Directional dynamics for cellular automata

Mathieu Sablik

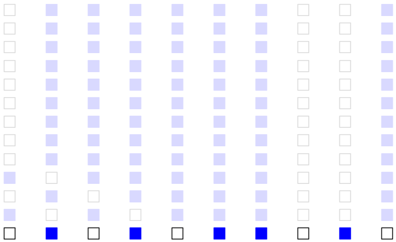
05 october 2007

e-mail : msablik@umpa.ens-lyon.fr

Problematic

Definition of CA

Cellular automata (CA) were introduced by von Neumann-1951 as simplified models of biological systems.



A cellular automaton is defined by :

- a finite alphabet : $\mathcal{A}$
- a semi-group : $\mathbb{M}$ (here $\mathbb{Z}$),
- a neighborhood : $\mathbb{U} = [r, s] \subset \mathbb{M}$,
- a local function : $\overline{F} : \mathcal{A}^{\mathbb{U}} \to \mathcal{A}$.

Definition

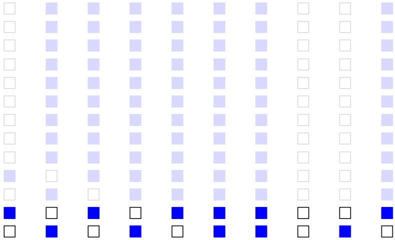
One defines $F : \mathcal{A}^{\mathbb{M}} \longrightarrow \mathcal{A}^{\mathbb{M}}$ *by :*

$$F(x)_m = \overline{F}((x_{m+u})_{u \in \mathbb{U}})$$

for all $m \in \mathbb{M}$ *and* $x \in \mathcal{A}^{\mathbb{M}}$.

Definition of CA

Cellular automata (CA) were introduced by von Neumann-1951 as simplified models of biological systems.



A cellular automaton is defined by :

- a finite alphabet : $\mathcal{A}$
- a semi-group : $\mathbb{M}$ (here $\mathbb{Z}$),
- a neighborhood : $\mathbb{U} = [r, s] \subset \mathbb{M}$,
- a local function : $\overline{F} : \mathcal{A}^{\mathbb{U}} \to \mathcal{A}$.

Definition

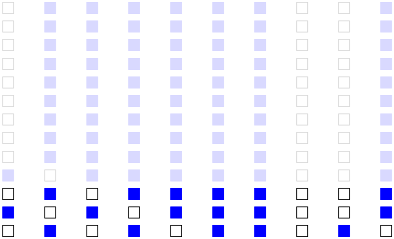
One defines $F : \mathcal{A}^{\mathbb{M}} \longrightarrow \mathcal{A}^{\mathbb{M}}$ *by :*

$$F(x)_m = \overline{F}((x_{m+u})_{u \in \mathbb{U}})$$

for all $m \in \mathbb{M}$ *and* $x \in \mathcal{A}^{\mathbb{M}}$.

Definition of CA

Cellular automata (CA) were introduced by von Neumann-1951 as simplified models of biological systems.



A cellular automaton is defined by :

- a finite alphabet : $\mathcal{A}$
- a semi-group : $\mathbb{M}$ (here $\mathbb{Z}$),
- a neighborhood : $\mathbb{U} = [r, s] \subset \mathbb{M}$,
- a local function : $\overline{F} : \mathcal{A}^{\mathbb{U}} \to \mathcal{A}$.

Definition

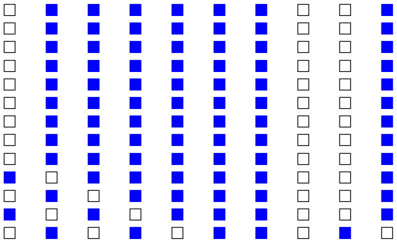
One defines $F : \mathcal{A}^{\mathbb{M}} \longrightarrow \mathcal{A}^{\mathbb{M}}$ *by :*

$$F(x)_m = \overline{F}((x_{m+u})_{u \in \mathbb{U}})$$

for all $m \in \mathbb{M}$ *and* $x \in \mathcal{A}^{\mathbb{M}}$.

Definition of CA

Cellular automata (CA) were introduced by von Neumann-1951 as simplified models of biological systems.



A cellular automaton is defined by :

- a finite alphabet : $\mathcal{A}$
- a semi-group : $\mathbb{M}$ (here $\mathbb{Z}$),
- a neighborhood : $\mathbb{U} = [r, s] \subset \mathbb{M}$,
- a local function : $\overline{F} : \mathcal{A}^{\mathbb{U}} \to \mathcal{A}$.

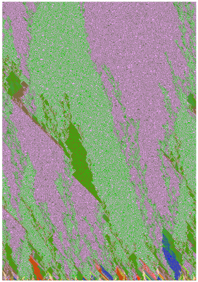
Definition

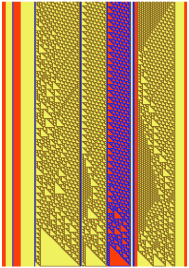
One defines $F : \mathcal{A}^{\mathbb{M}} \longrightarrow \mathcal{A}^{\mathbb{M}}$ by :

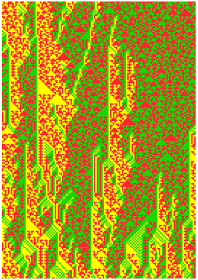
$$F(x)_m = \overline{F}((x_{m+u})_{u \in \mathbb{U}})$$

for all $m \in \mathbb{M}$ and $x \in \mathcal{A}^{\mathbb{M}}$.

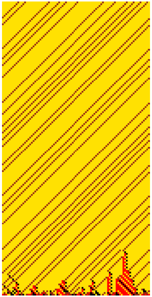
Some examples of space-time diagrams

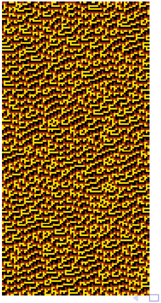


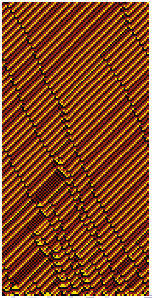




Classification of Wolfram (1982) :



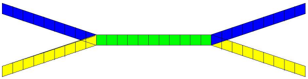




Topological characterisation

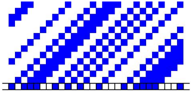
- $\mathcal{A}^{\mathbb{Z}}$ is compact for the product topology. One define the cantor distance as :

$$d_C(x,y) = 2^{-\min\{|i| \,:\, x_i \neq y_i\}}$$



- $\mathbb{Z}$ acts on $\mathcal{A}^{\mathbb{Z}}$ by shift defined for all $m \in \mathbb{Z}$ by :

$$\begin{array}{cccc} \sigma^m : & \mathcal{A}^{\mathbb{Z}} & \longrightarrow & \mathcal{A}^{\mathbb{Z}} \\ & (x_i)_{i\in\mathbb{Z}} & \longmapsto & (x_{i+m})_{i\in\mathbb{Z}}. \end{array}$$



Hedlund-69

A CA is a continuous function $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ which commutes with the shift σ.

Applications :

- Give a topological framework to study CA.
- Allows to show easly combinatory results.
- Allows to consider CA as dynamical systems...

Dynamics for the action of a semi-group $\mathbb{M}$

Definition

A **dynamical system** is a metric space (X, d) and a continuous $\mathbb{M}$-action T on X.

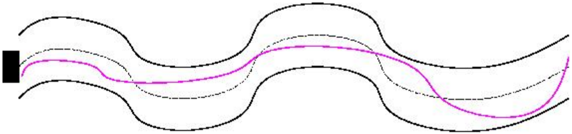
Let $B(x, \delta) = \{y \in X : d(x, y) < \delta\}$ and

$$E^{\mathbb{M}}(x, \varepsilon) = \{y \in X : d(T^m(x), T^m(y)) < \varepsilon, \ \forall m \in \mathbb{M}\}.$$

Definitions around the equicontinuity :

- $x \in Eq^{\mathbb{M}}(X, T) \iff \forall \varepsilon > 0, \exists \delta > 0, \ B(x, \delta) \subset E^{\mathbb{M}}(x, \varepsilon)$;
- (X, T) is $\mathbb{M}$-**equicontinuous** if

$$\forall \varepsilon > 0, \exists \delta > 0 \forall x \in X, \ B(x, \delta) \subset E^{\mathbb{M}}(x, \varepsilon);$$



Dynamics for the action of a semi-group $\mathbb{M}$

Definition

A **dynamical system** is a metric space (X, d) and a continuous $\mathbb{M}$-action T on X.

Let $B(x, \delta) = \{y \in X : d(x, y) < \delta\}$ and

$$E^{\mathbb{M}}(x, \varepsilon) = \{y \in X : d(T^m(x), T^m(y)) < \varepsilon, \ \forall m \in \mathbb{M}\}.$$

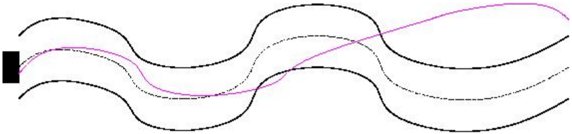
Definitions around the sensitivity :

- (X, T) is $\mathbb{M}$-**sensitive** if

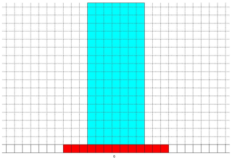
$$\exists \varepsilon > 0, \forall x \in X, \forall \delta > 0, \ \exists y \in B(x, \delta) \setminus E^{\mathbb{M}}(x, \varepsilon);$$

- (X, T) is $\mathbb{M}$-**expansive** if

$$\exists \varepsilon > 0, \forall x \in X, E^{\mathbb{M}}(x, \varepsilon) = \{x\}.$$



Dynamic of the $\mathbb{N}$-action F on $\mathcal{A}^{\mathbb{Z}}$

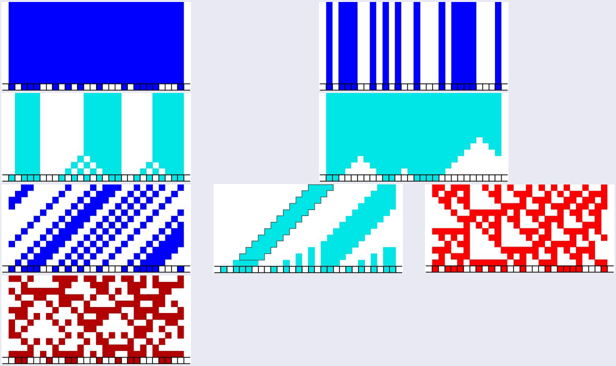


$$E^{\mathbb{N}}_{\mathcal{A}^{\mathbb{Z}}}(x,\varepsilon) = \{y \in \mathcal{A}^{\mathbb{Z}} : d_C(F^n(x), F^n(y)) < \varepsilon \ \forall n \in \mathbb{N}$$

$$B_{\mathcal{A}^{\mathbb{Z}}}(x,\delta) = \{y \in \mathcal{A}^{\mathbb{Z}} : d_C(x,y) < \delta\}$$

Theorem : Classification of CA of Kůrka-97

- $(\mathcal{A}^{\mathbb{Z}}, F)$ equicontinuous
- $\emptyset \subsetneq Eq^0(\mathcal{A}^{\mathbb{Z}}, F) \subsetneq \mathcal{A}^{\mathbb{Z}}$
- $(\mathcal{A}^{\mathbb{Z}}, F)$ sensitive
- $(\mathcal{A}^{\mathbb{Z}}, F)$ expansive



Directional dynamics for unidimensional CA

Study of the $\mathbb{Z} \times \mathbb{N}$-action (σ, F)

It is possible to consider the $\mathbb{Z} \times \mathbb{N}$-action (σ, F).
Classification of P. Kůrka : restriction of (σ, F) at $\{0\} \times \mathbb{N}$-action !

Question

Which sub-semi-group we must consider to study the $\mathbb{Z} \times \mathbb{N}$-action (σ, F) ?

Let $\mathbb{M}$ be a sub-semi-group of $\mathbb{Z} \times \mathbb{N}$. There is two options :

- $\mathbb{M}$ contains a sub-semi-group of $\mathbb{Z} \times \{0\}$: the $\mathbb{M}$-action (σ, F) contains the dynamic of a power of σ,

The dynamic is so strong ;

- $\mathbb{M} = p\mathbb{Z} \times q\mathbb{N}$ with $q \neq 0$: dynamics according to the slope $\alpha = \frac{p}{q}$.

Question

How is it possible to define dynamics according to every direction $\alpha \in \mathbb{R}$?

Study of the $\mathbb{Z} \times \mathbb{N}$-action (σ, F)

It is possible to consider the $\mathbb{Z} \times \mathbb{N}$-action (σ, F).
Classification of P. Kůrka : restriction of (σ, F) at $\{0\} \times \mathbb{N}$-action !

Question

Which sub-semi-group we must consider to study the $\mathbb{Z} \times \mathbb{N}$-action (σ, F) ?

Let $\mathbb{M}$ be a sub-semi-group of $\mathbb{Z} \times \mathbb{N}$. There is two options :

- $\mathbb{M}$ contains a sub-semi-group of $\mathbb{Z} \times \{0\}$: the $\mathbb{M}$-action (σ, F) contains the dynamic of a power of σ,

The dynamic is so strong ;

- $\mathbb{M} = p\mathbb{Z} \times q\mathbb{N}$ with $q \neq 0$: dynamics according to the slope $\alpha = \frac{p}{q}$.

Question

How is it possible to define dynamics according to every direction $\alpha \in \mathbb{R}$?

Study of the $\mathbb{Z} \times \mathbb{N}$-action (σ, F)

It is possible to consider the $\mathbb{Z} \times \mathbb{N}$-action (σ, F).
Classification of P. Kůrka : restriction of (σ, F) at $\{0\} \times \mathbb{N}$-action !

Question

Which sub-semi-group we must consider to study the $\mathbb{Z} \times \mathbb{N}$-action (σ, F) ?

Let $\mathbb{M}$ be a sub-semi-group of $\mathbb{Z} \times \mathbb{N}$. There is two options :

- $\mathbb{M}$ contains a sub-semi-group of $\mathbb{Z} \times \{0\}$: the $\mathbb{M}$-action (σ, F) contains the dynamic of a power of σ,

The dynamic is so strong ;

- $\mathbb{M} = p\mathbb{Z} \times q\mathbb{N}$ with $q \neq 0$: dynamics according to the slope $\alpha = \frac{p}{q}$.

Question

How is it possible to define dynamics according to every direction $\alpha \in \mathbb{R}$?

Study of the $\mathbb{Z} \times \mathbb{N}$-action (σ, F)

It is possible to consider the $\mathbb{Z} \times \mathbb{N}$-action (σ, F).
Classification of P. Kůrka : restriction of (σ, F) at $\{0\} \times \mathbb{N}$-action !

Question

Which sub-semi-group we must consider to study the $\mathbb{Z} \times \mathbb{N}$-action (σ, F) ?

Let $\mathbb{M}$ be a sub-semi-group of $\mathbb{Z} \times \mathbb{N}$. There is two options :

- $\mathbb{M}$ contains a sub-semi-group of $\mathbb{Z} \times \{0\}$: the $\mathbb{M}$-action (σ, F) contains the dynamic of a power of σ,

  The dynamic is so strong ;

- $\mathbb{M} = p\mathbb{Z} \times q\mathbb{N}$ with $q \neq 0$: dynamics according to the slope $\alpha = \frac{p}{q}$.

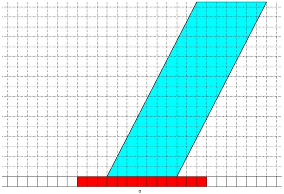
Question

How is it possible to define dynamics according to every direction $\alpha \in \mathbb{R}$?

Dynamic of slope α

Consider the suspension of (σ, F) defined for all $(m,n) \in \mathbb{R} \times \mathbb{R}^+$ by :

$$T^{(m,n)} : \begin{array}{ccc} \mathcal{A}^{\mathbb{Z}} \times \mathbb{T} \times \mathbb{T} & \longrightarrow & \mathcal{A}^{\mathbb{Z}} \times \mathbb{T} \times \mathbb{T} \\ (x, \beta_1, \beta_2) & \longmapsto & (\sigma^{\lfloor m+\beta_1 \rfloor} \circ F^{\lfloor n+\beta_2 \rfloor}(x), \{m+\beta_1\}, \{n+\beta_2\}) \end{array}$$



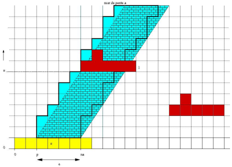
$$E^{\alpha}_{\Sigma}(x,\varepsilon) = \left\{ y \in \Sigma : d_C(\sigma^{\lfloor n\alpha \rfloor} \circ F^n(x), \sigma^{\lfloor n\alpha \rfloor} \circ F^n(y)) < \varepsilon \ \forall n \in \mathbb{N} \right\}$$

$$B_{\Sigma}(x,\delta) = \left\{ y \in \Sigma : d_C(x,y) < \delta \right\}$$

Definition

$$x \in Eq^{\alpha}(\Sigma, F) \iff \forall \varepsilon > 0, \ \exists \delta > 0 \quad B_{\Sigma}(x,\delta) \subset E^{\alpha}_{\Sigma}(x,\varepsilon)$$

Dynamic of slope α



$u \in \mathcal{L}_\Sigma$ is a Σ-blocking word of slope α if :

$$\forall x \in [u]_0 \cap \Sigma, \text{ one has } [u]_0 \subset E^\alpha_\Sigma(x, 2^{-\max\{|u|+|\alpha|:u\in\mathbb{U}$$

Characterisation of equicontinuous points

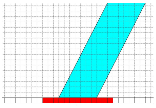
If Σ is a transitive subshift then :
$x \in Eq^\alpha(\Sigma, F) \iff \exists u \in \mathcal{L}_\Sigma$ which is a blocking word.

- Some recall :

-(Σ, σ) is *transitive* if $\forall u, v \in \mathcal{L}_\Sigma$, $\exists w \in \mathcal{L}_\Sigma$ such that $uwv \in \mathcal{L}_\Sigma$.

-(Σ, σ) is *weakly-specified* if $\exists N \in \mathbb{N}$ such that $\forall u, v \in \mathcal{L}_\Sigma$, $\exists n \leq N$ and $\exists w \in \mathcal{L}_\Sigma(n)$ such that $uwv \in \mathcal{L}_\Sigma$.

Dynamic of slope α



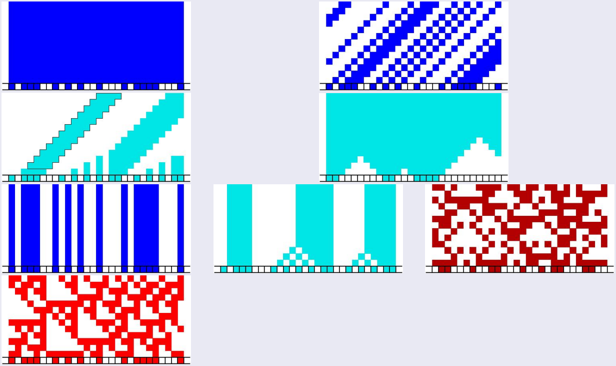
$$E^{\alpha}_{\Sigma}(x,\varepsilon) = \{y \in \Sigma : d_C(\sigma^{\lfloor n\alpha \rfloor} \circ F^n(x), \sigma^{\lfloor n\alpha \rfloor} \circ F^n(y)) < \varepsilon \ \forall n \in \mathbb{N}\}$$

$$B_{\Sigma}(x,\delta) = \{y \in \Sigma : d_C(x,y) < \delta\}$$

Theorem : Classification of CA under the slope α

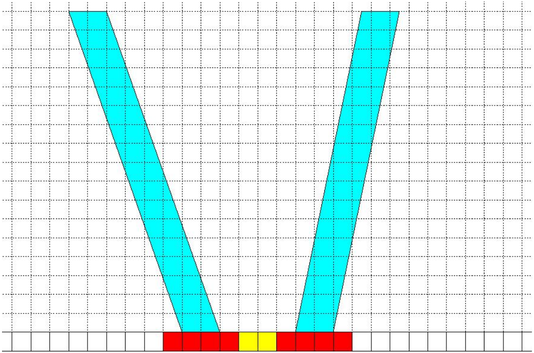
Let Σ be a transitive subshift. One of the following case holds :

- (Σ, F) equicontinuous of slope α
- $\emptyset \subsetneq Eq^{\alpha}(\Sigma, F) \subsetneq \mathcal{A}^{\mathbb{Z}}$
- (Σ, F) sensible of slope α
- (Σ, F) expansive of slope α



Convexity of $\mathbf{A}(\Sigma, F)$

$$\mathbf{A}(\Sigma, F) = \{\alpha \in \mathbb{R} : Eq^{\alpha}(\Sigma, F) \neq \emptyset\}$$

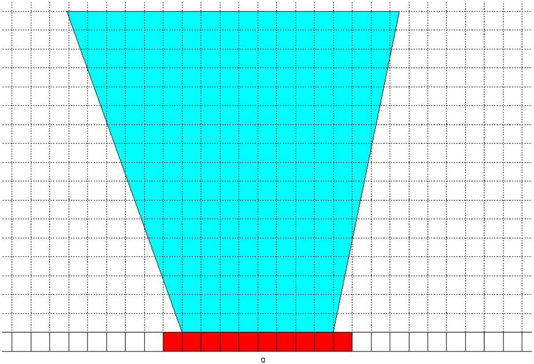


Let $\alpha', \alpha'' \in \mathbf{A}(\Sigma, F)$.

If Σ is transitive we can stick blocking word of slope α' and α''.

Convexity of $\mathbf{A}(\Sigma, F)$

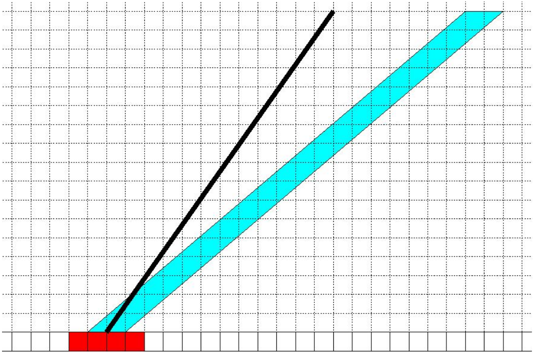
$$\mathbf{A}(\Sigma, F) = \{\alpha \in \mathbb{R} : Eq^{\alpha}(\Sigma, F) \neq \emptyset\}$$



Then $[\alpha', \alpha''] \subset \mathbf{A}(\Sigma, F)$.

$\mathbf{A}(\Sigma, F) \subset]-s, -r[$

$$\mathbf{A}(\Sigma, F) = \{\alpha \in \mathbb{R} : Eq^{\alpha}(\Sigma, F) \neq \emptyset\}$$

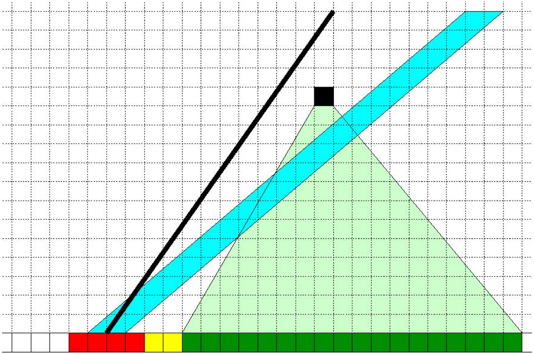


Let $\mathbb{U} = [r, s]$ the neighbourhood of F.

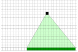
Let $\alpha > -r$ such that $\alpha \in \mathbf{A}(\Sigma, F)$.

$\mathbf{A}(\Sigma, F) \subset]-s, -r[$

$$\mathbf{A}(\Sigma, F) = \{\alpha \in \mathbb{R} : Eq^{\alpha}(\Sigma, F) \neq \emptyset\}$$

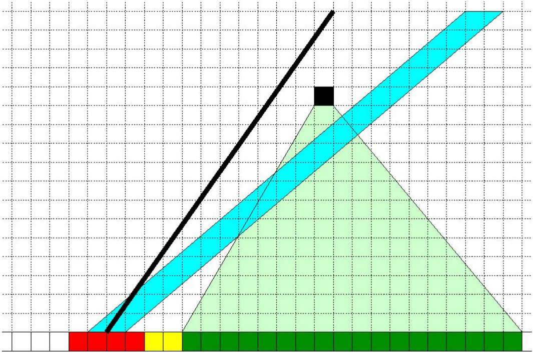


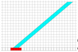
Let N be the constant of weakly-specification. Let $n \geq \frac{|u|+N}{\alpha+r}$.
We consider the local rule of the CA F^n of neighborhood $[nr, sn]$.

For every word $u \in \mathcal{A}^{[rn,sn]}$, one obtains .

$\mathbf{A}(\Sigma, F) \subset]-s, -r[$

$$\mathbf{A}(\Sigma, F) = \{\alpha \in \mathbb{R} : Eq^{\alpha}(\Sigma, F) \neq \emptyset\}$$



By weakly-specification, it is possible to stick every word of  with  sufisently near thanks to  of maximum length N.
Then F is "nilipotent" ($\exists c \in \mathcal{A}^{\mathbb{Z}}$ σ-periodic and $n \in \mathbb{N}$ such that $\forall x \in \mathcal{A}^{\mathbb{Z}}$ there exists k which verifies $F^n(x) = \sigma^k(c)$)

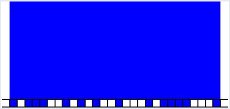
Directions with equicontinuous points

Theorem

Let Σ be a weakly specified subshift and $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA of neighborhood $\mathbb{U} = [r, s]$.

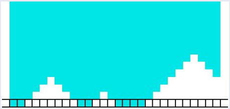
Four cases are possible for $\mathbf{A}(\Sigma, F) = \{\alpha \in \mathbb{R} : Eq^{\alpha}(\Sigma, F) \neq \emptyset\}$ **:**

- $\mathbf{A}(\Sigma, F) = \mathbb{R}$



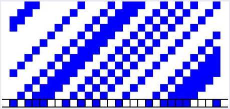
$F(x)_m = 1$

- $\mathbf{A}(\Sigma, F) = \underset{?}{(}\alpha', \alpha''\underset{?}{)} \subset [-s, -r]$



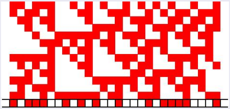
$F(x)_m = x_{m-1} \cdot x_m \cdot x_{m-1}$

- $\mathbf{A}(\Sigma, F) = \{\alpha\}$ $\alpha \in \mathbb{Q}$?



$F(x)_m = x_{m+1}$

- $\mathbf{A}(\Sigma, F) = \emptyset$



$F(x)_m = x_{m-1} + x_m + x_{m+1} \mod 2$

What happen if Σ is not weakly specified ?

Let $\mathcal{A} = \{0,1\}$ and $F(x)_i = x_{i-1} \cdot x_i \cdot x_{i+1}$. Consider $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ such that

$$\mathcal{L}_\Sigma \cap (\{0^m 1^n : f(n) \geq m\} \cup \{1^n 0^m : f(n) \geq m\}) = \emptyset.$$

For all $h : \mathbb{N} \to \mathbb{N}$ such as $f(n) \geq h(n) \geq f(n)$, one has

$$[100001] \subset E^h({}^\infty 0^\infty, 2^{-2})$$

where

$$E^h({}^\infty 0^\infty, 2^{-2}) = \{y \in \Sigma : d_C(\sigma^{h(n)} \circ F^n({}^\infty 0^\infty), \sigma^{h(n)} \circ F^n(y)) < \varepsilon \ \forall n \in \mathbb{N}\}$$

Remark

It is possible to define dynamics of slope $h : \mathbb{N} \to \mathbb{N}$ considering the following tube around the orbit of x :

$$E^h(x, \varepsilon) = \{y \in \Sigma : d_C(\sigma^{h(n)} \circ F^n(x), \sigma^{h(n)} \circ F^n(y)) < \varepsilon \ \forall n \in \mathbb{N}\}$$

Equicontinuous directions

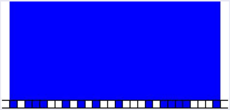
$$\mathbf{A'}(\Sigma, F) = \{\alpha \in \mathbb{R} : Eq^{\alpha}(\Sigma, F) = \Sigma\} \subset \mathbf{A}(\Sigma, F)$$

Theorem

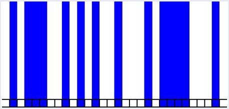
Let Σ be a weakly specifed subshift and F of neighborhood $\mathbb{U} = [r, s]$.

Three case are possible :

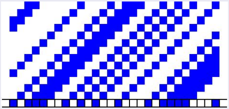
- $\mathbf{A'}(\Sigma, F) = \mathbb{R}$
- $\mathbf{A'}(\Sigma, F) = \{\alpha\}$ $\alpha \in \mathbb{Q} \cap [-s, -r]$
- $\mathbf{A'}(\Sigma, F) = \emptyset$



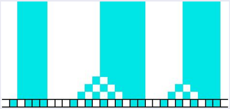
$F(x)_m = 1$



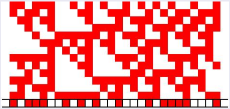
$F(x)_m = x_m$



$F(x)_m = x_{m+1}$



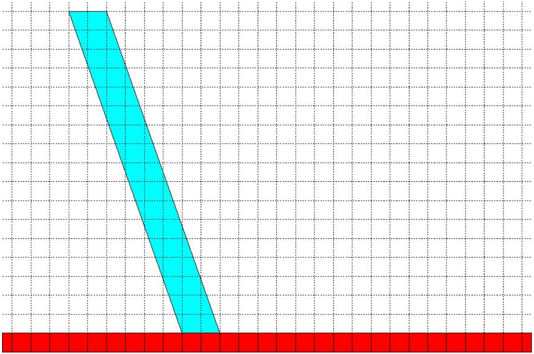
$F(x)_m = \text{majority}(x_{m-1}, x_m, x_{m+1})$

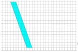


$F(x)_m = x_{m-1} + x_m + x_{m+1} \mod 2$

Cone of expansivity

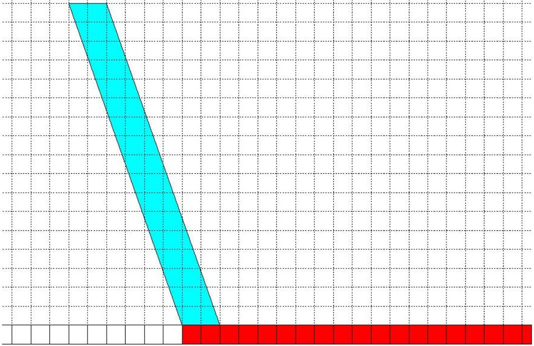
$$\mathbf{B}^{\mathbb{N}}(\Sigma, F) = \{\alpha \in \mathbb{R} : (\Sigma, F) \text{ expansive of slope } \alpha\}$$

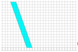
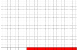


(Σ, F) expansive $\Longleftrightarrow$  codes 

Cone of right expansivity

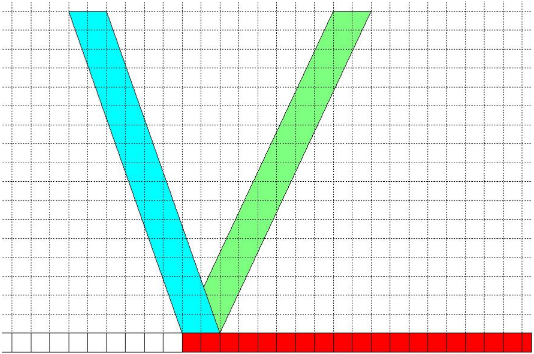
$$\mathbf{B}_d^{\mathbb{N}}(\Sigma, F) = \{\alpha \in \mathbb{R} : (\Sigma, F) \text{ right expansive of slope } \alpha\}$$

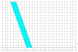
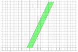


(Σ, F) right expansive $\Longleftrightarrow$  codes 

Cone of right expansivity

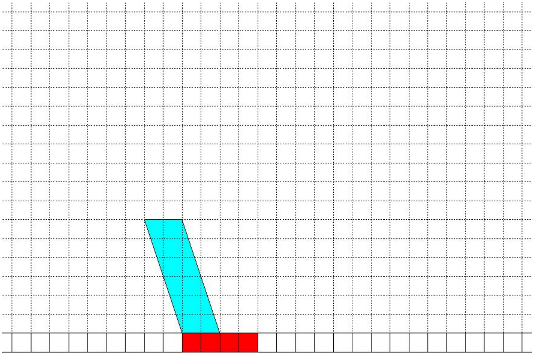
$$\mathbf{B}_d^{\mathbb{N}}(\Sigma, F) = \{\alpha \in \mathbb{R} : (\Sigma, F) \text{ right expansive of slope } \alpha\}$$



$\forall \alpha' \geq \alpha$ one has $\alpha' \in \mathbf{B}_d^{\mathbb{N}}(\Sigma, F)$ since  codes .

Cone of right expansivity

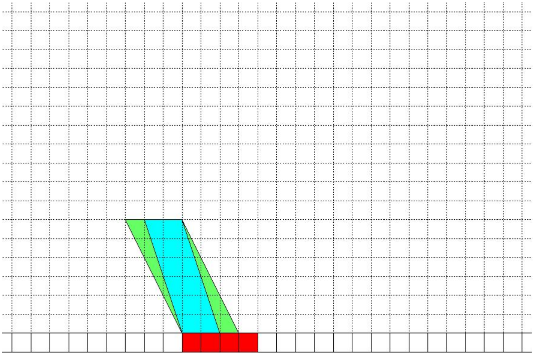
$$\mathbf{B}_d^{\mathbb{N}}(\Sigma, F) = \{\alpha \in \mathbb{R} : (\Sigma, F) \text{ right expansive of slope } \alpha\}$$



$\mathbf{B}_d^{\mathbb{N}}(\Sigma, F)$ is open

Cone of right expansivity

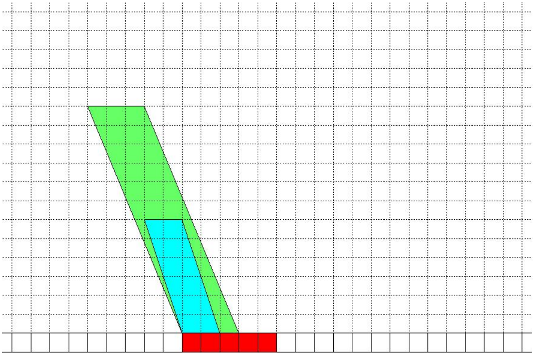
$$\mathbf{B}_d^{\mathbb{N}}(\Sigma, F) = \{\alpha \in \mathbb{R} : (\Sigma, F) \text{ right expansive of slope } \alpha\}$$



$\mathbf{B}_d^{\mathbb{N}}(\Sigma, F)$ is open

Cone of right expansivity

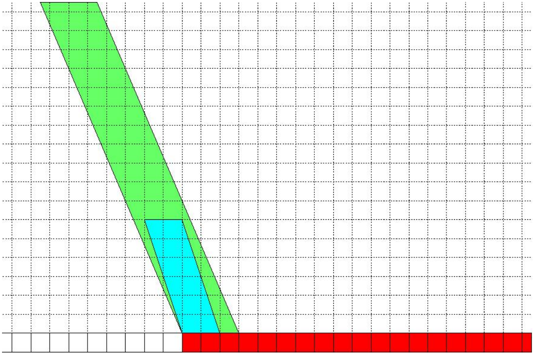
$$\mathbf{B}_d^{\mathbb{N}}(\Sigma, F) = \{\alpha \in \mathbb{R} : (\Sigma, F) \text{ right expansive of slope } \alpha\}$$



$\mathbf{B}_d^{\mathbb{N}}(\Sigma, F)$ is open

Cone of right expansivity

$$\mathbf{B}_d^{\mathbb{N}}(\Sigma, F) = \{\alpha \in \mathbb{R} : (\Sigma, F) \text{ right expansive of slope } \alpha\}$$



$\mathbf{B}_d^{\mathbb{N}}(\Sigma, F)$ is open

Directions of expansivity

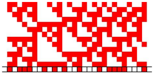
- $\mathbf{B}_g^{\mathbb{N}}(\Sigma, F) = \{\alpha \in \mathbb{R} : (\Sigma, F)$ expansive of slope $\alpha\}$.
- $\mathbf{B}_d^{\mathbb{N}}(\Sigma, F) = \{\alpha \in \mathbb{R} : (\Sigma, F)$ left expansive of slope $\alpha\}$.
- $\mathbf{B}^{\mathbb{N}}(\Sigma, F) = \{\alpha \in \mathbb{R} : (\Sigma, F)$ right expansive of slope$\alpha\}$.

Theorem

Let Σ be a subshift and $(\mathcal{A}^{\mathbb{Z}}, F)$ of neighborhood $\mathbb{U} = [r, s]$.

- $\mathbf{B}_d^{\mathbb{N}}(\Sigma, F) =]\alpha', +\infty[\subset] - s, +\infty[$. $\alpha' \in \mathbb{Q}$?
- $\mathbf{B}_g^{\mathbb{N}}(\Sigma, F) =] - \infty, \alpha''[\subset] - \infty, -r[$. $\alpha'' \in \mathbb{Q}$?

$$\mathbf{B}^{\mathbb{N}}(\Sigma, F) = \mathbf{B}_d^{\mathbb{N}}(\Sigma, F) \cap \mathbf{B}_g^{\mathbb{N}}(\Sigma, F) =]\alpha', \alpha''[\subset] - s, -r[.$$



Example : F is right permutative if $\forall u \in \mathcal{A}^{[r,s-1]}$, $\overline{F}(u\cdot) : \mathcal{A} \to \mathcal{A}$ is bijective.
One has $\mathbf{B}_d^{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) =] - s, +\infty[$.

There is other type of propagation of informations ?

In short

$$A = \{\alpha \in \mathbb{R} : \emptyset \subsetneq Eq^{\alpha}(F) \subsetneq \mathcal{A}^{\mathbb{Z}}\}$$

$$A' = \{\alpha \in \mathbb{R} : (\mathcal{A}^{\mathbb{Z}}, F) \text{ equicontinuous of slope } \alpha\}$$

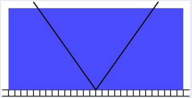
$$B = \{\alpha \in \mathbb{R} : (\mathcal{A}^{\mathbb{Z}}, F) \text{ expansif of slope } \alpha\}$$

right or left expansive directions

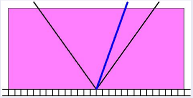
Sensitive directions

Theorem

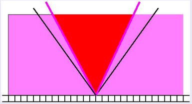
Let Σ be a weakly-specified subshift and $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA.



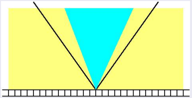
$A' = \mathbb{R}$



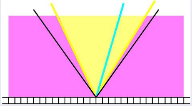
$A' = \{\alpha\} \subset \mathbb{Q}$



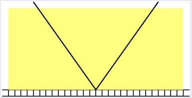
$A = \emptyset, \quad B =]\alpha', \alpha''[$



$A = [\alpha', \alpha'']$



$A = \{\alpha\}$



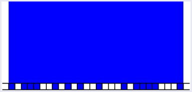
$A = \emptyset, \quad B = \emptyset$

In short

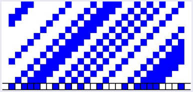
$$A = \{\alpha \in \mathbb{R} : \emptyset \subsetneq Eq^{\alpha}(F) \subsetneq \mathcal{A}^{\mathbb{Z}}\}$$
$$A' = \{\alpha \in \mathbb{R} : (\mathcal{A}^{\mathbb{Z}}, F) \text{ equicontinuous of slope } \alpha\}$$
$$B = \{\alpha \in \mathbb{R} : (\mathcal{A}^{\mathbb{Z}}, F) \text{ expansif of slope } \alpha\}$$

Theorem

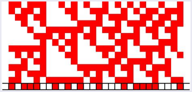
Let Σ be a weakly-specified subshift and $(\mathcal{A}^{\mathbb{Z}}, F)$ be a CA.



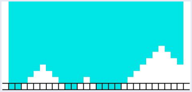
$A' = \mathbb{R}$



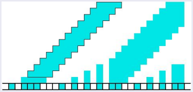
$A' = \{\alpha\} \subset \mathbb{Q}$



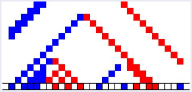
$A = \emptyset, \quad B =]\alpha', \alpha''[$



$A = [\alpha', \alpha'']$



$A = \{\alpha\}$



$A = \emptyset, \quad B = \emptyset$

Some applications

Different sight about directional dynamics

- Notion of directional attractors
- Notion of directional entropy
- (F, σ)-invariant measures

Notion of attractor

- **Limit set** of $Y \subset \mathcal{A}^{\mathbb{Z}}$ is :

$$\Lambda_F(Y) = \bigcap_{n \in \mathbb{N}} \overline{\cup_{m \geq n} F^m(Y)}.$$

- $Y \subset \mathcal{A}^{\mathbb{Z}}$ is an **attractor** if there exists an open set $U \subset \mathcal{A}^{\mathbb{Z}}$ such that :

$$F^n(\overline{U}) \subset U \ \forall n \in \mathbb{N} \quad \text{and} \quad Y = \Lambda_F(U).$$

Theorem : Attractor's classification of Kůrka and Hurley

A_1^0 $(\mathcal{A}^{\mathbb{Z}}, F)$ has a pair of disjoint attractors ;
A_2^0 $(\mathcal{A}^{\mathbb{Z}}, F)$ has a unique minimal quasi-attractor ;
A_3^0 $(\mathcal{A}^{\mathbb{Z}}, F)$ has a unique minimal attracteur different from $\Lambda_F(\mathcal{A}^{\mathbb{Z}})$;
A_4^0 $(\mathcal{A}^{\mathbb{Z}}, F)$ has a unique attracteur : $\Lambda_F(\mathcal{A}^{\mathbb{Z}})$;

Directional attractor

- **Limit set** of $Y \subset \mathcal{A}^{\mathbb{Z}}$ of slope α is :

$$\Lambda_F^{\alpha}(Y) = \bigcap_{n\in\mathbb{N}} \overline{\cup_{m\geq n} F^m \circ \sigma^{\lfloor m\alpha \rfloor}(Y)}.$$

- $Y \subset \mathcal{A}^{\mathbb{Z}}$ is an **attractor** of slope α if there exists an open set $U \subset \mathcal{A}^{\mathbb{Z}}$ such that :

$$F^n \circ \sigma^{\lfloor n\alpha \rfloor}(\overline{U}) \subset U \ \forall n \in \mathbb{N} \quad \text{and} \quad Y = \Lambda_F^{\alpha}(U).$$

Theorem : Classification according a direction

A_1^{α} $(\mathcal{A}^{\mathbb{Z}}, F)$ has a pair of disjoint attractors of slope α ;

A_2^{α} $(\mathcal{A}^{\mathbb{Z}}, F)$ has a unique minimal quasi-attractor of slope α ;

A_3^{α} $(\mathcal{A}^{\mathbb{Z}}, F)$ has a unique minimal attracteur of slope α different from $\Lambda_F^{\alpha}(\mathcal{A}^{\mathbb{Z}})$;

A_4^{α} $(\mathcal{A}^{\mathbb{Z}}, F)$ has a unique attracteur de pente α : $\Lambda_F^{\alpha}(\mathcal{A}^{\mathbb{Z}})$;

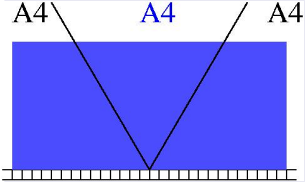
Links between sensitivity to initial conditions and attractors

Links according a direction Kůrka

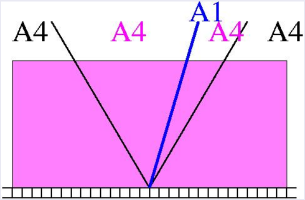
	A_1^0	A_2^0	A_3^0	A_4^0
$(\mathcal{A}^{\mathbb{Z}}, F)$ equicontinuous	OK	$\emptyset$	$\emptyset$	OK
$\emptyset \subsetneq Eq^0(F) \subsetneq \mathcal{A}^{\mathbb{Z}}$	OK	OK	OK	OK
$(\mathcal{A}^{\mathbb{Z}}, F)$ sensitive	OK	OK	OK	OK
$(\mathcal{A}^{\mathbb{Z}}, F)$ expansive	$\emptyset$	$\emptyset$	$\emptyset$	OK

Links between sensitivity to initial conditions and attractors

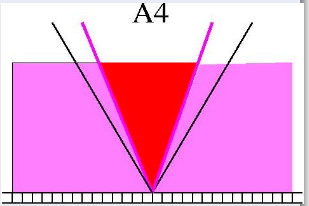
Theorem



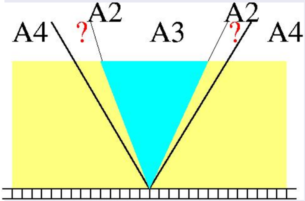
$A' = \mathbb{R}$



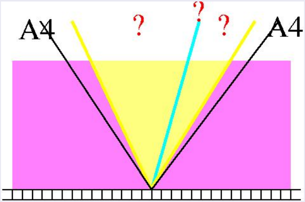
$A' = \{\alpha\} \subset \mathbb{Q}$



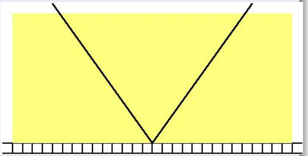
$A = \emptyset, \quad B =]\alpha', \alpha''[$



$A = [\alpha', \alpha'']$



$A = \{\alpha\}$



$A = \emptyset, \quad B = \emptyset$

Different sight about directional dynamics

- Notion of directional attractors
- Notion of directional entropy
- (F, σ)-invariant measures

Directionnal entropy

Definition and study of $\alpha \to h_{\text{top}}(F, \alpha)$ by Milnor-96 and Boyle-Lind-97.
Let $\mathcal{P} = \{U_1, ..., U_p\}$ be a partition :

$$H_{\text{top}}(\mathcal{P}) = \log(\min\{n \in \mathbb{N} : \exists i_1, ...i_n \in [1, p], \mathcal{A}^{\mathbb{Z}} = U_{i_1} \cup ... \cup U_{i_p}\}).$$

Definition

Let $\mathcal{P}_{[-l,l]}$ be the partition on centred words of length l.

$$h_{top}(F, \alpha) = \lim_{l \to \infty} \lim_{N \to \infty} \frac{1}{N} H_{top} \left(\bigvee_{n=0}^{N-1} F^{-n} \circ \sigma^{-\lfloor n\alpha \rfloor} \mathcal{P}_{[-l,l]} \right)$$

Liens avec la dynamique directionnelle

Majoration

$h_{top}(F,\alpha) \leq (\max(s+\alpha) - \min(r+\alpha,0))\, h_{top}(\sigma)$ where $\mathbb{U} = [r,s]$ is the neighbour of $(\mathcal{A}^{\mathbb{Z}}, F)$.
We have equality if F is bipermutative.

Ask

There is other case of equality ?

Some links with directional dynamics

- If $\alpha \in$ **A'**(Σ, F) then $h_{\text{top}}(F,\alpha) = 0$.
- $\alpha \to h_{\text{top}}(F,\alpha)$ is convexe on $\mathbf{B}_g^{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F) \cup \mathbf{B}_d^{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}, F)$.
- $h_{top}(\sigma) > 0$ iff $h_{top}(F,\alpha) > 0\ \forall \alpha \in \mathbf{B}_g^{\mathbb{N}}(F) \cup \mathbf{B}_d^{\mathbb{N}}(F)$.

Different sight about directional dynamics

- Notion of directional attractors
- Notion of directional entropy
- (F, σ)-invariant measures

(F,σ)-invariant measures

$$A = \{\alpha \in \mathbb{R} : \emptyset \subsetneq Eq^{\alpha}(F) \subsetneq \mathcal{A}^{\mathbb{Z}}\}$$

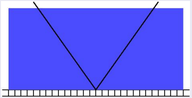
$$A' = \{\alpha \in \mathbb{R} : (\mathcal{A}^{\mathbb{Z}}, F) \text{ équicontinue de pente } \alpha\}$$

$$B = \{\alpha \in \mathbb{R} : (\mathcal{A}^{\mathbb{Z}}, F) \text{ expansif de pente } \alpha\}$$

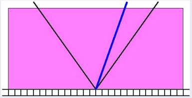
right or left expansive directions

Sensitive directions

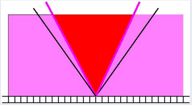
Soit $\mu \in \mathcal{M}^{\text{erg}}_{F,\sigma}(\mathcal{A}^{\mathbb{Z}})$



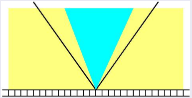
$\mu = \delta_{\infty a^\infty}$



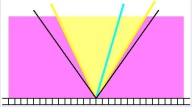
$\mathcal{M}_{F,\sigma} = \sum_{i=0}^{p-1} F^{m+i}\mathcal{M}_\sigma$



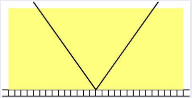
$Ah_\mu(\sigma) \leq h_\mu(F,\alpha) \leq Bh_\mu(\sigma)$



$\mu(B) > 0 \Rightarrow \mu = \delta_{\infty a^\infty}$



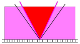
$h_\mu(F,\alpha) = 0$



???

The case of algebraic CA

A CA is said algebraic if $\mathcal{A}^{\mathbb{Z}}$ is a group and $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a morphism.

An algebraic CA is in the class . Moreover, one has :

$$h_\mu(F,\alpha) = (\max(s+\alpha,0) - \min(r+\alpha,0))\, h_\mu(\sigma)$$

There is a lot of rigidity results :

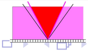
- General agebraic action : Furstenberg-67, Schmidt-95, Eisiendler-05
- Cellular automata : Host-Maass-Martínez-03,Pivato-05

Theorem Sablik-06

Let $(\mathcal{A}^{\mathbb{Z}}, F)$ be an algebraic CA, $\Sigma \subset \mathcal{A}^{\mathbb{Z}}$ a subgroup and $\mu \in \mathcal{M}_{\sigma,F}(\Sigma)$.

- μ (F,σ)-ergodic and $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{|\mathcal{A}|^{p_1}})$
- $h_\mu(F) > 0$
- $D_\infty(F) = \cup_{n\in\mathbb{N}}\mathrm{Ker}(F^n)$ has dense infinite subgroupes σ-invariants

Then $\mu = \lambda_{\mathcal{A}^{\mathbb{Z}}}$.

It is possible to obtain rigidity results for the class  ?

(F,σ)-invariant measures

$$A = \{\alpha \in \mathbb{R} : \emptyset \subsetneq Eq^{\alpha}(F) \subsetneq \mathcal{A}^{\mathbb{Z}}\}$$

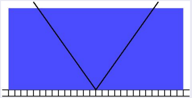
$$A' = \{\alpha \in \mathbb{R} : (\mathcal{A}^{\mathbb{Z}}, F) \text{ équicontinue de pente } \alpha\}$$

$$B = \{\alpha \in \mathbb{R} : (\mathcal{A}^{\mathbb{Z}}, F) \text{ expansif de pente } \alpha\}$$

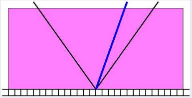
right or left expansive directions

Sensitive directions

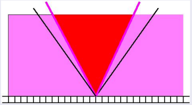
Soit $\mu \in \mathcal{M}^{\mathrm{erg}}_{F,\sigma}(\mathcal{A}^{\mathbb{Z}})$



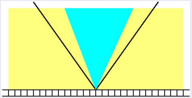
$\mu = \delta_{\infty a \infty}$



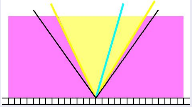
$\mathcal{M}_{F,\sigma} = \sum_{i=0}^{p-1} F^{m+i}\mathcal{M}_{\sigma}$



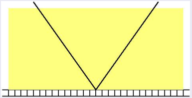
$Ah_{\mu}(\sigma) \leq h_{\mu}(F,\alpha) \leq Bh_{\mu}(\sigma)$



$\mu(B) > 0 \Rightarrow \mu = \delta_{\infty a \infty}$



$h_{\mu}(F,\alpha) = 0$



???

(F,σ)-invariant measures

$$A = \{\alpha \in \mathbb{R} : \emptyset \subsetneq Eq^{\alpha}(F) \subsetneq \mathcal{A}^{\mathbb{Z}}\}$$

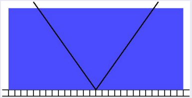
$$A' = \{\alpha \in \mathbb{R} : (\mathcal{A}^{\mathbb{Z}}, F) \text{ équicontinue de pente } \alpha\}$$

$$B = \{\alpha \in \mathbb{R} : (\mathcal{A}^{\mathbb{Z}}, F) \text{ expansif de pente } \alpha\}$$

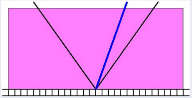
right or left expansive directions

Sensitive directions

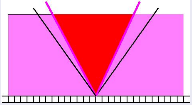
Soit $\mu \in \mathcal{M}^{\mathrm{erg}}_{F,\sigma}(\mathcal{A}^{\mathbb{Z}})$



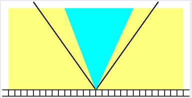
$\mu = \delta_{{}^{\infty}a^{\infty}}$



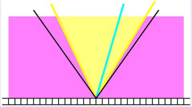
$\mathcal{M}_{F,\sigma} = \sum_{i=0}^{p-1} F^{m+i}\mathcal{M}_{\sigma}$



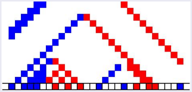
$Ah_{\mu}(\sigma) \leq h_{\mu}(F,\alpha) \leq Bh_{\mu}(\sigma)$



$\mu(B) > 0 \Rightarrow \mu = \delta_{{}^{\infty}a^{\infty}}$



$h_{\mu}(F,\alpha) = 0$



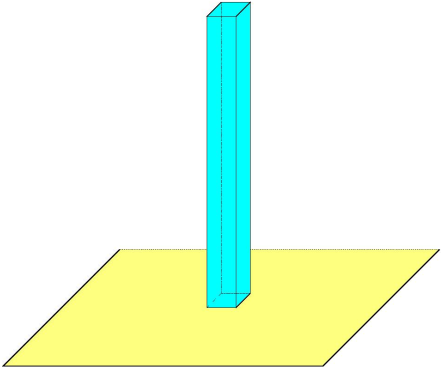
$\mathcal{M}_{\sigma}(\{\square, \blacksquare\}^{\mathbb{Z}}) \cup \mathcal{M}_{\sigma}(\{\square, \blacksquare\}^{\mathbb{Z}})$

What happen in other dimensions ?

Joint work with G. Theyssier

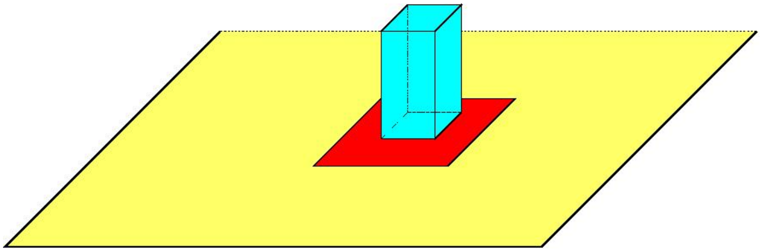
Action of F sur $\mathcal{A}^{\mathbb{Z}^d}$

$$E_{\Sigma}^{\mathbb{N}}(x,\varepsilon) = \left\{y \in \Sigma : \forall n \in \mathbb{N} \text{ on a } d_C(F^n(x), F^n(y)) < \varepsilon\right\}$$

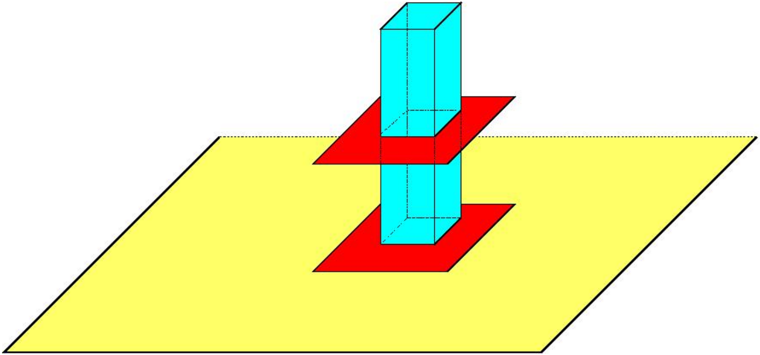


$$B_{\Sigma}(x,\delta)\{y \in \Sigma : d_C(x,y) < \delta\}$$

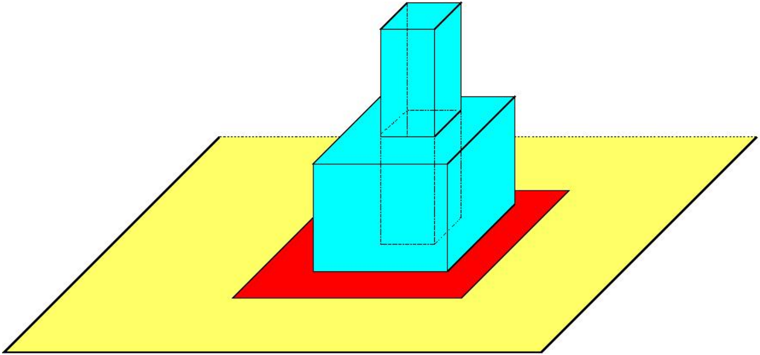
$F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ cannot be expansive



$F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ cannot be expansive



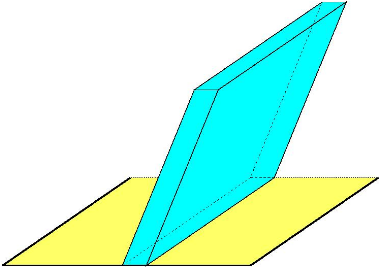
$F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ cannot be expansive



Expansivity as a $Z^d \times \mathbb{N}$-action

Let Γ be a *sub-vectorial space* of $\mathbb{R}^d \times \mathbb{R}_+$.
Denote $\Gamma^T = \{t \in \mathbb{R}^d \times \mathbb{R}_+ : \exists t' \in \Gamma \text{ tel que } ||t - t'|| < 1\}$.

$$E_\Sigma^\Gamma(x,\varepsilon) = \left\{ y \in \Sigma : \forall n \in \Gamma^T \cap \mathbb{Z}^d \times \mathbb{N} \quad d_C((\sigma,F)^n(x),(\sigma,F)^n(y)) < \varepsilon \right\}$$



Definition

(Σ, F) is expansive of slope Γ if $\exists \varepsilon > 0$ such that

$$\forall x \in \Sigma \quad E_\Sigma^\Gamma(x,\varepsilon) = \{x\}.$$

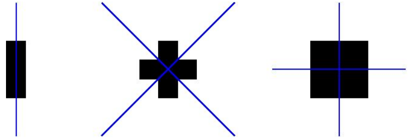
The direction of expansivity is defined by :

- the base, denoted $\Gamma_0 = \Gamma \cap \mathbb{R}^d \times \{0\}$
- the angle according the direction of the CA

Some properties

Such examples :

$\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$ and $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is defined as the addition according the following neighborhood :



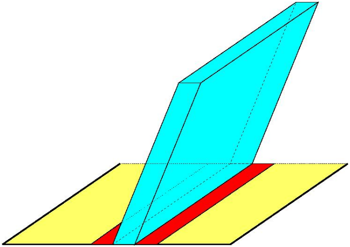
Some properties

- If a base is fixed, one obtains the results of unidimensional CA.
- Expansivity is possible just according a slpoe of codim 1
- The set of expansive direction is open.

Which directions are possible for the bases ?

CA with equicontinuous points and sensitive CA

$$E^{\Gamma}_{\Sigma}(x,\varepsilon) = \{y \in \Sigma : \forall n \in \Gamma^{T} \cap \mathbb{Z}^d \times \mathbb{N} \quad d_C((\sigma,F)^n(x),(\sigma,F)^n(y)) < \varepsilon\}$$



$$B^{\Gamma_0}_{\Sigma}(x,\delta) = \{y \in \Sigma : \forall n \in \Gamma^{T}_0 \cap \mathbb{Z}^d \times \mathbb{N} \quad d_C((\sigma,F)^n(x),(\sigma,F)^n(y)) < \delta\}$$

Définition

- $x \in Eq^{\Gamma}(\Sigma,F) \iff \forall\varepsilon > 0\ \exists\delta$ such that $B^{\Gamma_0}(x,\delta) \subset E^{\Gamma}_{\Sigma}(x,\varepsilon)$.
- (Σ,F) is sensitive if $\exists\varepsilon > 0, \quad \forall\delta > 0, \exists y \in B^{\Gamma_0}(x,\delta) \cap E^{\Gamma}_{\Sigma}(x,\varepsilon)$.

Some properties

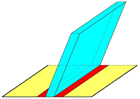
Let Γ be a sub-vectorial space. One defines :

- $\mathcal{E}^\Gamma$ the set of CA which have equicontinuous points according to Γ,
- $\mathcal{S}^\Gamma$ the set of sensitive CA according to Γ,
- $\mathcal{N}^\Gamma$ the set of CA which are neither in $\mathcal{E}^\Gamma$ nor in $\mathcal{S}^\Gamma$.

$\mathrm{codim}(\Gamma) = 1$	$\mathrm{codim}(\Gamma) \geq 2$
• $\mathcal{N}^\Gamma = \emptyset$ • $\mathcal{E}^\Gamma$ and $\mathcal{S}^\Gamma$ are neither r.e. nor co-r.e. • If $F \in \mathcal{S}^\Gamma$ then the sensitivity constant is recursive.	• $\mathcal{N}^\Gamma \neq \emptyset$ • $\mathcal{E}^\Gamma$, $\mathcal{S}^\Gamma$ and $\mathcal{N}^\Gamma$ are neither r.e. nor co-r.e. • If $F \in \mathcal{S}^\Gamma$ then the sensitive constant cannot be recursive.

Equicontinuous CA as a $Z^d \times \mathbb{N}$-action

$$E_{\Sigma}^{\Gamma}(x,\varepsilon) = \{y \in \Sigma : \forall n \in \Gamma^T \cap \mathbb{Z}^d \times \mathbb{N} \quad d_C((\sigma,F)^n(x),(\sigma,F)^n(y)) < \varepsilon\}$$



$$B_{\Sigma}^{\Gamma_0}(x,\delta) = \{y \in \Sigma : \forall n \in \Gamma_0^T \cap \mathbb{Z}^d \times \mathbb{N} \quad d_C((\sigma,F)^n(x),(\sigma,F)^n(y)) < \delta\}$$

Définition

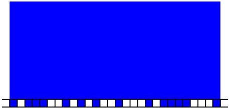
- (Σ, F) equicontinuous of slope Γ if and only if

$\Longleftrightarrow \forall \varepsilon > 0 \quad \exists \delta$ tel que $\forall x \in \Sigma \qquad B^{\Gamma_0}(x,\delta) \subset E_{\Sigma}^{\Gamma}(x,\varepsilon)$.

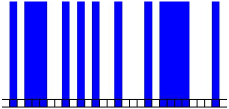
Some properties for equicontinuity of slope Γ :

- If (Σ, F) is equicontinuous of slope Γ then (Σ, F) is equicontinuous of slope Γ' for every sub-vectorial space $\Gamma' \supset \Gamma$.
- If F is an equicontinuous CA according to a Γ (Γ maximal) then Γ is a rational subvectorial space.

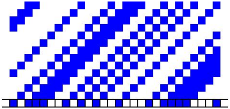
Some examples



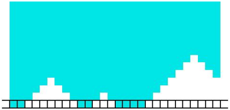
$F(x)_m = 1$



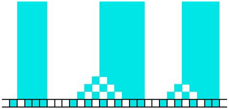
$F(x)_m = x_m$



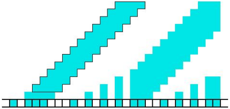
$F(x)_m = x_{m-1}$



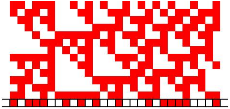
$F(x)_m = x_{m-1} \cdot x_m \cdot x_{m+1}$



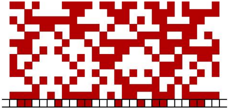
$F(x)_m =$
$\max(x_{m-1}, x_m, x_{m+1})$



$F(x)_m =$
$\max(x_m, x_{m-1}, x_{m-2})$



$F(x)_m = x_m + x_{m+1}$
mod 2



$F(x)_m = x_{m-1} + x_m + x_{m+1}$
mod 2