A REMARKABLE INTEGER SEQUENCE RELATED TO π AND $\sqrt{2}$

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Dedicated to Jeff Shallit on the occasion of his 60th birthday

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Abstract

We prove that five ways to define entry **A086377** in the OEIS do lead to the same integer sequence.

1. Introduction

In September of 2003 Benoit Cloitre contributed a sequence to the On-Line Encyclopedia of Integer Sequences [4], defined by him as $a_1 = 1$, and for $n \geq 2$ by

$$a_n = \begin{cases} a_{n-1} + 2 & \text{if } n \text{ is in the sequence,} \\ a_{n-1} + 2 & \text{if } n \text{ and } n-1 \text{ are not in the sequence,} \\ a_{n-1} + 3 & \text{if } n \text{ is not in the sequence, but } n-1 \text{ is in the sequence.} \end{cases} \tag{1}$$

The first 25 values of this sequence are

$$1, 4, 6, 8, 11, 13, 16, 18, 21, 23, 25, 28, 30, 33, 35, 37, 40, 42, 45, 47, 49, 52, 54, 57, 59.$$

The purpose of this paper is to prove equivalence of five ways to define this integer sequence, most of them already conjecturally stated in the OEIS article on **A086377**. Besides a simplified recursion, the alternatives involve statements in terms of a morphic sequence, of a Beatty sequence, and of approximation properties linking a classical continued fraction of $\frac{4}{\pi}$ to that of $\sqrt{2}$.

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2. The theorem

Theorem 1. *The following five definitions produce the same integer sequence:*

(a_n) *defined by* $a_1 = 1$ *and for* $n \geq 2$*:*

$$a_n = \begin{cases} a_{n-1} + 2 & \text{if } n \text{ is in the sequence,} \\ a_{n-1} + 2 & \text{if } n \text{ and } n-1 \text{ are not in the sequence,} \\ a_{n-1} + 3 & \text{if } n \text{ is not in the sequence, but } n-1 \text{ is in the sequence;} \end{cases}$$

(b_n) *defined by* $b_1 = 1$ *and for* $n \geq 2$*:*

$$b_n = \begin{cases} b_{n-1} + 2 & \text{if } n-1 \text{ is not in the sequence,} \\ b_{n-1} + 3 & \text{if } n-1 \text{ is in the sequence;} \end{cases}$$

(c_n) *for* $n \geq 1$ *defined as the position of the* n*-th zero in the fixed point of the morphism*

$$\phi : \begin{cases} 0 \mapsto 011 \\ 1 \mapsto 01 \end{cases};$$

(d_n) *defined by* $d_n = \left\lfloor (1 + \sqrt{2}) \cdot n - \frac{1}{2}\sqrt{2} \right\rfloor$ *for* $n \geq 1$*;*

(e_n) *defined by* $e_n = \lceil r_n \rfloor = \lfloor r_n + \frac{1}{2} \rfloor$*, with* $r_1 = \dfrac{4}{\pi}$ *and* $r_{n+1} = \dfrac{n^2}{r_n - (2n-1)}$*, for* $n \geq 1$*.*

At first we found it hard to believe the equivalence of these definitions, but a verification of the first 130000 terms ($a_{130000} = 313847$) convinced us to look for proofs.

3. Simplification and a morphic sequence

To show that (b_n) defines the same sequence as (a_n), simply note that $a_n - a_{n-1} \geq 2$ for all n: hence if n is in the sequence then $n-1$ is not, and we can combine the first two cases in Equation (1).

In a comment to sequence **A086377**, Clark Kimberling asked if the integers in this sequence coincide with the positions of the zeroes in sequence **A189687**, which is the fixed point of the substitution

$$\phi : \begin{cases} 0 \mapsto 011 \\ 1 \mapsto 01 \end{cases},$$

defining the sequence (c_n) in the Theorem. It is not hard to see that this indeed produces the same as sequence (b_n); repeatedly applying the morphism ϕ to 0 produces after a few steps the initial segment

$$011010101101011010110101011010110101011010110101011010110101011\cdots.$$

The position c_n of the n-th zero is 2 ahead of c_{n-1} precisely when the latter is followed by a single 1, that is, when there is a 1 at position $n-1$, and it is 3 ahead of c_{n-1} if that zero is followed by 11, which means that there was a 0 at position $n-1$. Thus the rule is exactly that defining (b_n).

4. Beatty sequence

Every pair of real numbers α and β determines a Beatty sequence by

$$\mathrm{B}(\alpha,\beta)_n := \lfloor n\alpha+\beta\rfloor, \qquad n=1,2,\ldots.$$

The numbers α and β also determine sequences by

$$\mathrm{St}(\alpha,\beta)_n := \lfloor (n+1)\alpha+\beta\rfloor - \lfloor n\alpha+\beta\rfloor, \qquad n=1,2,\ldots,$$

which is a Sturmian sequence (of slope α), over the alphabet $\{0,1\}$, provided that $0\le\alpha<1$.

Thus Sturmian sequences are first differences of Beatty sequences (when $0\le\alpha<1$), but Beatty sequences and Sturmian sequences are also linked in another way.

Lemma 1. *Let $\alpha>1$ be irrational, and let $(s_n)_{n\ge1}$ be given by $s_n=\mathrm{St}(\frac{1}{\alpha},-\frac{\beta}{\alpha})_n$, for some real number β with $\alpha+\beta>1$ and such that $k\alpha+\beta\notin\mathbb{Z}$ for all positive integers k. Then $\mathrm{B}(\alpha,\beta)$ is the sequence of positions of 1 in (s_n).*

Proof. This is a generalization of Lemma 9.1.3 in [1], from homogeneous to inhomogeneous Sturmian sequences. The proof also generalizes:

$$\begin{aligned}
&\text{there exists } k\ge1:\ n=\lfloor k\alpha+\beta\rfloor && \text{if and only if}\\
&\text{there exists } k\ge1:\ n\le k\alpha+\beta<n+1 && \text{if and only if}\\
&\text{there exists } k\ge1:\ \frac{n-\beta}{\alpha}\le k<\frac{n+1-\beta}{\alpha} && \text{if and only if}\\
&\text{there exists } k\ge1:\ \left\lfloor\frac{n-\beta}{\alpha}\right\rfloor=k-1 \text{ and } \left\lfloor\frac{n-\beta}{\alpha}+\frac{1}{\alpha}\right\rfloor=k && \text{if and only if}\\
&\left\lfloor\frac{n+1}{\alpha}-\frac{\beta}{\alpha}\right\rfloor-\left\lfloor\frac{n}{\alpha}-\frac{\beta}{\alpha}\right\rfloor=1 && \text{if and only if}\\
&\mathrm{St}\Big(\frac{1}{\alpha},-\frac{\beta}{\alpha}\Big)_n=1.
\end{aligned}$$

□

Our goal in this section is to prove that $(c_n) = (d_n)$. Let ψ be the morphism $\psi : \begin{cases} 0 \to 10 \\ 1 \to 100 \end{cases}$, and let w be the fixed point. Then

$$w = 1001010100101001010010101001010010101001010010101001010010100\cdots,$$

which is obtained by exchanging 0s and 1s in the fixed point of ϕ, i.e., $\psi = E\phi E$, with E the exchange morphism given by $E(0) = 1, E(1) = 0$. So the positions of 0 in the fixed point of ϕ correspond to the positions of 1 in the fixed point w of ψ.

Let $\alpha_d = 1 + \sqrt{2}$ and $\beta_d = -\frac{1}{2}\sqrt{2}$; then $d_n = \mathrm{B}(\alpha_d, \beta_d)_n$, for $n \geq 1$.

Applying Lemma 1, we deduce that d_n also equals the position of the n-th 1 in the Sturmian sequence $\mathrm{St}(\alpha, \beta)$, generated by

$$\alpha = \frac{1}{\alpha_d} = \sqrt{2} - 1,\ \beta = \frac{-\beta_d}{\alpha_d} = 1 - \frac{1}{2}\sqrt{2}.$$

Lemma 2. $\mathrm{St}\big(\sqrt{2} - 1, 1 - \frac{1}{2}\sqrt{2}\big) = w$.

Proof. This was already proved by Nico de Bruijn in 1981 ([2]), where it is the main example. See also Lemma 2 in [6]. Note, however, that our Sturmian sequences start at $n = 1$.

For a 'modern' proof as suggested by [3, Section 4], let ψ_1 and ψ_2 be the elementary morphisms given by $\psi_1(0) = 01, \psi_1(1) = 0$, and $\psi_2(0) = 10, \psi_2(1) = 0$. Then $\psi = \psi_2\psi_1 E$. This implies that the fixed point w of ψ is a Sturmian word (see [5, Corollary 2.2.19]). To find its parameters (α, β), use the 2D fractional linear maps that describe how the parameters of a Sturmian word change when one applies an elementary morphism. For Sturmian words starting at $n = 0$, the maps for E, ψ_1 and ψ_2 are[2] respectively (see [5, Lemma 2.2.17, Lemma 2.2.18, Exercise 2.2.6])

$$T_0(x, y) = (1-x, 1-y),\ T_1(x, y) = \left(\frac{1-x}{2-x}, \frac{1-y}{2-x}\right),\ T_2(x, y) = \left(\frac{1-x}{2-x}, \frac{2-x-y}{2-x}\right).$$

The change of parameters by applying ψ is therefore the composition

$$T_{210}(x, y) := T_2T_1T_0(x, y) = \left(\frac{1}{2+x}, \frac{2+x-y}{2+x}\right).$$

But the parameters α and β of w do *not* change when one applies ψ. This means that (α, β) is a fixed point of T_{210}, and one easily computes $\alpha = \sqrt{2} - 1$, and then $\beta = \frac{1}{2}\sqrt{2}$. Since our Sturmian words start at $n = 1$, we have to subtract α from β and obtain that $w = \mathrm{St}\big(\sqrt{2} - 1, 1 - \frac{1}{2}\sqrt{2}\big)$. □

[2]Actually there is a subtlety here involving the ceiling representation of a Sturmian sequence, but that does not apply in our case since $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$.

5. Converging recurrence

In a comment to entry **A086377**, Joseph Biberstine conjectured a beautiful connection with the infinite continued fraction expansion

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \cfrac{5^2}{11 + \cfrac{6^2}{\ddots}}}}}},$$

derived from the arctangent function expansion. If we define R_n for $n \geq 1$ by

$$R_n = 2n - 1 + \cfrac{n^2}{2n + 1 + \cfrac{(n+1)^2}{2n + 3 + \cfrac{(n+2)^2}{2n + 5 + \cfrac{(n+3)^2}{\ddots}}}},$$

then $R_1 = 4/\pi$ and $R_n = 2n - 1 + \dfrac{n^2}{R_{n+1}}$. We see that

$$\frac{R_n}{n}\frac{R_{n+1}}{n+1} - \frac{2n-1}{n}\frac{R_{n+1}}{n+1} - \frac{n^2}{n(n+1)} = 0.$$

This implies that if R_n/n converges, for $n \to \infty$, then it does so to a (positive) zero of $x^2 - 2x - 1$, that is, to $1 + \sqrt{2}$; cf. Lemma 3 below.

We consider now, conversely and slightly more generally, for any real $h \geq 1$, a sequence of positive numbers r_n satisfying

$$r_n = hn - 1 + \frac{n^2}{r_{n+1}} \tag{2}$$

for $n \geq 1$. We first show that this sequence is unique, i.e., there is a unique $r_1 > 0$ such that $r_n > 0$ for all $n \geq 1$, and give estimates for its terms.

Lemma 3. *For each $h \geq 1$, there is a unique sequence of positive real numbers $(r_n)_{n\geq 1}$ satisfying the recurrence* (2)*. Moreover, we have for this sequence, for all $n \geq 1$,*

$$0 < r_n - \alpha n + c < \frac{(\alpha - c)(c-1)}{\alpha n} \tag{3}$$

with $\alpha = \dfrac{h + \sqrt{h^2+4}}{2}$ and $c = \dfrac{1+\alpha}{2\alpha - h} = \dfrac{1}{2} + \dfrac{h+2}{2\sqrt{h^2+4}}$.

Proof. Let $f_n(x) = hn - 1 + n^2/x$. Suppose that a sequence of positive numbers r_n satisfies (2), i.e., that $f_n(r_{n+1}) = r_n$ for all $n \geq 1$. Then we have $r_n > hn - 1$ and thus $r_n < (h + 1/h)n$ for all $n \geq 1$. We deduce that there exists some $\delta > 0$ and $N \geq 1$ such that $r_n > (h + \delta)n$ for all $n \geq N$. Suppose that there is another sequence of positive numbers $\tilde{r}_n$ satisfying (2). Since $|f_n'(x)| = |n/x|^2 < 1/(h + \delta)$ for all $x > (h + \delta)n$, we have

$$\begin{aligned}
|r_N - \tilde{r}_N| &= |f_N f_{N+1} \cdots f_{n-1}(r_n) - f_N f_{N+1} \cdots f_{n-1}(\tilde{r}_n)| \\
&< \frac{|r_n - \tilde{r}_n|}{(h+\delta)^{n-N}} < \frac{n/h}{(h+\delta)^{n-N}}
\end{aligned}$$

for all $n \geq N$, hence $r_N = \tilde{r}_N$, which implies that $r_n = \tilde{r}_n$ for all $n \geq 1$.

Next we show that

$$f_n\big(\alpha(n+1) - c\big) < \alpha n - c + \frac{(\alpha - c)(c-1)}{\alpha n}$$

and

$$f_n\Big(\alpha(n+1) - c + \frac{(\alpha - c)(c-1)}{(n+1)\alpha}\Big) > \alpha n - c.$$

Indeed, using that $\alpha^2 = h\alpha + 1$ and $2\alpha c - hc = 1 + \alpha$, we have

$$\begin{aligned}
(\alpha n + \alpha - c)\, f_n\big(\alpha(n+1) - c\big) &= (hn-1)(\alpha n + \alpha - c) + n^2 \\
&= (h\alpha + 1)n^2 + (h\alpha - hc - \alpha)n - (\alpha - c) \\
&< \alpha^2 n^2 + (\alpha^2 - 2\alpha c)n - (\alpha - c) + \frac{(\alpha - c)^2(c-1)}{\alpha n} \\
&= (\alpha n + \alpha - c)\Big(\alpha n - c + \frac{(\alpha - c)(c-1)}{\alpha n}\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\alpha n + \alpha - c + \frac{(\alpha - c)(c-1)}{\alpha(n+1)}\Big)(\alpha n - c) \\
&\quad< \alpha^2 n^2 + (\alpha^2 - 2\alpha c)n - (\alpha - c) - \frac{c(\alpha - c)(c-1)}{\alpha(n+1)} \\
&\quad< (h\alpha + 1)n^2 + (h\alpha - hc - \alpha)n - (\alpha - c) - \frac{(\alpha - c)(c-1)}{\alpha(n+1)} \\
&\quad< (hn-1)\Big(\alpha n + \alpha - c + \frac{(\alpha - c)(c-1)}{\alpha(n+1)}\Big) + n^2 \\
&\quad= \Big(\alpha n + \alpha - c + \frac{(\alpha - c)(c-1)}{\alpha(n+1)}\Big)\, f_n\Big(\alpha(n+1) - c + \frac{(\alpha - c)(c-1)}{\alpha(n+1)}\Big).
\end{aligned}$$

As f_n is monotonically decreasing for $x > 0$, we deduce that

$$0 < f_n(x) - \alpha n + c < \frac{(\alpha - c)(c-1)}{\alpha n}$$

for all x with $0 \le x - \alpha(n+1) + c \le \frac{(\alpha-c)(c-1)}{\alpha(n+1)}$. Then we also have

$$0 < f_n f_{n+1} \cdots f_{n+k-1}\big(\alpha(n+k) - c + x\big) - \alpha n + c < \frac{(\alpha-c)(c-1)}{\alpha n}$$

for all $k, n \ge 1$, $0 \le x - \alpha(n+k) + c \le \frac{(\alpha-c)(c-1)}{\alpha(n+k)}$. As f_n is contracting for $x \ge \alpha(n+1)-c$, the intervals $[f_1 f_2 \cdots f_n(\alpha(n+1)-c), f_1 f_2 \cdots f_n(\alpha(n+1)-c+\frac{(\alpha-c)(c-1)}{\alpha(n+1)})]$ converge to a point r_1. Then the numbers r_n given by (2) satisfy (3) for all $n \ge 1$. By the first paragraph of the proof, this is the unique sequence of positive numbers satisfying (2). □

Now consider when $\alpha n - c + \frac{1}{2}$ is close to $\lceil \alpha n - c + \frac{1}{2} \rceil$. Let p_k/q_k be the convergents of the regular continued fraction $\alpha = [h; h, h, \ldots]$, i.e., $q_{-1} = 0$, $q_0 = 1$, $q_{k+1} = hq_k + q_{k-1}$ for $k \ge 1$, $p_k = q_{k+1}$. Then we have

$$q_k = \frac{\alpha^{k+1} + (-1)^k/\alpha^{k+1}}{\alpha + 1/\alpha}$$

and thus

$$q_k\alpha - p_k = \frac{(-1)^k}{\alpha^{k+1}}. \tag{4}$$

Lemma 4. *Let h be a positive integer and $\alpha = \dfrac{h+\sqrt{h^2+4}}{2}$. Then we have*

$$\lceil \alpha n \rceil - \alpha n = \begin{cases} j/\alpha^{2k} & \text{if } n = jq_{2k-1},\ k \ge 1,\ 1 \le j < \alpha^{2k}, \\ (\alpha-1)/\alpha^{2k+1} & \text{if } n = q_{2k-1} + q_{2k},\ k \ge 0, \\ (\alpha+1)/\alpha^{2k+2} & \text{if } n = q_{2k+1} - q_{2k},\ k \ge 0, \end{cases}$$

and $n(\lceil \alpha n \rceil - \alpha n) \ge 1$ for all other $n \ge 1$.

Proof. The formulas for $n = jq_{2k-1}$, $n = q_{2k-1} + q_{2k}$ and $n = q_{2k+1} - q_{2k}$ are immediate from (4). By [7, Ch. 2, §5, Theorem 2], we have $n(\lceil \alpha n \rceil - \alpha n) \ge 1$ for all $n \ge 1$ that are not of the form jq_k, $1 \le j < \alpha/\sqrt{h}$, $q_k + q_{k-1}$ or $q_k - q_{k-1}$. Since $\alpha q_{2k} - \lfloor \alpha q_{2k} \rfloor = 1/\alpha^{2k+1}$, $\alpha(q_{2k} + q_{2k+1}) - \lfloor \alpha(q_{2k} + q_{2k+1}) \rfloor = (\alpha-1)/\alpha^{2k+2}$ and $\alpha(q_{2k} - q_{2k-1}) - \lfloor \alpha(q_{2k} - q_{2k-1}) \rfloor = (\alpha+1)/\alpha^{2k+1}$, we have $\lceil \alpha n \rceil - \alpha n > 1/2$ for $n = jq_{2k}$, $n = q_{2k} + q_{2k+1}$ and $n = q_{2k} - q_{2k-1}$. If moreover $n \ge 2$, then we have thus $n(\lceil \alpha n \rceil - \alpha n) \ge 1$ for these n as well. Since $q_0 + q_{-1} = 1$, the case $n = 1$ has already been treated. □

We obtain that

$$n\big(\lceil \alpha n \rceil - \alpha n\big) = \begin{cases} \dfrac{j^2(1 - 1/\alpha^{4k})}{\sqrt{h^2+4}} & \text{if } n = jq_{2k-1},\ k \ge 1,\ 1 \le j < \alpha^{2k}, \\ \dfrac{h - (\alpha-1)^2/\alpha^{4k+2}}{\sqrt{h^2+4}} & \text{if } n = q_{2k-1} + q_{2k},\ k \ge 0, \\ \dfrac{h - (\alpha+1)^2/\alpha^{4k+4}}{\sqrt{h^2+4}} & \text{if } n = q_{2k+1} - q_{2k},\ k \ge 0. \end{cases}$$

The worst case for $n = q_{2k-1} + q_{2k}$ or $n = q_{2k+1} - q_{2k}$ is given by $n = q_{-1} + q_0 = 1$, hence

$$n\big(\lceil \alpha n \rceil - \alpha n\big) \geq h + 1 - \alpha = 1 - \frac{1}{\alpha}$$

for all $n \geq 1$ such that $n \neq q_{2k-1}$ for all $k \geq 1$.

Now we come back to the case $h = 2$ and consider the distance of $\alpha n - c + \frac{1}{2}$ to the nearest integer above $\alpha n - c + \frac{1}{2}$. Note that $c - \frac{1}{2} = \frac{1}{\sqrt{2}}$. We have

$$\begin{aligned} 2\Big(\Big\lceil \alpha n - \frac{1}{\sqrt{2}} \Big\rceil - \alpha n + \frac{1}{\sqrt{2}}\Big) &= 2\Big\lceil \alpha n - \frac{1}{\sqrt{2}} \Big\rceil - 1 - \alpha(2n-1) \\ &\geq \lceil \alpha(2n-1) \rceil - \alpha(2n-1) > \frac{\alpha - 1}{2\alpha n}, \end{aligned}$$

where we have used that q_{2k-1} is even for all $k \geq 1$; thus

$$\Big\lceil \alpha n - \frac{1}{\sqrt{2}} \Big\rceil - \alpha n + \frac{1}{\sqrt{2}} > \frac{\alpha - 1}{4\alpha n}.$$

Since $(\alpha - c)(c - 1) = \dfrac{1}{4\alpha}$, we have

$$\alpha n - \frac{1}{\sqrt{2}} < r_n + \frac{1}{2} < \alpha n - \frac{1}{\sqrt{2}} + \frac{1}{4\alpha n} < \alpha n - \frac{1}{\sqrt{2}} + \frac{\alpha - 1}{4\alpha n} < \Big\lceil \alpha n - \frac{1}{\sqrt{2}} \Big\rceil$$

for all $n \geq 1$, thus $d_n = e_n$. This completes the proof of Theorem 1.

We remark that $h = 2$ cannot be replaced by an arbitrary positive integer in the previous paragraph. For example, for $h = 1$, we have $\alpha = \frac{1+\sqrt{5}}{2}$, $c = \frac{\alpha^2}{\sqrt{5}}$, $\lfloor 137\alpha - c + \frac{1}{2} \rfloor = 220$ and $\lfloor r_{137} + \frac{1}{2} \rfloor = 221$. However, computer simulations suggest that (for any h) we always have $\lfloor \alpha n - c \rfloor = \lfloor r_n \rfloor$.

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