A Q-ANALOG OF THE MARKOFF
INJECTIVITY CONJECTURE HOLDS

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Abstract. The elements of Markoff triples are given by coefficients in cer-
tain matrix products defined by Christoffel words, and the Markoff injectivity
conjecture, a long-standing open problem, is then equivalent to injectivity on
Christoffel words. A $q$-analog of these matrix products has been proposed
recently by Leclere and Morier-Genoud, and we prove that injectivity on
Christoffel words holds for this $q$-analog. The proof is based on the eval-
uation at $q = \exp(2\pi i/6)$. Other roots of unity provide some information on
the original problem, which corresponds to the case $q = 1$. We also extend the
problem to arbitrary words and provide a large family of pairs of words where
injectivity does not hold.

1. Introduction

Christoffel words are words over the alphabet $\{0, 1\}$ defined recursively as follows:
0, 1 and 01 are Christoffel words and if $u, v, uv \in \{0, 1\}^*$ are Christoffel words then
$uuv$ and $uvv$ are Christoffel words [BLRS09]. The shortest Christoffel words are:
0, 1, 01, 001, 011, 0011, 00101, 01011, 0111, 0001, 001001, 00100101, 0010101, · · ·
Note that these are usually named lower Christoffel words.

A Markoff triple is a positive solution of the Diophantine equation
$x^2 + y^2 + z^2 = 3xyz$ [Mar80, Mar79]. Markoff triples can be defined recursively as follows: $(1, 1, 1),
(1, 2, 1)$ and $(1, 5, 2)$ are Markoff triples and if $(x, y, z)$ is a Markoff triple with $y \geq x
and y \geq z$, then $(x, 3xy - z, y)$ and $(y, 3yz - x, z)$ are Markoff triples. A list of small
Markoff numbers (elements of a Markoff triple) is
1, 2, 5, 13, 29, 34, 89, 194, 233, 433, 610, 985, 1325, 1597, 2897, 4181, · · ·
referred as sequence A002559 in OEIS [OEI22].

It is known that each Markoff number can be expressed in terms of a Christoffel
word. More precisely, let $\mu$ be the monoid homomorphism $\{0, 1\}^* \to \text{GL}_2(\mathbb{Z})$ defined by
$\mu(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mu(1) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$.
Each Markoff number is equal to $\mu(w)_{12}$ for some Christoffel word $w$ [Reu09], where
$M_{12}$ denotes the element above the diagonal in a matrix $M = (M_{11}, M_{12}, M_{21}, M_{22}) \in \text{GL}_2(\mathbb{Z})$.

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For example, the Markoff number 194 is associated with the Christoffel word 00101 as it is the entry at position (1, 2) in the matrix
\[
\mu(00101) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 463 & 194 \\ 284 & 119 \end{pmatrix}.
\]

Whether the map \( w \mapsto \mu(w)_{12} \) provides a bijection between Christoffel words and Markoff numbers is a question (stated differently in [Fro13]) that has remained open for more than 100 years [Aig13]. The conjecture can be expressed in terms of the injectivity of the map \( w \mapsto \mu(w)_{12} \) [Ren19] §3.3.

**Conjecture 1.1** (Markoff Injectivity Conjecture). The map \( w \mapsto \mu(w)_{12} \) is injective on the set of Christoffel words.

In [LL22], a \( q \)-analog of the Markoff Injectivity Conjecture was considered based on the \( q \)-analog of \( \mu(0) \) and \( \mu(1) \) proposed in [LMG21], which in terms of

\[
L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix} \quad \text{and} \quad R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}
\]

can be written as

\[
\mu_q(0) = R_qL_q = \begin{pmatrix} q + q^2 & 1 \\ q & 1 \end{pmatrix},
\]

\[
\mu_q(1) = R_qR_qL_qL_q = \begin{pmatrix} q + 2q^2 + q^3 + q^4 & 1 + q \\ q + q^2 & 1 \end{pmatrix}.
\]

It extends to a morphism of monoids \( \mu_q : \{0, 1\}^* \to \GL_2(\mathbb{Z}[q^{\pm1}]). \) The \( q \)-analog satisfies that \( \mu_1(w) = \mu(w) \) for every \( w \in \{0, 1\}^*. \) Thus if \( w \) is a Christoffel word, then the entry above the diagonal \( \mu_q(w)_{12} \) is a polynomial of indeterminate \( q \) with nonnegative integer coefficients such that it is a Markoff number when evaluated at \( q = 1. \) For example,

\[
\mu_q(00101)_{12} = 1 + 4q + 10q^2 + 18q^3 + 27q^4 + 33q^5 + 33q^6 + 29q^7 + 21q^8 + 12q^9 + 5q^{10} + q^{11}
\]

which, when evaluated at \( q = 1, \) is equal to

\[
\mu_1(00101)_{12} = 1 + 4 + 10 + 18 + 27 + 33 + 33 + 29 + 21 + 12 + 5 + 1 = 194.
\]

In this work, we prove the \( q \)-analog of the Markoff Injectivity Conjecture.

**Theorem 1.2.** The map \( w \mapsto \mu_q(w)_{12} \) is injective on the set of Christoffel words.

**Theorem 1.2** is proved in Section 3. In Section 3, we give examples where the map \( w \mapsto \mu_q(w)_{12} \) is not injective when considered on the language \( \{0, 1\}^*. \)

## 2. Proof of Theorem 1.2

The main idea of this section is to evaluate the polynomial \( \mu_q(w)_{12} \) at primitive root of unity \( \zeta_k = \exp(2\pi i / k), \) in particular when \( k = 6. \)

First, we observe that when \( w \in \{0, 1\}^*, \) the matrix \( \mu_{\zeta_6}(w) \) can be expressed in terms of \( \zeta_6, \) the length \(|w|\) of \( w \) and the number \(|w|_1\) of occurrences of 1 in \( w. \)

**Lemma 2.1.** For every \( w \in \{0, 1\}^*, \) we have

\[
(1) \quad \mu_{\zeta_6}(w) = \zeta_6^{|w|+|w|_1} \begin{pmatrix} |w| & -|w| & |w|_1 \\ -|w|_1 & -|w| & 1 \end{pmatrix} \zeta_6 + \begin{pmatrix} |w|_1 & |w|_1 & |w| \\ |w|_1 & |w|_1 & -|w|_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Proof. The proof is done by recurrence on the length of $w$. We have $\mu_{\zeta_6}(\varepsilon) = (1 \ 0 \ 0)$. Thus, the formula works for $w = \varepsilon$. If (1) holds for $w$, then we have

$$
\mu_{\zeta_6}(w0) = \zeta_6^{[w] + |w|+1} \left( |w|_1 + 1 + |w| \zeta_6 |w| - (|w|+|w|_1) \zeta_6 \begin{pmatrix} 1+\zeta_6 & 1-\zeta_6 \\ 1 & 1 \end{pmatrix} \right) \zeta_6^{1+\zeta_6} \begin{pmatrix} 1 & 1-\zeta_6 \\ 1 & 1 \end{pmatrix}
$$

$$
= \zeta_6^{[w] + |w|+1} \left( |w|_1 + 1 + (|w|+1) \zeta_6 |w| + 1 - (|w|+|w|_1+1) \zeta_6 \begin{pmatrix} 1 & 1-\zeta_6 \\ 1 & 1 \end{pmatrix} \right),
$$

$$
\mu_{\zeta_6}(w1) = \zeta_6^{[w] + |w|+1} \left( |w|_1 + 1 + |w| \zeta_6 |w| - (|w|+|w|_1) \zeta_6 \begin{pmatrix} 1+\zeta_6 & 1-\zeta_6 \\ 1 & 1 \end{pmatrix} \right) \zeta_6^{2+\zeta_6} \begin{pmatrix} 2+\zeta_6 & 1-2\zeta_6 \\ 2 & 1 \end{pmatrix}
$$

$$
= \zeta_6^{[w] + |w|+2} \left( |w|_1 + 2 + (|w|+1) \zeta_6 |w| + 1 - (|w|+|w|_1+1) \zeta_6 \begin{pmatrix} 1 & 1-\zeta_6 \\ 1 & 1 \end{pmatrix} \right),
$$

hence (1) holds for $w0$ and $w1$. □

In particular, Equation (1) implies that the entry above the diagonal is

$$
\mu_{\zeta_6}(w)_{12} = \zeta_6^{[w] + |w|} (|w| - (|w| + |w|_1) \zeta_6) \in \mathbb{C}.
$$

The next result shows that when $w \in \{0, 1\}^* \setminus \{\varepsilon\}$, the number $\mu_{\zeta_6}(w)_{12}$ lies in one of the six cones of angle $\pi/3$ that partition the complex plane according to the value of $|w| + |w|_1$, see Figure 1.

Lemma 2.2. For every $w \in \{0, 1\}^* \setminus \{\varepsilon\}$, we have

$$
\mu_{\zeta_6}(w)_{12} \in \{ \rho \cdot e^{i\theta} \mid \rho > 0, (|w| + |w|_1 + 4)\frac{\pi}{3} < \theta \leq (|w| + |w|_1 + 5)\frac{\pi}{3} \}.
$$

Moreover, $w = \varepsilon$ if and only if $\mu_{\zeta_6}(w)_{12} = 0$.

Proof. Let $w \in \{0, 1\}^* \setminus \{\varepsilon\}$. Since $|w| + |w|_1 \geq |w| > 0$, then observe that

$$
|w| - (|w| + |w|_1) \zeta_6 \in \{ \rho \cdot e^{i\theta} \mid \rho > 0, \frac{4\pi}{3} < \theta \leq \frac{5\pi}{3} \}.
$$

Figure 1. A partition of the complex plane $\mathbb{C} \setminus \{0\}$ into six disjoint cones spanned by the vectors $\zeta_6^k$ and $\zeta_6^{k+1}$, $k \in \{0, 1, 2, 3, 4, 5\}$. For every $w \in \{0, 1\}^* \setminus \{\varepsilon\}$, $\mu_{\zeta_6}(w)_{12}$ lies in the cone corresponding to $|w| + |w|_1 \mod 6$. For instance, $\mu_{\zeta_6}(011) = \zeta_6^3(3-5\zeta_6) = 3\zeta_6^5 - 5$ and $|w| + |w|_1 \equiv 5 \mod 6$. 

Proof.
Since \( \zeta_6 = e^{i\pi/2} \), from Equation (2), we have

\[
\mu_{\zeta_6}(w)_{12} = \zeta_6^{\lfloor |w| + |w_1| \rfloor} \left( (|w| - (|w| + |w_1|)\zeta_6 \right)
\]

\[
e^{\frac{\pi i}{2}(|w| + |w_1|)} \left\{ \rho \cdot e^{i\theta} | \rho > 0, \frac{4\pi}{3} < \theta \leq \frac{5\pi}{3} \right\} = \left\{ \rho \cdot e^{i\theta} | \rho > 0, \frac{(|w| + |w_1| + 4)\pi}{3} < \theta \leq \frac{(|w| + |w_1| + 5)\pi}{3} \right\}.
\]

We have \( \mu_{\zeta_6}(w)_{12} = 0 \) if \( w = \varepsilon \) and, from above, \( \mu_{\zeta_6}(w)_{12} \neq 0 \) if \( w \in \{0, 1\}^* \setminus \{\varepsilon\} \).

Thus, if \( \mu_{\zeta_6}(w)_{12} = 0 \), then \( w = \varepsilon \).

The next result shows that we can recover the number of 0’s and 1’s occurring in a word \( w \in \{0, 1\}^* \) from the the polynomial \( \mu_q(w)_{12} \) evaluated at \( q = \zeta_6 \).

**Proposition 2.3.** Let \( w, w' \in \{0, 1\}^* \). If \( \mu_{\zeta_6}(w)_{12} = \mu_{\zeta_6}(w')_{12} \), then \( |w|_0 = |w'|_0 \) and \( |w|_1 = |w'|_1 \).

**Proof.** If \( \mu_{\zeta_6}(w)_{12} = \mu_{\zeta_6}(w')_{12} = 0 \), then from Lemma 2.2 we have \( w = \varepsilon = w' \), thus \( |w|_0 = 0 = |w'|_0 \) and \( |w|_1 = 0 = |w'|_1 \). Now, assume that \( \mu_{\zeta_6}(w)_{12} = \mu_{\zeta_6}(w')_{12} \neq 0 \). From Lemma 2.2 we have

\[
\mu_{\zeta_6}(w)_{12} = \left\{ \rho \cdot e^{i\theta} | \rho > 0, \frac{|w| + |w_1 + 4\pi}{3} < \theta \leq \frac{|w'| + |w'|_1 + 5\pi}{3} \right\},
\]

\[
\mu_{\zeta_6}(w')_{12} = \left\{ \rho \cdot e^{i\theta} | \rho > 0, \frac{|w'| + |w'|_1 + 4\pi}{3} < \theta \leq \frac{|w| + |w_1| + 5\pi}{3} \right\},
\]

which are two disjoint cones in the complex plane when \( |w| + |w_1| \not\equiv |w'| + |w'|_1 \mod 6 \). Since \( \mu_{\zeta_6}(w)_{12} = \mu_{\zeta_6}(w')_{12} \), the two cones must intersect and be equal. Therefore, we have \( |w| + |w|_1 = |w'| + |w'|_1 \mod 6 \). From Lemma 2.1 we have

\[
|w'|-\lfloor |w'|+|w'|_1 \rfloor \zeta_6 = \zeta_6^{\lfloor |w'|-|w'|_1 \rfloor} \mu_{\zeta_6}(w)_{12}
\]

\[
= \zeta_6^{\lfloor |w'|-|w'|_1 \rfloor} \mu_{\zeta_6}(w)_{12}
\]

\[
= \zeta_6^{\lfloor |w'|-|w'|_1 \rfloor} \cdot \frac{\zeta_6^{\lfloor |w|+|w_1| \rfloor} - \lfloor |w|+|w_1| \rfloor \zeta_6}{\zeta_6^{\lfloor |w|+|w_1| \rfloor} - \lfloor |w|+|w_1| \rfloor \zeta_6}.
\]

This implies that \( |w'| = |w| \) and \( |w'| + |w'|_1 = |w| + |w|_1 \). Then \( |w|_1 = |w'|_1 \) and \( |w|_0 = |w| - |w|_1 = |w'| - |w'|_1 = |w'|_0 \).

We may now prove the main result.

**Proof of Theorem 1.2.** We want to show the injectivity of the map \( w \mapsto \mu_q(w)_{12} \) over the set of Christoffel words. Let \( w, w' \in \{0, 1\}^* \) be two Christoffel words such that \(\mu_q(w)_{12} = \mu_q(w')_{12} \). In particular, we have \( \mu_{\zeta_6}(w)_{12} = \mu_{\zeta_6}(w')_{12} \). From Proposition 2.3 \( |w|_0 = |w'|_0 \) and \( |w|_1 = |w'|_1 \). Thus \( w = w' \) because there exists a unique Christoffel word with a fixed number of 0’s and 1’s. Therefore the map \( w \mapsto \mu_{\zeta_6}(w)_{12} \) is injective over the set of Christoffel words, and so is the map \( w \mapsto \mu_q(w)_{12} \).

**Remark 2.4.** Note that the monoid generated by \( \mu_{\zeta_6}(0) \) and \( \mu_{\zeta_6}(1) \) is a finite group when \( k \in \{2, 3, 4, 5\} \), and we observe the following relations between the residue class of \( \mu_{\zeta_6}(w)_{12} \mod k \) and \( \mu_{\zeta_6}(w)_{12} \) for \( w \in \{0, 1\}^* \) (we leave the proof to the reader):

- \( \mu_1(w)_{12} \mod 2 \equiv \begin{cases} 0 & \text{if and only if } \mu_{-1}(w)_{12} = 0, \\ 1 & \text{if and only if } \mu_{-1}(w)_{12} \in \{-1, 1\}, \end{cases} \)
• $\mu_1(w)_{12} \pmod{3} \equiv \begin{cases} 0 & \text{if and only if } \mu_{\zeta_3}(w)_{12} = 0, \\ 1 & \text{if and only if } \mu_{\zeta_3}(w)_{12} \in \{1, \zeta_3, \zeta_3^2\}, \\ 2 & \text{if and only if } \mu_{\zeta_3}(w)_{12} \in \{-1, -\zeta_3, -\zeta_3^2\}, \end{cases}$

• $\mu_1(w)_{12} \pmod{4} \equiv \begin{cases} 0 & \text{if and only if } \mu_i(w)_{12} = 0, \\ 1 \text{ or } 3 & \text{if and only if } \mu_i(w)_{12} \in \{\pm 1, \pm i\}, \\ 2 & \text{if and only if } \mu_i(w)_{12} \in \{1 \pm i, -1 \pm i\}. \end{cases}$

The value $\mu_1(w)_{12} \pmod{5}$ can also be deduced from $\mu_{\zeta_5}(w)_{12}$, which takes 31 distinct values in the complex plane, see Figure 2. For $k \geq 6$, the monoid generated by $\mu_{\zeta_k}(0)$ and $\mu_{\zeta_k}(1)$ is infinite, and we have not found relations between the residue class of $\mu_1(w)_{12} \pmod{k}$ and $\mu_{\zeta_k}(w)_{12}$.

**Figure 2.** For $w \in \{0,1\}^*$, $\mu_{\zeta_5}(w)_{12}$ takes 31 different values. The set $A_k = \{\mu_{\zeta_5}(w)_{12} : \mu_1(w)_{12} \equiv k \pmod{5}, w \in \{0,1\}^*\}$ consists of the vertices of a regular pentagon when $k \in \{1, 2, 3, 4\}$, of the vertices of a regular decagon and the origin when $k = 0$.

3. $w \mapsto \mu_q(w)_{12}$ is not injective on $\{0,1\}^*$

In this section, we provide a list of pairs of words over the alphabet $\{0,1\}$ for which $w \mapsto \mu_q(w)_{12}$ is not injective. For example, 00011 and 01001 have the same image as we have

$$\mu_q(00011)_{12} = 1 + 4q + 10q^2 + 19q^3 + 27q^4 + 33q^5 + 34q^6 + 29q^7 + 21q^8 + 12q^9 + 5q^{10} + q^{11} = \mu_q(01001)_{12}.$$
The section contains two results: Theorem 3.1 and Theorem 3.2. All pairs of words we know of are of form of Equation (4) or Equation (7). So we believe they completely describe the pairs of words $x, y \in \{0, 1\}^*$ such that $\mu_q(x)_{12} = \mu_q(y)_{12}$.

### 3.1. First result.
To state the results, we need the two involutions $w \mapsto \bar{w}$ and $w \mapsto \overline{w}$ on $\{0, 1\}^*$ which are defined by $\bar{w} = w_k \cdots w_1$ and $\overline{w} = \overline{w_k} \cdots \overline{w_1}$ if $w = w_1 \cdots w_k$, with $\overline{1} = 1$, $\overline{0} = 0$, i.e., $\bar{w}$ is the mirror image of $w$ and $\overline{w}$ is obtained from $\bar{w}$ by exchanging $0$ and $1$. Also, more generally, we consider images of the homomorphism

$$ M_q : \{0, 1\}^* \rightarrow \text{GL}_2(\mathbb{Z}[q^{\pm1}]), \quad 0 \mapsto L_q, \quad 1 \mapsto R_q $$

which will be used to prove identities for $\mu_q$ since $\mu_q(0) = M_q(10)$ and $\mu_q(1) = M_q(1100)$.

**Theorem 3.1.** For all $w \in \{0, 1\}^*$, $k, m, n \geq 0$, we have

$$(3) \quad M_q(0^k w 10^m)_{12} = M_q(0^k \overline{w} 10^m)_{12},$$

$$(4) \quad \mu_q(0w1)_{12} = \mu_q(0\bar{w}1)_{12}. $$

**Proof.** We have $M_q(0^k w 0^m)_{12} = \sigma^k M_q(w)_{12}$ for all $w \in \{0, 1\}^*$, $k, m \geq 0$, because $(1, 0) L_q = (q, 0)$ and $L_q(1, 0) = (1, 0)$. Hence, it suffices to prove (3) for $k = m = n = 0$. Since

$$ Q_q L_q Q_q^{-1} = t R_q, \quad \text{and} \quad Q_q R_q Q_q^{-1} = t L_q, \quad \text{with} \quad Q_q = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, $$

we have, for $w = w_1 \cdots w_k \in \{0, 1\}^*$,

$$(5) \quad Q_q M_q(w) Q_q^{-1} = t M_q(\overline{w}) = t M_q(\overline{w}) = t M_q(\overline{w})$$

and thus

$$ M_q(1w1)_{12} = (R_q Q_q^{-1} M_q(\overline{w}) Q_q R_q)_{12} = (1, 1) t M_q(\overline{w}) \cdot (q, 1) = (q, 1) M_q(\overline{w}) \cdot (1, 1) = M_q(10110)_{12},$$

using that $1 \times 1$ matrices are invariant under transposition. This proves (3).

Let $\sigma : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be the homomorphism given by $\sigma(0) = 10$ and $\sigma(1) = 1100$. Then we have $\mu_q(w) = M_q(\sigma(w))$ and $\sigma(\bar{w}) = \sigma(\overline{w})$ for all $w \in \{0, 1\}^*$, thus

$$ \mu_q(0w1)_{12} = M_q(10\sigma(w)1100)_{12} = M_q(10\sigma(\overline{w})1100)_{12} = M_q(10\sigma(\overline{w})1100)_{12} = \mu_q(0\bar{w}1)_{12},$$

where we have used (3) and $\overline{w} = 0\overline{w}$ for the second equation. \hfill $\square$

Recall that if $0w1 \in \{0, 1\}^*$ is a Christoffel word, then $w$ is a palindrome. Therefore Theorem 3.1 is compatible with Theorem 1.2.

### 3.2. Second result.
We obtain more identities by images of the homomorphisms with $w \in \{0, 1\}^*$

$$ \varphi_w : \{0, 1, 2, 3\}^* \rightarrow \{0, 1\}^*, \quad \varphi_w(0) = w011010101, \quad \varphi_w(1) = w100110101, \quad \varphi_w(2) = w011010101, \quad \varphi_w(3) = w100110101,$$

$$ \psi_w : \{0, 1, 2, 3\}^* \rightarrow \{0, 1\}^*, \quad \psi_w(0) = w010w01, \quad \psi_w(1) = w10\overline{w}10, \quad \psi_w(2) = w010w01, \quad \psi_w(3) = w10\overline{w}10.$$

We extend the involution $w \mapsto \overline{w}$ to $\{0, 1, 2, 3\}^*$ by setting $\overline{2} = 3$ and $\overline{3} = 2$. 
Theorem 3.2. For all \( w \in \{0,1\}^* \), \( v \in \{0,1,2,3\}^* \), \( k, m, n \geq 0 \), we have
\[
M_q\left(0^k1\varphi_w(v)w10^n\right)_{12} = M_q\left(0^k1\varphi_w(\tau)vw10^n\right)_{12}.
\]
(6)
\[
\mu_q(0\varphi_w(\tau)v1)_{12} = \mu_q(0\varphi_w(v)w1)_{12}.
\]
(7)

For the proof of the theorem, we decompose \( \varphi_w = \eta_w \circ \tau \) with
\[
\eta_w : \{0,1,2,3\}^* \to \{0,1\}^* \quad \text{and} \quad \tau : \{0,1,2,3\}^* \to \{0,1,2,3\}^*.
\]
and we use the homomorphism
\[
\eta_w' : \{0,1,2,3\}^* \to \{0,1\}^* \quad \text{and} \quad \tau : \{0,1,2,3\}^* \to \{0,1,2,3\}^*.
\]
satisfying \( \varphi_w(\tau)v = \eta_w(\tau(v))w = \eta_w'\tau(v) \). We have to show that the difference
\[
\Delta_w(v) = M_q(1\eta_w(v)w1)_{12} - M_q(1\eta_w'(\tau(v))1)_{12}
\]
is zero for all \( v \in \tau(\{0,1,2,3\}^*) = \{02,03,12,13\}^* = \{0,1\}\{2,3\}^* \).

Lemma 3.3. Let \( a \in \{2,3\} \), \( v \in \{\{0,1\}\{2,3\}\}^* \), \( w \in \{0,1\}^* \). If \( \Delta_w(v) = 0 \), then
\[
\Delta_w(u0\overline{av}) = \Delta_w(u1\overline{av})
\]
for all \( u \in \{\{0,1\}\{2,3\}\}^* \) and
\[
\Delta_w(u2\overline{av}) = \Delta_w(u3\overline{av})
\]
for all \( u \in \{\{0,1\}\{2,3\}\}^* \{0,1\} \).

Proof. Assume first that \( |u| \) is even. Then
\[
\Delta_w(u0\overline{av}) - \Delta_w(u1\overline{av})
\]
\[
= M_q(1\eta_w(u0\overline{av})w1)_{12} - M_q(1\eta_w(u1\overline{av})v1)_{12}
\]
\[
+ M_q(1\eta_w'(\overline{v0u})1)_{12} - M_q(1\eta_w'(\overline{v1u})1)_{12}
\]
\[
= \left(M_q(1\eta_w(u)v)\right) \left(M_q(0110) - M_q(1001)\right)M_q(1\eta_w'(\overline{v0u})w1)_{12}
\]
\[
+ \left(M_q(1\eta_w'(\overline{v0u}))\right) \left(M_q(0110) - M_q(1001)\right)M_q(1\eta_w'(\overline{v0u})1)_{12}
\]
\[
= (q^3 + 1) \left(M_q(1\eta_w(u)v)S \overline{Qq}M_q(1\eta_w'(\overline{v0u})w1)_{12}
\]
\[
+ (q^3 + 1) \left(M_q(1\eta_w'(\overline{v0u}))S \overline{Qq}M_q(1\eta_w'(\overline{v0u})1)_{12}
\]
\[
= (q^3 + 1) q^{1|\eta_w(u)v|} \left(R_qS \overline{Qq}M_q(1\eta_w'(\overline{v0u})w1)_{12}
\]
\[
+ (q^3 + 1) q^{1|\eta_w'(\overline{v0u})|} \left(M_q(1\eta_w'(\overline{v0u})S \overline{Qq}R_q)_{12}
\]
\[
= (q^3 + 1) q^{1|\eta_w(u)v|} d_a \left(M_q(1\eta_w(v)w1)_{12} - M_q(1\eta_w'(\overline{v0u})1)_{12}
\]
\[
= (q^3 + 1) q^{1|\eta_w(u)v|} d_a \Delta_w(v),
\]
with \( d_2 = -q \) and \( d_3 = q^4 \). Here, we use for the third equation that
\[
M_q(0110) = M_q(1001) + (q^3+1) SQ_q, \quad \text{where} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Q_q = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}.
\]
For the fourth equation, we use that, by [3],
\[
M_q(z) SQ_q M_q(\overline{z}) = M_q(z) S^4 M_q(z) Q_q = \det(M_q(z)) SQ_q = q^{1|z|} SQ_q
\]
for all \( z \in \{0,1\}^* \), in particular for \( z = \eta_w(u)w \) (with \( \tau = \eta_w(\tau)\bar{w} \)) and for \( z = \bar{w} \eta_w(u) \) (with \( \tau = w \eta_w(\tau) \)).

For the fifth equation, we use that
\[
(1,0) R_q SQ_q M_q(\bar{w}^{-1} \eta_q(2)) = (1,0) R_q SQ_q M_q(0110) = -(q^2, q) = -(q, 0) R_q,
\]
\[
(1,0) R_q SQ_q M_q(\bar{w}^{-1} \eta_q(3)) = (1,0) R_q SQ_q M_q(1001) = (q^5, q^4) = q^4(1,0) M_q(1),
\]
\[
M_q(\eta_q(3)\bar{w}^{-1}) SQ_q R_q(0,1) = M_q(1001) SQ_q R_q(0,1) = (q, q) = q M_q(1) q(0,1),
\]
\[
M_q(\eta_q(2)\bar{w}^{-1}) SQ_q R_q(0,1) = M_q(0110) SQ_q R_q(0,1) = -(q^4, q^4) = -q^4 R_q(0,1).
\]

Therefore, \( \Delta_w(v) = 0 \) implies that \( \Delta_w(u 0 \bar{u} a v) = \Delta_w(u 1 \bar{u} a v). \)

The proof of \( \Delta_w(u 2 \bar{u} a v) = \Delta_w(u 3 \bar{u} a v) \) for odd \(|u|\) runs along the same lines. \( \square \)

**Lemma 3.4.** For all \( v \in \{(0,1)\} \{2,3\} \), \( w \in \{0,1\}^* \), we have \( \Delta_w(v) = 0 \).

**Proof.** We proceed by induction on the length of \( v \). The statement is trivially true for \(|v| = 0\). Suppose that it is true up to length \( k - 1 \) and consider it for length \( k \).

We claim that the value of \( \Delta_w(v_1 \cdots v_{2k}) \) does not depend on the choice of \( v_1 \cdots v_j \), for any \( j \leq k \). The claim is true for \( j = 1 \), by Lemma 3.3 with \( u = \epsilon \) and the induction hypothesis. If the claim is true up to \( j - 1 \), then it gives together with Lemma 3.3 for any \( u_1 \cdots u_j \in \{(0,1)\} \{2,3\} \) \( \cup \{(0,1)\} \{2,3\} \) \{0,1\}, that
\[
\Delta_w(u_1 \cdots u_j v_{j+1} \cdots v_{2k}) = \Delta_w(v_{j+1} \cdots v_{2j-1} u_j v_{j+1} \cdots v_{2k}) = \Delta_w(v_1 \cdots v_{2k}).
\]

This proves the claim.

Since \( \eta_w(\tau)w = w \eta'_w(\tau) \bar{w} \) for all \( w \in \{(0,1)\} \{2,3\} \) \( \cup \{(0,1)\} \{2,3\} \) \{0,1\}, we have \( \Delta_w(v_1 \cdots v_{2k}) = 0 \) for all \( v_1 \cdots v_{2k} \in \{(0,1)\} \{2,3\} \) \{0,1\}. \( \square \)

**Proof of Theorem 3.1.** As for (6), it suffices to prove (1) for \( k = m = n = 0 \). Since \( \varphi_w(v) = \eta_w(\tau(v)) \) and \( \varphi_w(\tau)w = \eta_w(\bar{w}) \) for all \( w \in \{0,1\}^* \), \( v \in \{0,1,2,3\}^* \), Lemma 3.4 implies that (6) holds.

Let \( \sigma \) be as in the proof of Theorem 3.1. Then
\[
\mu_q(0 \psi_w(v)w1)_{12} = M_q(1 \eta_0 \sigma(w)1 \bar{v} \sigma(w)1100) = M_q(1 \eta_0 \sigma(w)1 \bar{v} \sigma(w)1100) = \mu_q(0 \psi_w(\bar{v})w1)_{12},
\]
using that \( \sigma(\psi_w(v)) \bar{v} = \eta_0 \sigma(w)1 \bar{v} \), and using (6) for the second equation. \( \square \)

The equation \( M_q(x)_{12} = M_q(y)_{12} \) has many solutions \( x, y \in \{0,1\}^* \) which are not of the form of Equation (3) or (6), for example
\[
M_q(1100000011)_{12} = 1+2q+3q^2+4q^3+4q^4+4q^5+3q^6+2q^7+q^8 = M_q(1000000011)_{12},
\]
but we believe that Equations (4) and (7) are complete.

**Question 3.5.** Do there exist \( x, y \in \{0,1\}^* \) satisfying \( \mu_q(x)_{12} = \mu_q(y)_{12} \) which are not given by Equation (4) or (7)?

**References**


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