A Q-ANALOG OF THE MARKOFF INJECTIVITY CONJECTURE HOLDS

SÉBASTIEN LABBÉ, MÉLODIE LAPOINTE, AND WOLFGANG STEINER

Abstract. The elements of Markoff triples are given by coefficients in certain matrix products defined by Christoffel words, and the Markoff injectivity conjecture, a long-standing open problem (also known as the uniqueness conjecture), is then equivalent to injectivity on Christoffel words. A q-analog of these matrix products has been proposed recently, and we prove that injectivity on Christoffel words holds for this q-analog. The proof is based on the evaluation at $q = \exp(i\pi/3)$. Other roots of unity provide some information on the original problem, which corresponds to the case $q = 1$. We also extend the problem to arbitrary words and provide a large family of pairs of words where injectivity does not hold.

1. Introduction

Christoffel words are words over the alphabet $\{\mathtt{0},\mathtt{1}\}$ that can be defined recursively as follows: $\mathtt{0}$, $\mathtt{1}$ and $\mathtt{01}$ are Christoffel words and if $u, v, uv \in \{\mathtt{0},\mathtt{1}\}^*$ are Christoffel words then uuv and uvv are Christoffel words [BLRS09]. The shortest Christoffel words are:

$$\mathtt{0}, \mathtt{1}, \mathtt{01}, \mathtt{001}, \mathtt{011}, \mathtt{0001}, \mathtt{00101}, \mathtt{01011}, \mathtt{0111}, \mathtt{00001}, \mathtt{0001001}, \mathtt{00100101}, \mathtt{0010101}, \ldots$$

Note that these are also named *lower* Christoffel words.

A Markoff triple is a positive solution of the Diophantine equation $x^2+y^2+z^2 = 3xyz$ [Mar80, Mar79]. Markoff triples can be defined recursively as follows: $(1, 1, 1)$, $(1, 2, 1)$ and $(1, 5, 2)$ are Markoff triples and if (x, y, z) is a Markoff triple with $y \geq x$ and $y \geq z$, then $(x, 3xy - z, y)$ and $(y, 3yz - x, z)$ are Markoff triples. A list of small Markoff numbers (elements of a Markoff triple) is

$$1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, 2897, 4181, \ldots$$

referenced as sequence A002559 in OEIS [OEI22].

It is known that each Markoff number can be expressed in terms of a Christoffel word. More precisely, let μ be the monoid homomorphism $\{\mathtt{0},\mathtt{1}\}^* \to \mathrm{GL}_2(\mathbb{Z})$ defined by

$$\mu(\mathtt{0}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mu(\mathtt{1}) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

Each Markoff number is equal to $\mu(w)_{12}$ for some Christoffel word w [Reu09], where M_{12} denotes the element above the diagonal in a matrix $M = \left(\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$.

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For example, the Markoff number 194 is associated with the Christoffel word $\mathtt{00101}$ as it is the entry at position $(1,2)$ in the matrix

$$\mu(\mathtt{00101}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 463 & 194 \\ 284 & 119 \end{pmatrix}.$$

Whether the map $w \mapsto \mu(w)_{12}$ provides a bijection between Christoffel words and Markoff numbers is a question (stated differently in [Fro13]) that has remained open for more than 100 years [Aig13]. The conjecture can be expressed in terms of the injectivity of the map $w \mapsto \mu(w)_{12}$ [Reu19, §3.3].

Conjecture 1.1 (Markoff Injectivity Conjecture)**.** *The map $w \mapsto \mu(w)_{12}$ is injective on the set of Christoffel words.*

In [MGO20], a q-analog of rational numbers and of continued fractions were introduced. This was the inspiration for several advances [MSS21, Ovs21, CV22, BMS22, KOR23] and among them a q-analog of Markoff triples [Kog20]. A q-analog of the matrices $\mu(\mathtt{0})$ and $\mu(\mathtt{1})$ was proposed in [LMG21], which in terms of

$$L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix} \quad \text{and} \quad R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix},$$

can be written as

$$\begin{aligned} \mu_q(\mathtt{0}) &= R_q L_q = \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix}, \\ \mu_q(\mathtt{1}) &= R_q R_q L_q L_q = \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}. \end{aligned}$$

It extends to a morphism of monoids $\mu_q : \{\mathtt{0},\mathtt{1}\}^* \to \mathrm{GL}_2(\mathbb{Z}[q^{\pm 1}])$. This q-analog satisfies that $\mu_1(w) = \mu(w)$ for every $w \in \{\mathtt{0},\mathtt{1}\}^*$. Thus if w is a Christoffel word, then the entry above the diagonal $\mu_q(w)_{12}$ is a polynomial of indeterminate q with nonnegative integer coefficients such that it is a Markoff number when evaluated at $q = 1$. For example,

$$\mu_q(\mathtt{00101})_{12} = 1+4q+10q^2+18q^3+27q^4+33q^5+33q^6+29q^7+21q^8+12q^9+5q^{10}+q^{11}$$

which, when evaluated at $q = 1$, is equal to

$$\mu_1(\mathtt{00101})_{12} = 1+4+10+18+27+33+33+29+21+12+5+1 = 194.$$

In [LL22], a q-analog of the Markoff Injectivity Conjecture was considered based on the map $w \mapsto \mu_q(w)_{12}$. It was proved that the map is injective over the language of any fixed Christoffel word, extending a result proved when $q = 1$ [LR21]. In this work, we go one step further and prove a q-analog of the Markoff Injectivity Conjecture.

Theorem 1.2. *The map $w \mapsto \mu_q(w)_{12}$ is injective on the set of Christoffel words.*

Theorem 1.2 is proved in Section 2. In Section 3, we give examples where the map $w \mapsto \mu_q(w)_{12}$ is not injective when considered on the language $\{\mathtt{0},\mathtt{1}\}^*$.

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2. Proof of Theorem 1.2

The main idea of this section is to evaluate the polynomial $\mu_q(w)_{12}$ at primitive root of unity $\zeta_k = \exp(2\pi i/k)$, in particular when $k = 6$.

First, we observe that when $w \in \{\mathtt{0},\mathtt{1}\}^*$, the matrix $\mu_{\zeta_6}(w)$ can be expressed in terms of ζ_6, the length $|w|$ of w and the number $|w|_\mathtt{1}$ of occurrences of $\mathtt{1}$ in w.

Lemma 2.1. *For every $w \in \{\mathtt{0},\mathtt{1}\}^*$, we have*

$$\mu_{\zeta_6}(w) = \zeta_6^{|w|+|w|_\mathtt{1}} \left[\begin{pmatrix} |w| & -|w|-|w|_\mathtt{1} \\ -|w|_\mathtt{1} & -|w| \end{pmatrix}\zeta_6 + \begin{pmatrix} |w|_\mathtt{1} & |w| \\ |w|+|w|_\mathtt{1} & -|w|_\mathtt{1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]. \tag{1}$$

Proof. The proof is done by recurrence on the length of w. We have $\mu_{\zeta_6}(\varepsilon) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Thus, the formula works for $w = \varepsilon$. If (1) holds for w, then we have

$$\begin{aligned}
\mu_{\zeta_6}(w\mathtt{0}) &= \zeta_6^{|w|+|w|_\mathtt{1}} \begin{pmatrix} |w|_\mathtt{1} + 1 + |w|\,\zeta_6 & |w| - (|w|+|w|_\mathtt{1})\,\zeta_6 \\ |w| + |w|_\mathtt{1} - |w|_\mathtt{1}\,\zeta_6 & 1 - |w|_\mathtt{1} - |w|\,\zeta_6 \end{pmatrix} \zeta_6 \begin{pmatrix} 1+\zeta_6 & 1-\zeta_6 \\ 1 & 1-\zeta_6 \end{pmatrix} \\
&= \zeta_6^{|w|+|w|_\mathtt{1}+1} \begin{pmatrix} |w|_\mathtt{1} + 1 + (|w|+1)\,\zeta_6 & |w| + 1 - (|w|+|w|_\mathtt{1}+1)\,\zeta_6 \\ |w| + |w|_\mathtt{1} + 1 - |w|_\mathtt{1}\,\zeta_6 & 1 - |w|_\mathtt{1} - (|w|+1)\,\zeta_6 \end{pmatrix}, \\
\mu_{\zeta_6}(w\mathtt{1}) &= \zeta_6^{|w|+|w|_\mathtt{1}} \begin{pmatrix} |w|_\mathtt{1} + 1 + |w|\,\zeta_6 & |w| - (|w|+|w|_\mathtt{1})\,\zeta_6 \\ |w| + |w|_\mathtt{1} - |w|_\mathtt{1}\,\zeta_6 & 1 - |w|_\mathtt{1} - |w|\,\zeta_6 \end{pmatrix} \zeta_6^2 \begin{pmatrix} 2+\zeta_6 & 1-2\zeta_6 \\ 2-\zeta_6 & -\zeta_6 \end{pmatrix} \\
&= \zeta_6^{|w|+|w|_\mathtt{1}+2} \begin{pmatrix} |w|_\mathtt{1} + 2 + (|w|+1)\,\zeta_6 & |w| + 1 - (|w|+|w|_\mathtt{1}+2)\,\zeta_6 \\ |w| + |w|_\mathtt{1} + 2 - (|w|_\mathtt{1}+1)\,\zeta_6 & -|w|_\mathtt{1} - (|w|+1)\,\zeta_6 \end{pmatrix},
\end{aligned}$$

hence (1) holds for $w\mathtt{0}$ and $w\mathtt{1}$. □

In particular, Equation (1) implies that the entry above the diagonal is

$$\mu_{\zeta_6}(w)_{12} = \zeta_6^{|w|+|w|_\mathtt{1}} \left(|w| - (|w| + |w|_\mathtt{1})\zeta_6\right) \in \mathbb{C}. \tag{2}$$

The next result shows that when $w \in \{\mathtt{0},\mathtt{1}\}^* \setminus \{\varepsilon\}$, the number $\mu_{\zeta_6}(w)_{12}$ lies in one of the six cones of angle $\frac{\pi}{3}$ that partition the complex plane according to the value of $|w| + |w|_\mathtt{1}$, see Figure 1.

Lemma 2.2. *For every $w \in \{\mathtt{0},\mathtt{1}\}^* \setminus \{\varepsilon\}$, we have*

$$\mu_{\zeta_6}(w)_{12} \in \left\{\rho \cdot e^{i\theta} \mid \rho > 0,\ (|w| + |w|_\mathtt{1} + 4)\tfrac{\pi}{3} < \theta \leq (|w| + |w|_\mathtt{1} + 5)\tfrac{\pi}{3}\right\}.$$

Moreover, $w = \varepsilon$ if and only if $\mu_{\zeta_6}(w)_{12} = 0$.

Proof. Let $w \in \{\mathtt{0},\mathtt{1}\}^* \setminus \{\varepsilon\}$. Since $|w| + |w|_\mathtt{1} \geq |w| > 0$, then observe that

$$|w| - (|w| + |w|_\mathtt{1})\zeta_6 \in \left\{\rho \cdot e^{i\theta} \mid \rho > 0,\ \tfrac{4\pi}{3} < \theta \leq \tfrac{5\pi}{3}\right\}.$$

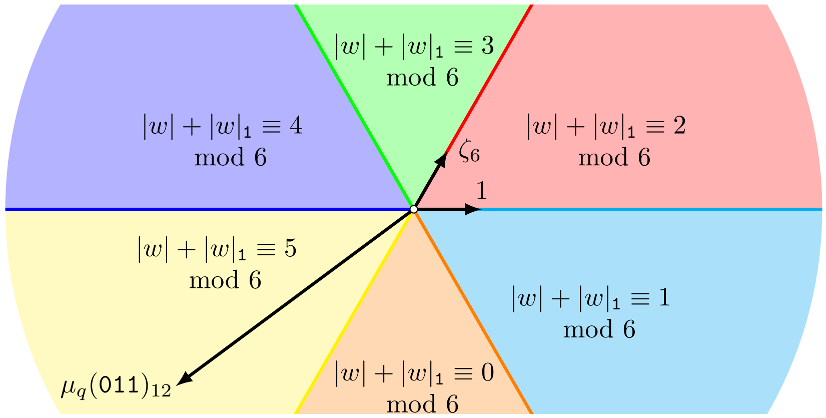

FIGURE 1. A partition of the complex plane $\mathbb{C} \setminus \{0\}$ into six disjoint cones spanned by the vectors ζ_6^k and ζ_6^{k+1}, $k \in \{0,1,2,3,4,5\}$. For every $w \in \{\mathtt{0},\mathtt{1}\}^* \setminus \{\varepsilon\}$, $\mu_q(w)_{12}$ lies in the cone corresponding to $|w|+|w|_\mathtt{1} \mod 6$. For instance, $\mu_q(\mathtt{011}) = \zeta_6^5(3-5\zeta_6) = 3\zeta_6^5 - 5$ and $|w| + |w|_\mathtt{1} = |\mathtt{011}| + |\mathtt{011}|_\mathtt{1} = 3 + 2 \equiv 5 \mod 6$.

Since $\zeta_6 = e^{\frac{i\pi}{3}}$, from Equation (2), we have

$$\begin{aligned}
\mu_{\zeta_6}(w)_{12} &= \zeta_6^{|w|+|w|_\mathtt{1}} \big(|w| - (|w| + |w|_\mathtt{1})\zeta_6\big) \\
&\in e^{\frac{i\pi}{3}(|w|+|w|_\mathtt{1})} \left\{\rho \cdot e^{i\theta} \mid \rho > 0,\ \tfrac{4\pi}{3} < \theta \leq \tfrac{5\pi}{3}\right\} \\
&= \left\{\rho \cdot e^{i\theta} \mid \rho > 0,\ \tfrac{(|w|+|w|_\mathtt{1}+4)\pi}{3} < \theta \leq \tfrac{(|w|+|w|_\mathtt{1}+5)\pi}{3}\right\}.
\end{aligned}$$

We have $\mu_{\zeta_6}(w)_{12} = 0$ if $w = \varepsilon$ and, from above, $\mu_{\zeta_6}(w)_{12} \neq 0$ if $w \in \{\mathtt{0},\mathtt{1}\}^* \setminus \{\varepsilon\}$. Thus, if $\mu_{\zeta_6}(w)_{12} = 0$, then $w = \varepsilon$. □

The next result shows that we can recover the number of $\mathtt{0}$'s and $\mathtt{1}$'s occurring in a word $w \in \{\mathtt{0},\mathtt{1}\}^*$ from the polynomial $\mu_q(w)_{12}$ evaluated at $q = \zeta_6$.

Proposition 2.3. *Let $w, w' \in \{\mathtt{0},\mathtt{1}\}^*$. If $\mu_{\zeta_6}(w)_{12} = \mu_{\zeta_6}(w')_{12}$, then $|w|_\mathtt{0} = |w'|_\mathtt{0}$ and $|w|_\mathtt{1} = |w'|_\mathtt{1}$.*

Proof. If $\mu_{\zeta_6}(w)_{12} = \mu_{\zeta_6}(w')_{12} = 0$, then from Lemma 2.2, we have $w = \varepsilon = w'$, thus $|w|_\mathtt{0} = 0 = |w'|_\mathtt{0}$ and $|w|_\mathtt{1} = 0 = |w'|_\mathtt{1}$. Now, assume that $\mu_{\zeta_6}(w)_{12} = \mu_{\zeta_6}(w')_{12} \neq 0$. From Lemma 2.2, we have

$$\begin{aligned}
\mu_{\zeta_6}(w)_{12} &\in \left\{\rho \cdot e^{i\theta} \mid \rho > 0,\ (|w| + |w|_\mathtt{1} + 4)\tfrac{\pi}{3} < \theta \leq (|w'| + |w'|_\mathtt{1} + 5)\tfrac{\pi}{3}\right\}, \\
\mu_{\zeta_6}(w')_{12} &\in \left\{\rho \cdot e^{i\theta} \mid \rho > 0,\ (|w'| + |w'|_\mathtt{1} + 4)\tfrac{\pi}{3} < \theta \leq (|w'| + |w'|_\mathtt{1} + 5)\tfrac{\pi}{3}\right\},
\end{aligned}$$

which are two disjoint cones in the complex plane when $|w| + |w|_\mathtt{1} \not\equiv |w'| + |w'|_\mathtt{1} \mod 6$. Since $\mu_{\zeta_6}(w)_{12} = \mu_{\zeta_6}(w')_{12}$, the two cones must intersect and be equal. Therefore, we have $|w| + |w|_\mathtt{1} \equiv |w'| + |w'|_\mathtt{1} \mod 6$. From Lemma 2.1, we have

$$\begin{aligned}
|w'| - (|w'| + |w'|_\mathtt{1})\zeta_6 &= \zeta_6^{-|w'|-|w'|_\mathtt{1}} \mu_{\zeta_6}(w')_{12} \\
&= \zeta_6^{-|w'|-|w'|_\mathtt{1}} \mu_{\zeta_6}(w)_{12} \\
&= \zeta_6^{-|w'|-|w'|_\mathtt{1}} \zeta_6^{|w|+|w|_\mathtt{1}} \big(|w| - (|w| + |w|_\mathtt{1})\zeta_6\big) \\
&= |w| - (|w| + |w|_\mathtt{1})\zeta_6.
\end{aligned}$$

This implies that $|w'| = |w|$ and $|w'| + |w'|_1 = |w| + |w|_1$. Then $|w|_1 = |w'|_1$ and $|w|_0 = |w| - |w|_1 = |w'| - |w'|_1 = |w'|_0$. □

We may now prove the main result. It is based on the isomorphism between the tree of Christoffel words and the Stern–Brocot tree, a tree of positive rational numbers. Indeed, the set of Christoffel words has the structure of a binary tree: if $u, v, uv \in \{0,1\}^*$ are Christoffel words, then uuv and uvv are the left and right children of the node uv [BLRS09, §3.2]. The Christoffel tree is isomorphic to the Stern–Brocot tree via the map that associates to a vertex w of the Christoffel tree the fraction $\frac{|w|_1}{|w|_0}$ [BLRS09, Proposition 7.6].

Proof of Theorem 1.2. We want to show the injectivity of the map $w \mapsto \mu_q(w)_{12}$ over the set of Christoffel words. Let $w, w' \in \{0,1\}^*$ be two Christoffel words such that $\mu_q(w)_{12} = \mu_q(w')_{12}$. In particular, we have $\mu_{\zeta_6}(w)_{12} = \mu_{\zeta_6}(w')_{12}$. From Proposition 2.3, $|w|_0 = |w'|_0$ and $|w|_1 = |w'|_1$.

Suppose by contradiction that $w \neq w'$. This implies that the fraction $\frac{|w|_1}{|w|_0} = \frac{|w'|_1}{|w'|_0}$ appears twice in the Stern–Brocot tree. This is a contradiction because every positive rational number appears in the Stern–Brocot tree exactly once [GKP94, §4.5]. Thus $w = w'$. Therefore the map $w \mapsto \mu_{\zeta_6}(w)_{12}$ is injective over the set of Christoffel words, and so is the map $w \mapsto \mu_q(w)_{12}$. □

Remark 2.4. The monoid generated by $\mu_{\zeta_k}(0)$ and $\mu_{\zeta_k}(1)$ is a finite group if and only if $k \in \{2,3,4,5\}$. Indeed, the monoid generated by $\zeta_k^{-1}\mu_{\zeta_k}(0)$ and $\zeta_k^{-2}\mu_{\zeta_k}(1)$ is isomorphic to the cyclic group C_3 when $k = 2$, the quaternion group Q_8 when $k = 3$, the special linear groups $\mathrm{SL}_2(\mathbb{F}_3)$ and $\mathrm{SL}_2(\mathbb{F}_5)$ when $k = 4$ and $k = 5$ respectively; since $\zeta_k^k = 1$, the corresponding monoids generated by $\mu_{\zeta_k}(0)$ and $\mu_{\zeta_k}(1)$ are also finite groups. For $k = 6$, the monoid is by Lemma 2.1 isomorphic to the abelianization of $\{0,1\}^*$, i.e., $(\mathbb{N}^2, +)$. For $k \geq 7$, the matrix $\zeta_k^{-1}\mu_{\zeta_k}(0)$ has an eigenvalue > 1 since $\zeta_k + \zeta_k^{-1} > 1$ and the characteristic polynomial of $q^{-1}\mu_q(0)$ is $x^2 - (q+1+q^{-1})x + 1$, which implies that the generated monoid is infinite, thus the monoid generated by $\mu_{\zeta_k}(0)$ is also infinite.

For $k \in \{2,3,4,5\}$, we also observe the following relations between the residue class of $\mu_1(w)_{12} \pmod k$ and $\mu_{\zeta_k}(w)_{12}$ for $w \in \{0,1\}^*$ (these relations hold not only for the 12-coefficient but for all coefficients of $\mu_1(w)$ and $\mu_{\zeta_k}(w)$ and can be verified by induction on the length of w):

- $\mu_1(w)_{12} \pmod 2 \equiv \begin{cases} 0 \text{ if and only if } \mu_{-1}(w)_{12} = 0, \\ 1 \text{ if and only if } \mu_{-1}(w)_{12} \in \{-1, 1\}, \end{cases}$
- $\mu_1(w)_{12} \pmod 3 \equiv \begin{cases} 0 \text{ if and only if } \mu_{\zeta_3}(w)_{12} = 0, \\ 1 \text{ if and only if } \mu_{\zeta_3}(w)_{12} \in \{1, \zeta_3, \zeta_3^2\}, \\ 2 \text{ if and only if } \mu_{\zeta_3}(w)_{12} \in \{-1, -\zeta_3, -\zeta_3^2\}, \end{cases}$
- $\mu_1(w)_{12} \pmod 4 \equiv \begin{cases} 0 \text{ if and only if } \mu_i(w)_{12} = 0, \\ 1 \text{ or } 3 \text{ if and only if } \mu_i(w)_{12} \in \{\pm 1, \pm i\}, \\ 2 \text{ if and only if } \mu_i(w)_{12} \in \{1 \pm i, -1 \pm i\}. \end{cases}$

The value $\mu_1(w)_{12} \mod 5$ can also be deduced from $\mu_{\zeta_5}(w)_{12}$, which takes 31 distinct values in the complex plane, see Figure 2. For $k \geq 6$, we have not found relations between the residue class of $\mu_1(w)_{12} \pmod k$ and $\mu_{\zeta_k}(w)_{12}$.

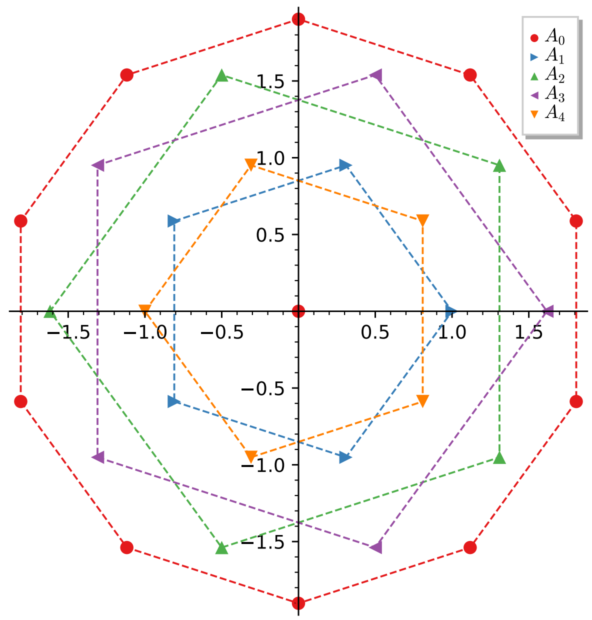

FIGURE 2. For $w \in \{\mathtt{0},\mathtt{1}\}^*$, $\mu_{\zeta_5}(w)_{12}$ takes 31 different values. The set $A_k = \{\mu_{\zeta_5}(w)_{12} \mid \mu_1(w)_{12} \equiv k \mod 5,\, w \in \{\mathtt{0},\mathtt{1}\}^*\}$ consists of the vertices of a regular pentagon when $k \in \{1,2,3,4\}$, of the vertices of a regular decagon and the origin when $k = 0$.

3. $w \mapsto \mu_q(w)_{12}$ is not injective on $\{\mathtt{0},\mathtt{1}\}^*$

In this section, we provide a list of pairs of words over the alphabet $\{\mathtt{0},\mathtt{1}\}$ for which $w \mapsto \mu_q(w)_{12}$ is not injective. For example, 00011 and 01001 have the same image as we have

$$\begin{aligned}\mu_q(\mathtt{00011})_{12} &= 1{+}4q{+}10q^2{+}19q^3{+}27q^4{+}33q^5{+}34q^6{+}29q^7{+}21q^8{+}12q^9{+}5q^{10}{+}q^{11}\\ &= \mu_q(\mathtt{01001})_{12}.\end{aligned}$$

The section contains two results: Theorem 3.1 and Theorem 3.2. All pairs of words we know of are of form of Equation (4) or Equation (7). So we believe they completely describe the pairs of words $x, y \in \{\mathtt{0},\mathtt{1}\}^*$ such that $\mu_q(x)_{12} = \mu_q(y)_{12}$.

3.1. **First result.** To state the results, we need the two involutions $w \mapsto \widetilde{w}$ and $w \mapsto \overline{w}$ on $\{\mathtt{0},\mathtt{1}\}^*$ which are defined by $\widetilde{w} = w_k \cdots w_1$ and $\overline{w} = \overline{w_k} \cdots \overline{w_1}$ if $w = w_1 \cdots w_k$, with $\overline{\mathtt{0}} = \mathtt{1}$, $\overline{\mathtt{1}} = \mathtt{0}$, i.e., $\widetilde{w}$ is the mirror image of w and $\overline{w}$ is obtained from $\widetilde{w}$ by exchanging 0 and 1. Also, more generally, we consider images of the homomorphism

$$M_q : \{\mathtt{0},\mathtt{1}\}^* \to \mathrm{GL}_2(\mathbb{Z}[q^{\pm 1}]), \quad \mathtt{0} \mapsto L_q,\ \mathtt{1} \mapsto R_q$$

which will be used to prove identities for μ_q since $\mu_q(\mathtt{0}) = M_q(\mathtt{10})$ and $\mu_q(\mathtt{1}) = M_q(\mathtt{1100})$.

Theorem 3.1. *For all $w \in \{\mathtt{0},\mathtt{1}\}^*$, $k,m,n \geq 0$, we have*

$$M_q\big(\mathtt{0}^k\mathtt{1}w\mathtt{1}\mathtt{0}^m\big)_{12} = M_q\big(\mathtt{0}^k\mathtt{1}\overline{w}\mathtt{1}\mathtt{0}^n\big)_{12}, \tag{3}$$

$$\mu_q(\mathtt{0}w\mathtt{1})_{12} = \mu_q(\mathtt{0}\widetilde{w}\mathtt{1})_{12}. \tag{4}$$

Proof. We have $M_q(\mathtt{0}^k w \mathtt{0}^m)_{12} = q^k M_q(w)_{12}$ for all $w \in \{\mathtt{0},\mathtt{1}\}^*$, $k,m \geq 0$, because $(1,0)\,L_q = (q,0)$ and $L_q{}^t(1,0) = {}^t(1,0)$. Hence, it suffices to prove (3) for $k = m = n = 0$. Since

$$Q_q\,L_q\,Q_q^{-1} = {}^tR_q, \quad \text{and} \quad Q_q\,R_q\,Q_q^{-1} = {}^tL_q, \quad \text{with } Q_q = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix},$$

we have, for $w = w_1 \cdots w_\ell \in \{\mathtt{0},\mathtt{1}\}^*$,

$$Q_q\,M_q(w)\,Q_q^{-1} = {}^tM_q(\overline{w_1}) \cdots {}^tM_q(\overline{w_\ell}) = {}^tM_q(\overline{w_\ell} \cdots \overline{w_1}) = {}^tM_q(\overline{w}) \tag{5}$$

and thus

$$\begin{aligned} M_q(\mathtt{1}w\mathtt{1})_{12} &= \big(R_q Q_q^{-1}\,{}^tM_q(\overline{w}) Q_q R_q\big)_{12} = (1,1)\,{}^tM_q(\overline{w})\,{}^t(q,1) = (q,1)\,M_q(\overline{w})\,{}^t(1,1) \\ &= M_q(\mathtt{1}\overline{w}\mathtt{1})_{12}, \end{aligned}$$

using that 1×1 matrices are invariant under transposition. This proves (3).

Let $\sigma : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*$ be the homomorphism given by $\sigma(\mathtt{0}) = \mathtt{10}$ and $\sigma(\mathtt{1}) = \mathtt{1100}$. Then we have $\mu_q(w) = M_q(\sigma(w))$ and $\overline{\sigma(w)} = \sigma(\widetilde{w})$ for all $w \in \{\mathtt{0},\mathtt{1}\}^*$, thus

$$\begin{aligned} \mu_q(\mathtt{0}w\mathtt{1})_{12} &= M_q\big(\mathtt{10}\sigma(w)\mathtt{1100}\big)_{12} = M_q\big(\mathtt{10}\overline{\sigma(w)}\mathtt{1100}\big)_{12} = M_q\big(\mathtt{10}\sigma(\widetilde{w})\mathtt{1100}\big)_{12} \\ &= \mu_q(\mathtt{0}\widetilde{w}\mathtt{1})_{12}, \end{aligned}$$

where we have used (3) and $\overline{\mathtt{0}w\mathtt{1}} = \mathtt{0}\overline{w}\mathtt{1}$ for the second equation. $\square$

Recall that if $\mathtt{0}w\mathtt{1} \in \{\mathtt{0},\mathtt{1}\}^*$ is a Christoffel word, then w is a palindrome [Reu19, Theorem 2.3.1]. Therefore Theorem 3.1 is compatible with Theorem 1.2.

3.2. **Second result.** We obtain more identities by images of the homomorphisms (with $w \in \{\mathtt{0},\mathtt{1}\}^*$)

$$\begin{aligned} \varphi_w : \{\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3}\}^* \to \{\mathtt{0},\mathtt{1}\}^*, \quad & \mathtt{0} \mapsto w\mathtt{0110}\overline{w}\mathtt{0110}, \quad \mathtt{2} \mapsto w\mathtt{0110}\overline{w}\mathtt{1001}, \\ & \mathtt{1} \mapsto w\mathtt{1001}\overline{w}\mathtt{1001}, \quad \mathtt{3} \mapsto w\mathtt{1001}\overline{w}\mathtt{0110}, \\ \psi_w : \{\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3}\}^* \to \{\mathtt{0},\mathtt{1}\}^*, \quad & \mathtt{0} \mapsto w\mathtt{01}\widetilde{w}\mathtt{01}, \quad \mathtt{2} \mapsto w\mathtt{01}\widetilde{w}\mathtt{10}, \\ & \mathtt{1} \mapsto w\mathtt{10}\widetilde{w}\mathtt{10}, \quad \mathtt{3} \mapsto w\mathtt{10}\widetilde{w}\mathtt{01}, \end{aligned}$$

and the involution $w \mapsto \widehat{w}$ on $\{\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3}\}^*$ defined by $\widehat{w} = \widehat{w_k} \cdots \widehat{w_1}$ if $w = w_1 \cdots w_k$, with $\widehat{\mathtt{0}} = \mathtt{1}$, $\widehat{\mathtt{1}} = \mathtt{0}$, $\widehat{\mathtt{2}} = \mathtt{2}$, $\widehat{\mathtt{3}} = \mathtt{3}$. Note that $\widehat{w} = \overline{w}$ for $w \in \{\mathtt{0},\mathtt{1}\}^*$. For the proof of the following theorem, we also use the extension of $w \mapsto \overline{w}$ to $\{\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3}\}^*$ defined by $\overline{\mathtt{2}} = \mathtt{3}$, $\overline{\mathtt{3}} = \mathtt{2}$.

Theorem 3.2. *For all $w \in \{\mathtt{0},\mathtt{1}\}^*$, $v \in \{\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3}\}^*$, $k,m,n \geq 0$, we have*[1]

$$M_q\big(\mathtt{0}^k\mathtt{1}\varphi_w(v)w\mathtt{1}\mathtt{0}^m\big)_{12} = M_q\big(\mathtt{0}^k\mathtt{1}\varphi_w(\widehat{v})w\mathtt{1}\mathtt{0}^n\big)_{12}. \tag{6}$$

$$\mu_q\big(\mathtt{0}\psi_w(v)w\mathtt{1}\big)_{12} = \mu_q\big(\mathtt{0}\psi_w(\widehat{v})w\mathtt{1}\big)_{12}. \tag{7}$$

[1]In the version of the article that is published in *Algebraic Combinatorics*, the two equations in Theorem 3.2 are erroneously stated with $\overline{v}$ instead of $\widehat{v}$.

For the proof of the theorem, we decompose $\varphi_w = \eta_w \circ \tau$ with

$$\begin{aligned} \eta_w : \{0,1,2,3\}^* &\to \{0,1\}^*, & 0 &\mapsto w0110, & 2 &\mapsto \overline{w}0110,\\ & & 1 &\mapsto w1001, & 3 &\mapsto \overline{w}1001,\\ \tau : \{0,1,2,3\}^* &\to \{0,1,2,3\}^*, & 0 &\mapsto 02, & 2 &\mapsto 03,\\ & & 1 &\mapsto 13, & 3 &\mapsto 12, \end{aligned}$$

and we use the homomorphism

$$\begin{aligned} \eta'_w : \{0,1,2,3\}^* &\to \{0,1\}^*, & 0 &\mapsto 0110w, & 2 &\mapsto 0110\overline{w},\\ & & 1 &\mapsto 1001w, & 3 &\mapsto 1001\overline{w}. \end{aligned}$$

Then

$$\varphi_w(\widehat{v})w = \eta_w(\tau(\widehat{v}))w = w\eta'_w\big(\overline{\tau(v)}\big)$$

for all $v \in \{0,1,2,3\}^*$; recall that $\overline{w}$ is defined by reversing w and exchanging 0 and 1 as well as 2 and 3, while $\widehat{w}$ only exchanges 0 and 1 after reversing w.

We have to show that the difference

$$\Delta_w(v) = M_q\big(1\eta_w(v)w1\big)_{12} - M_q\big(1w\eta'_w(\overline{v})1\big)_{12}$$

is zero for all $v \in \tau(\{0,1,2,3\}^*) = \{02,03,12,13\}^* = (\{0,1\}\{2,3\})^*$.

Lemma 3.3. *Let $a \in \{2,3\}$, $v \in (\{0,1\}\{2,3\})^*$, $w \in \{0,1\}^*$. If $\Delta_w(v) = 0$, then*

$$\Delta_w(u0\overline{u}av) = \Delta_w(u1\overline{u}av)$$

for all $u \in (\{0,1\}\{2,3\})^*$ and

$$\Delta_w(u2\overline{u}av) = \Delta_w(u3\overline{u}av)$$

for all $u \in (\{0,1\}\{2,3\})^*\{0,1\}$.

Proof. Assume first that $|u|$ is even. Then

$$\begin{aligned} &\Delta_w(u0\bar{u}av) - \Delta_w(u1\bar{u}av)\\ &= M_q\big(1\eta_w(u0\bar{u}av)w1\big)_{12} - M_q\big(1\eta_w(u1\bar{u}av)w1\big)_{12}\\ &\quad + M_q\big(1w\eta'_w(\bar{v}\bar{a}u0\bar{u})1\big)_{12} - M_q\big(1w\eta'_w(\bar{v}\bar{a}u1\bar{u})1\big)_{12}\\ &= \Big(M_q\big(1\eta_w(u)w\big)\big(M_q(0110) - M_q(1001)\big)M_q\big(\eta_w(\bar{u}av)w1\big)\Big)_{12}\\ &\quad + \Big(M_q\big(1w\eta'_w(\bar{v}\bar{a}u)\big)\big(M_q(0110) - M_q(1001)\big)M_q\big(w\eta'_w(\bar{u})1\big)\Big)_{12}\\ &= (q^3+1)\,\Big(M_q\big(1\eta_w(u)w\big)SQ_qM_q\big(\eta_w(\bar{u}av)w1\big)\Big)_{12}\\ &\quad + (q^3+1)\,\Big(M_q\big(1w\eta'_w(\bar{v}\bar{a}u)\big)SQ_qM_q\big(w\eta'_w(\bar{u})1\big)\Big)_{12}\\ &= (q^3+1)\,q^{|\eta_w(u)w|}\Big(R_qSQ_qM_q\big(\overline{w}^{-1}\eta_w(av)w1\big)\Big)_{12}\\ &\quad + (q^3+1)\,q^{|\eta_w(u)w|}\Big(M_q\big(1w\eta'_w(\bar{v}\bar{a})\overline{w}^{-1}\big)SQ_qR_q\Big)_{12}\\ &= (q^3+1)\,q^{|\eta_w(u)w|}d_a\,\Big(M_q\big(1\eta_w(v)w1\big)_{12} - M_q\big(1w\eta'_w(\bar{v})1\big)_{12}\Big)\\ &= (q^3+1)\,q^{|\eta_w(u)w|}d_a\,\Delta_w(v), \end{aligned}$$

with $d_2 = -q$ and $d_3 = q^4$. Here, we use for the third equation that

$$M_q(0110) = M_q(1001) + (q^3{+}1)\,SQ_q, \quad \text{where} \quad S = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},\ Q_q = \begin{pmatrix} q & 0\\ 0 & 1\end{pmatrix}.$$

For the fourth equation, we use that, by (5),

$$M_q(z)\,S\,Q_q\,M_q(\overline{z}) = M_q(z)\,S\,{}^tM_q(z)\,Q_q = \det(M_q(z))\,SQ_q = q^{|z|}SQ_q$$

for all $z \in \{\mathtt{0},\mathtt{1}\}^*$, in particular for $z = \eta_w(u)w$ (with $\overline{z} = \eta_w(\overline{u})\overline{w}$) and for $z = \overline{w}\eta'_w(u)$ (with $\overline{z} = w\eta'_w(\overline{u})$). For the fifth equation, we use that

$$(1,0)\,R_qSQ_qM_q(\overline{w}^{-1}\eta_w(\mathtt{2})) = (1,0)\,R_qSQ_qM_q(\mathtt{0110}) = -(q^2,q) = -q\,(1,0)\,R_q,$$
$$(1,0)\,R_qSQ_qM_q(\overline{w}^{-1}\eta_w(\mathtt{3})) = (1,0)\,R_qSQ_qM_q(\mathtt{1001}) = (q^5,q^4) = q^4(1,0)\,M_q(\mathtt{1}),$$
$$M_q(\eta'_w(\mathtt{3})\overline{w}^{-1})SQ_qR_q{}^t(0,1) = M_q(\mathtt{1001})SQ_qR_q{}^t(0,1) = {}^t(q,q) = q\,M_q(\mathtt{1})\,{}^t(0,1),$$
$$M_q(\eta'_w(\mathtt{2})\overline{w}^{-1})SQ_qR_q{}^t(0,1) = M_q(\mathtt{0110})SQ_qR_q{}^t(0,1) = -{}^t(q^4,q^4) = -q^4R_q{}^t(0,1).$$

Therefore, $\Delta_w(v) = 0$ implies that $\Delta_w(u\mathtt{0}\bar{u}av) = \Delta_w(u\mathtt{1}\bar{u}av)$.

The proof of $\Delta_w(u\mathtt{2}\bar{u}av) = \Delta_w(u\mathtt{3}\bar{u}av)$ for odd $|u|$ runs along the same lines. $\square$

Lemma 3.4. *For all $v \in (\{\mathtt{0},\mathtt{1}\}\{\mathtt{2},\mathtt{3}\})^*$, $w \in \{\mathtt{0},\mathtt{1}\}^*$, we have $\Delta_w(v) = 0$.*

Proof. We proceed by induction on the length of v. The statement is trivially true for $|v| = 0$. Suppose that it is true up to length $k-1$ and consider it for length k.

We claim that the value of $\Delta_w(v_1\cdots v_{2k})$ does not depend on the choice of $v_1\cdots v_j$, for any $j \le k$. The claim is true for $j = 1$, by Lemma 3.3 with $u = \varepsilon$ and the induction hypothesis. If the claim is true up to $j-1$, then it gives together with Lemma 3.3, for any $u_1\cdots u_j \in (\{\mathtt{0},\mathtt{1}\}\{\mathtt{2},\mathtt{3}\})^* \cup (\{\mathtt{0},\mathtt{1}\}\{\mathtt{2},\mathtt{3}\})^*\{\mathtt{0},\mathtt{1}\}$, that

$$\begin{aligned}\Delta_w(u_1\cdots u_jv_{j+1}\cdots v_{2k}) &= \Delta_w(\overline{v_{j+1}\cdots v_{2j-1}}u_jv_{j+1}\cdots v_{2k})\\ &= \Delta_w(\overline{v_{j+1}\cdots v_{2j-1}}v_jv_{j+1}\cdots v_{2k}) = \Delta_w(v_1\cdots v_{2k}).\end{aligned}$$

This proves the claim.

Since $\eta_w(\overline{u}u)w = w\eta'_w(u\overline{u})$ for all $u \in (\{\mathtt{0},\mathtt{1}\}\{\mathtt{2},\mathtt{3}\})^* \cup (\{\mathtt{0},\mathtt{1}\}\{\mathtt{2},\mathtt{3}\})^*\{\mathtt{0},\mathtt{1}\}$, we have $\Delta_w(\overline{v_{k+1}\cdots v_{2k}}v_{k+1}\cdots v_{2k}) = 0$, thus $\Delta_w(v_1\cdots v_{2k}) = 0$ for all $v_1\cdots v_{2k} \in (\{\mathtt{0},\mathtt{1}\}\{\mathtt{2},\mathtt{3}\})^*$. $\square$

Proof of Theorem 3.2. As for (3), it suffices to prove (6) for $k = m = n = 0$. Since $\varphi_w(v) = \eta_w(\tau(v))$ and $\varphi_w(\overline{v})w = w\eta'_w\big(\overline{\tau(v)}\big)$ for all $w \in \{\mathtt{0},\mathtt{1}\}^*$, $v \in \{\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3}\}^*$, Lemma 3.4 implies that (6) holds.

Let σ be as in the proof of Theorem 3.1. Then

$$\begin{aligned}\mu_q\big(\mathtt{0}\psi_w(v)w\mathtt{1}\big)_{12} &= M_q(\mathtt{1}\eta_{\mathtt{0}\sigma(w)\mathtt{1}}(v)\mathtt{0}\sigma(w)\mathtt{1100}) = M_q(\mathtt{1}\eta_{\mathtt{0}\sigma(w)\mathtt{1}}(\widehat{v})\mathtt{0}\sigma(w)\mathtt{1100})\\ &= \mu_q\big(\mathtt{0}\psi_w(\widehat{v})w\mathtt{1}\big)_{12},\end{aligned}$$

using that $\mathtt{0}\sigma(\psi_w(v)) = \eta_{\mathtt{0}\sigma(w)\mathtt{1}}(v)\mathtt{0}$, and using (6) for the second equation. $\square$

The equation $M_q(x)_{12} = M_q(y)_{12}$ has many solutions $x, y \in \{\mathtt{0},\mathtt{1}\}^*$ which are not of the form of Equation (3) or (6), for example

$$M_q(\mathtt{110000011})_{12} = 1+2q+3q^2+4q^3+4q^4+4q^5+3q^6+2q^7+q^8 = M_q(\mathtt{100010001})_{12},$$

but we believe that Equations (4) and (7) are complete.

Question 3.5. Do there exist $x, y \in \{\mathtt{0},\mathtt{1}\}^*$ satisfying $\mu_q(x)_{12} = \mu_q(y)_{12}$ which are not given by Equation (4) or (7)?

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(S. Labbé) Univ. Bordeaux, CNRS, Bordeaux INP, LaBRI, UMR 5800, F-33400, Talence, France

Email address: sebastien.labbe@labri.fr

URL: http://www.slabbe.org/

(M. Lapointe) Université de Moncton, Département de mathématiques et de statistique, 18 avenue Antonine-Maillet, Moncton NB E1A 3E9, Canada

Email address: melodie.lapointe@umoncton.ca

URL: http://lapointemelodie.github.io/

(W. Steiner) Université Paris Cité, CNRS, IRIF, F–75006 Paris, France

Email address: steiner@irif.fr

URL: https://www.irif.fr/~steiner