D(0, 1) \).

\[ \mathcal{G} := \{ \xi \in D(0, 1) \mid f(\xi) = 0 \text{ for some } f \in \mathcal{B} \} \]

and

\[ \mathcal{G}^{-1} := \{ \xi^{-1} \mid \xi \in \mathcal{G} \}. \]

External boundary \( \partial \mathcal{G}^{-1} \) of \( \mathcal{G}^{-1} \) : curve with a cusp at \( z = 1 \), a spike on the negative real axis, \( = \left[ -\frac{1+\sqrt{5}}{2}, -1 \right] \), and is fractal at an infinite number of points.

\[ \Omega := \mathcal{G}^{-1} \cup \overline{D(0, 1)}. \]
Theorem (Solomyak)

The Galois conjugates \( (\neq \beta) \) and the beta-conjugates of all Parry numbers \( \beta \) belong to \( \Omega \), occupy it densely, and

\[
P\cap \Omega = \emptyset.
\]

\[
f_\beta(z) = -1 + \sum_{i=1}^{\infty} t_i z^i = (-1 + \beta z)(1 + \sum_{j=1}^{\infty} T_\beta^j(1)z^j), \quad |z| < 1,
\]

- the zeros \( \neq \beta^{-1} \) of \( f_\beta(z) \) are those of \( 1 + \sum_{j=1}^{\infty} T_\beta^j(1)z^j \);
and \( 1 + \sum_{j=1}^{\infty} T_\beta^j(1)z^j \) belongs to \( \mathcal{B} \).
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$\beta := \text{a Parry number},$

$\mathbb{K} := \text{algebraic number field generated by } \beta, \text{ its Galois and beta-conjugates (assumed with multiplicity 1), so that } \mathbb{K} \supset \mathbb{Q}(\beta)$.

Weighted sum of Dirac measures:

$$\Delta_{\beta} := \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{\sigma : \mathbb{K} \to \mathbb{C}} \delta_{\{\sigma(\beta)\}}$$

where (images are Galois- or beta-conjugates):

$$\sigma (\neq \text{Id}) : \beta \to \beta^{(i)} \quad \text{or} \quad \sigma : \beta \to \xi_j.$$

**Definition:**

a sequence of Parry numbers $(\beta_i)$ is convergent if the sequence

$$(\Delta_{\beta_i})_i$$

has a unique accumulation point.
Topology: a sequence of probability measures \( \{\mu_k\} \) on a metric space \( S \) weakly converges to \( \mu \) if for any bounded continuous function \( f : S \rightarrow \mathbb{R} \) we have \( (f, \mu_k) \rightarrow (f, \mu) \) as \( k \rightarrow \infty \).

Need: subspace of standard continuous functions with compact support in Solomyak’s fractal set \( \Omega \).

Absolute logarithmic height of a Parry number \( \beta \):

\[
h(\beta) := \frac{1}{[K : \mathbb{Q}]} \sum_v [K_v : \mathbb{Q}_v] \max(0, \log |\beta|_v)
\]
**Theorem**

Let \((\beta_i)_{i \geq 1}\) be a strict sequence of Parry numbers, for which the beta-conjugates have multiplicity one, which satisfies

\[
\lim_{i \to \infty} h(\beta_i) \to 0.
\]

Then

\[
\lim_{i \to \infty} \Delta \beta_i = \nu\{|z|=1\} \quad \text{Haar measure.}
\]

**Strict**: A sequence \(\{\alpha_k\}\) of points in \(\overline{\mathbb{Q}}^*\) is strict if any proper algebraic subgroup of \(\overline{\mathbb{Q}}^*\) contains \(\alpha_k\) for only finitely many values of \(k\).
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An analog of Bilu’s Equidistribution Limit Theorem

Proof: ingredients: Erdős - Turán’s Theory, improved by Amoroso and Mignotte. Ex.: if \((\beta_i)_i\) tends to \(\theta\), the family of Parry polynomials \((n_{\beta_i})_i\) has CONSTANT height (up to \(\pm 1\)).

Possible generalizations: to general convergent sequences of Parry numbers with

\[
\lim_{i \to +\infty} d_{P,i} = +\infty \quad \text{and} \quad \lim_{i \to +\infty} \frac{\log \beta_i}{d_{P,i}} = 0,
\]

Need: \(p\)-adic control of the beta-conjugates to have convergence property for the measure: given by the forms of irreducible factors in the factorization of the Parry polynomials.

Rumely: reformulation in terms of Potential Theory, equilibrium measures, \(\rightarrow\) A. Granville Theorem. Like in electrostatics, repulsive effects between conjugates...
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Let $k \geq 2$. $\beta_k$ is the dominant root of the minimal polynomial

$$P_{\beta_k}(X) = X^3 - (k + 2)X^2 + 2kX - k.$$  

We have : $k < \beta_k < k + 1$ and $\lim_{k \to +\infty}(\beta_k - k) = 0$. The length of $d_{\beta_k}(1)$ is $2k + 2 = d_P$;

$$f_{\beta_k}(z) = -1 + k z + \sum_{i=2}^{k-1} ((i-1)z^i + (k-i+1)z^{k+i+1}) + kz^k + z^{k+1} + kz^{2k+2}$$

is minus the reciprocal polynomial of the Parry polynomial $n_\beta(X)$. Convergence condition : $(\log \beta_k)/(2k + 2) \to 0$.  
$k = 30$ : the beta-conjugates are the roots of

$$\phi_2(X)\phi_3(X)\phi_5(X)\phi_6(X)\phi_{10}(X)\phi_{15}(X)\phi_{30}(X)\phi_{31}(X).$$
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Ex. : Bassino’s convergent family of cubic Pisot numbers

\[ \beta = 30.0356 \ldots, \text{dominant root of } X^3 - 32X^2 + 60X - 30. \]
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Ex.: Bassino's convergent family of cubic Pisot numbers

Fig.: $k = 30$
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Ex.: Bassino’s convergent family of cubic Pisot numbers

Fig.: $k = 31$
Conjugates of Parry numbers near the unit circle

Ex. : Bassino’s convergent family of cubic Pisot numbers

\[ F_{\text{ig.}:} k = 32 \]
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Ex.: Bassino's convergent family of cubic Pisot numbers

Fig.: $k = 33$
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Ex.: Bassino’s convergent family of cubic Pisot numbers

Fig.: $k = 34$
Conjugates of Parry numbers near the unit circle

Ex. : Bassino’s convergent family of cubic Pisot numbers

Fig.: $k = 35$
Conjugates of Parry numbers near the unit circle

Ex.: Bassino’s convergent family of cubic Pisot numbers

\[ k = 36 \]
Conjugates of Parry numbers near the unit circle

Ex.: Bassino’s convergent family of cubic Pisot numbers

\[ F_{\mathbf{G}} \]

\[ k = 37 \]

**Figure:** $k = 37$
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Ex.: Bassino's convergent family of cubic Pisot numbers

Fig.: \( k = 38 \)
Conjugates of Parry numbers near the unit circle

Ex.: Bassino’s convergent family of cubic Pisot numbers

Figure: $k = 39$
Conjugates of Parry numbers near the unit circle

Ex.: Bassino’s convergent family of cubic Pisot numbers

\[ \text{Fig.: } k = 40 \]
Conjugates of Parry numbers near the unit circle

Ex.: Bassino’s convergent family of cubic Pisot numbers

\[ k = 200 \]
Conjugates of Parry numbers near the unit circle

Ex. : Bassino’s convergent family of cubic Pisot numbers

*Fig.*: $k = 400$
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Ex.: Bassino’s convergent family of cubic Pisot numbers

Fig.: $k = 600$