MULTIDIMENSIONAL CONTINUED FRACTIONS AND SYMBOLIC CODINGS OF TORAL TRANSLATIONS

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Abstract. It has been a long standing problem to find good symbolic codings for translations on the $d$-dimensional torus that enjoy the beautiful properties of Sturmian sequences like low factor complexity and good local discrepancy properties. Inspired by Rauzy’s approach we construct such codings in terms of multidimensional continued fraction algorithms that are realized by sequences of substitutions. In particular, given any strongly convergent continued fraction algorithm, these sequences lead to renormalization schemes which produce symbolic codings and bounded remainder sets at all scales in a natural way.

As strong convergence of a continued fraction algorithm results in a Pisot type property of the attached symbolic dynamical systems, our approach provides a systematic way to confirm purely discrete spectrum results for wide classes of dynamical systems. Indeed, as our examples illustrate, we are able to confirm the Pisot conjecture for many well-known families of sequences of substitutions. These examples comprise classical algorithms like the Jacobi–Perron, Brun, Cassaigne–Selmer, and Arnoux–Rauzy algorithms.

As a consequence, we gain symbolic codings of almost all translations of the 2-dimensional torus having factor complexity $2^n+1$ that are balanced on all factors, and hence, have multiscale bounded remainder sets. Using the Brun algorithm, we also give symbolic codings of almost all 3-dimensional toral translations having multiscale bounded remainder sets.

1. Introduction

One of the classical motivations of symbolic dynamics is to provide representations of dynamical systems as subshifts made of infinite sequences which code itineraries through suitable choices of partitions. In the present paper we focus on symbolic models for toral translations. More precisely, for a given toral translation, we provide symbolic realizations based on multidimensional continued fraction algorithms. These realizations have strong dynamical and arithmetic properties. In particular, they define bounded remainder sets for toral translations with a natural subdivision structure governed by the underlying continued fraction algorithm. We recall that bounded remainder sets are defined as sets having bounded local discrepancy. In ergodic terms, these are sets for which the Birkhoff sums of their characteristic function have bounded deviations. Their study started with the work of W. M. Schmidt in his series of papers on irregularities of distributions (see for instance [Sch74]) and has led to many important contributions (cf. e.g. [GL15] for the according references).

Our approach is inspired by the seminal example of Sturmian dynamical systems, introduced by M. Morse and G. Hedlund in [MH40]. There is an impressive literature devoted to their study and to possible generalizations in word combinatorics [Fog02], and also in digital geometry [RK01]. The importance of Sturmian dynamical systems is due to several reasons. For instance, they provide symbolic codings for the simplest arithmetic dynamical systems, namely for irrational translations on the circle, they code discrete lines, and they are one-dimensional models of quasicrystals [BG13]. The scale invariance of Sturmian dynamical systems allows them to be described by using a
renormalization scheme governed by classical continued fractions which in turn can be interpreted as Poincaré sections of the geodesic flow acting on the modular surface. This admits important generalizations in the study of interval exchange transformations in relation with the Teichmüller flow and renormalization schemes that can often be interpreted as continued fractions \[Yoc06\]. The basic combinatorial elements for the understanding of Sturmian dynamical systems together with their renormalization scheme are substitutions which are symbolic versions of induction steps (i.e., of first return maps).

In order to get symbolic models, in the present work we rely on substitutive dynamical systems as well as on the more general $S$-adic dynamical systems. A substitution is a rule, either combinatorial or geometric, that replaces a letter by a word, or a tile by a patch of tiles. Substitutions are used to define substitutive dynamical systems which play a fundamental role in symbolic dynamics, as emphasized e.g. in the monographs \[BG13\], \[Fog02\], \[Que10\]. In particular, Pisot substitutions are of importance in this context since they create hierarchical structures with a significant amount of long range order \[ABB+15\]. Substitutive dynamical systems defined in terms of Pisot substitutions are conjectured to have purely discrete spectrum, that is, to be isomorphic (in the measure-theoretic sense) to a translation on a compact abelian group. The fact that this so-called Pisot substitution conjecture is still open (even though it is solved for beta-numeration in \[Bar18\]) shows that important parts of the picture are still to be developed.

More generally, $S$-adic dynamical systems are defined in terms of words that are generated by iterating sequences of substitutions, rather than iterating just a single substitution, much the same way like multidimensional continued fraction algorithms in general produce sequences of matrices, and not just powers of a single one. A survey on $S$-adic dynamical systems is provided in \[BD14\]. The $S$-adic formalism offers representations similar to the Bratteli–Vershik systems related to Markov compacta, and to representations by Rohlin towers as studied for instance in \[DHS99\] or \[BR10\] Chapter 6. In \[BST19\], we extend the Pisot conjecture to $S$-adic dynamical systems, which enables us to go beyond algebraicity. Since $S$-adic dynamical systems are defined in terms of sequences of substitutions, they can be regarded as non-abelian versions of multidimensional continued fraction algorithms. The Pisot condition used in the substitutive case is replaced here by the requirement that the second Lyapunov exponent of the system is negative. Under this condition $S$-adic dynamical systems are conjectured to have purely discrete spectrum. In \[BST19\] we prove that this extended Pisot conjecture holds for large families of $S$-adic dynamical systems based on well-known continued fraction algorithms, such as the Brun or the Arnoux–Rauzy algorithm. As a striking outcome, this yields symbolic codings for almost every translation of the torus \[T^2\] \[BST19\], paving the way for the development of equidistribution results for the associated two-dimensional Kronecker sequences.

In the present paper we extend this study to higher dimensions and handle many well-known continued fraction algorithms. For instance, our new theory works for generalized continued fraction algorithms including the Brun, Selmer, and Jacobi–Perron algorithm. To each strongly convergent continued fraction algorithm we attach a shift-invariant set of $S$-adic sequences which generically leads to $S$-adic dynamical systems having purely discrete spectrum. This shows that $S$-adic dynamical systems are measurably conjugate to minimal translations on the torus. In other words, we provide symbolic representations of toral translations, i.e., symbolic dynamical systems that code toral translations in the measure-theoretic sense, as well as symbolic representations for multidimensional continued fractions. In particular, we gain symbolic codings of almost all translations of the 2-dimensional torus having factor complexity $2n + 1$ that are balanced on all factors (and, hence, bounded remainder sets at all scales); see Corollary 6.3. Using the Brun algorithm, we also give symbolic codings of almost all 3-dimensional toral translations with multiscale bounded remainder sets; see Theorem 6.8.

We use two main ingredients. Firstly, the above-mentioned Pisot type property which is formulated in terms of the second Lyapunov exponent and can be seen as a strong convergence property in the setting of continued fractions. Secondly, the existence of a single substitutive dynamical system that “behaves well” which corresponds to a periodic sequence in the set of $S$-adic sequences under consideration. We mention that some of our results on the purely discrete spectrum of $S$-adic dynamical systems do not require “coincidence type” conditions in their assumptions which
so far were commonly needed in this context in order to get purely discrete spectrum; see Theorems 3.3 and 3.6. Indeed, we can prove that each algorithm that satisfies the Pisot condition has an acceleration that leads to toral translations almost surely by using the existence of arbitrarily large blocks of Pisot substitutions in the set of $S$-adic sequences.

Figure 1. An $S$-adic Rauzy fractal and its subdivision (cf. Section 2.4) whose directive sequence $(\sigma_n)_{n \in \mathbb{N}}$ starts with $\sigma_0 = \cdots = \sigma_7$ and $\sigma_8 = \cdots = \sigma_{15}$, where $\sigma_0$, defined by $1 \mapsto 13$, $2 \mapsto 12$, $3 \mapsto 2$, is a Cassaigne substitution (see Section 6.1), and $\sigma_8$ is the classical Tribonacci substitution $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$.

In our proofs, we also heavily rely on the theory of $S$-adic Rauzy fractals which has been developed in [BST19]. For an illustration of such a Rauzy fractal, see Figure 1. Rauzy fractals have been introduced in [Rau82] for the so-called Tribonacci substitution; see also [Thu89]. One motivation for Rauzy’s construction was to exhibit explicit factors of the substitutive dynamical system as translations on compact abelian groups, under the Pisot hypothesis. The formalism of $S$-adic Rauzy fractals allows us to verify the Pisot conjecture on sequences of substitutions for wide families of systems of Pisot type, thereby extending the results in [BST19, FN20]; see Theorems 3.1 and 3.5. Already in [BST19], for the Brun algorithm as well as the Arnoux–Rauzy algorithm, purely discrete spectrum results have been shown. Parallel to our work, [FN20] proved results on purely discrete spectrum of $S$-adic dynamical systems coming from continued fraction algorithms with special emphasis on the Cassaigne–Selmer algorithm. However, the conditions we have to assume in our main results are easy to check effectively and our results (stated in Section 3) are more general than the ones in [BST19, FN20]. This allows us to treat the Arnoux–Rauzy algorithm in arbitrary dimensions as well as (multiplicative) continued fraction algorithms like the Jacobi–Perron algorithm (which requires to work with $S$-adic dynamical systems based on infinitely many substitutions). Another novelty of our results is that they can be used to establish multiscale bounded remainder sets and multiscale natural codings for almost all translations on the torus; see Theorem 3.8. Note that the constructions of bounded remainder sets given in [GL15, HKK17] do not offer such a scalability.

As applications for the present results, we want to mention the recent paper [CDFG20], where our present results are used in the framework of Schrödinger operators with quasi-periodic multi-frequency potentials based on toral translations. In particular, they use our theory to produce Cantor spectra of zero Lebesgue measure for these potentials. Moreover, we are currently considering higher-dimensional versions of the three-distance theorem in [ABK+20] where the involved shapes are generated by symbolic and geometric versions of continued fraction algorithms (related again to $S$-adic Rauzy fractals). More generally, we would like to deduce global discrepancy estimates for multidimensional Kronecker sequences from the local study of bounded remainder sets and thanks to the symbolic codings considered here. This is in the spirit of the one-dimensional results obtained in [Ada04a]. In [ABM+20] we also consider Markov partitions for nonstationary hyperbolic toral automorphisms (as defined in [AF05]) related to continued fraction algorithms.
We thereby develop symbolic models as nonstationary subshifts of finite type and Markov partitions for sequences of toral automorphisms. The pieces of the corresponding Markov partitions are fractal sets (and more precisely $S$-adic Rauzy fractals) defined by associating substitutions to (incidence) matrices, or in terms of Bratteli diagrams, obtained by constructing suspensions via two-sided Markov compacta [Bru13].

Outline of the paper. After recalling basic notation and definitions in Section 2, Section 3 is devoted to the statement of our main results on purely discrete spectrum including their consequences on natural codings of translations and bounded remainder sets. The concepts needed in the proofs of our results are provided in Section 4. In particular, we recall the required background on Rauzy fractals. These proofs are then given in Section 5, Section 6 is devoted to the detailed discussion of some examples which provide codings of a.e. translation on $T^2$ and $T^3$ that lead to bounded remainder sets of all scales.

2. Mise en scène

2.1. Multidimensional continued fraction algorithms. There are several formalisms for defining multidimensional continued fractions, see e.g. [AL18, Bre81, BAG01, KLDM06, Lag93, Lag94, Sch00]. In this paper, a $(d-1)$-dimensional continued fraction algorithm $(\Delta, T, A)$ is defined on a set

$$\Delta \subseteq \{x \in [0, 1]^d : \|x\|_1 = 1\}$$

by a map

$$A : \Delta \to \text{GL}(d, \mathbb{Z})$$

satisfying $\mathbf{t}A(x)^{-1}x \in \mathbb{R}_{\geq 0}^d$ for all $x \in \Delta$, together with the associated transformation

$$T : \Delta \to \Delta, \quad x \mapsto \mathbf{t}A(x)^{-1}x / \|\mathbf{t}A(x)^{-1}x\|_1. \quad (2.1)$$

Here $\mathbf{t}M$ denotes the transpose of a matrix $M$. The map $A$ is usually piecewise constant which entails that $T$ is piecewise continuous. These algorithms are called linear simplex-splitting in [Lag93, Section 2], and their iteration produces convergent matrices used for simultaneous Diophantine approximation. The matrices $\mathbf{t}A(x)$ are called partial quotient matrices. This class of algorithms contains prominent examples like the classical algorithms of Brun [Bru19, Bru20, Bru58, Jacobi–Perron [Ber71, HJ68, Per07, Sch73], and Selmer [Sel61], which are discussed in Section 6. In the present paper the transition from the linear homogeneous version of the algorithm given by the transpose of the map $A$ to its projectivized version (2.1) is performed by a normalization by the $1$-norm. This choice allows one to work with a symmetric version of the algorithm, as e.g. in [AL18].

A multidimensional continued fraction algorithm $(\Delta, T, A)$ is called positive if $A(x)$ is a non-negative matrix for all $x \in \Delta$, i.e., if $A(\Delta)$ is contained in

$$\mathcal{M}_d = \{M \in \mathbb{N}^{d \times d} : |\det M| = 1\},$$

with $\mathbb{N} = \{0, 1, 2, \ldots\}$. It is additive if the set of produced matrices $A(\Delta)$ is finite, multiplicative otherwise. Setting

$$A^{(n)}(x) = A(T^{n-1}x) \cdots A(Tx) A(x),$$

$A$ is a linear cocycle for $T$, i.e., it fulfills the cocycle property $A^{(m+n)}(x) = A^{(m)}(T^n x) A^{(n)}(x)$; this is the reason for defining $T$ by the transpose of $A$.

The column vectors $\mathbf{y}_i^{(n)}$, $1 \leq i \leq d$, of the convergent matrices $\mathbf{t}A^{(n)}(x)$ produce $d$ sequences of rational convergents $(\mathbf{y}_i^{(n)}/\|\mathbf{y}_i^{(n)}\|_1)_{n \in \mathbb{N}}$ that are supposed to converge to $x$. More precisely,

- $T$ converges weakly at $x \in \Delta$ if $\lim_{n \to \infty} \mathbf{y}_i^{(n)}/\|\mathbf{y}_i^{(n)}\|_1 = x$ holds for all $i \in \{1, \ldots, d\}$;
- $T$ converges strongly at $x \in \Delta$ if $\lim_{n \to \infty} \|\mathbf{y}_i^{(n)} - \mathbf{y}_i^{(n)} x\|_1 = 0$ holds for all $i \in \{1, \ldots, d\}$;
- $T$ converges exponentially at $x \in \Delta$ if there are positive constants $\kappa, \delta \in \mathbb{R}$ such that $\|\mathbf{y}_i^{(n)} - \mathbf{y}_i^{(n)} x\|_1 < \kappa e^{-\delta n}$ holds for all $i \in \{1, \ldots, d\}$ and all $n \in \mathbb{N}$. 


An important role is played by the following condition, which entails almost everywhere strong (and even exponential) convergence of the algorithm; see [Lag93 Equation (4.21)].

**Definition 2.1 (Pisot condition, cf. [BD14, BST19])**. Let \((X, T, \nu)\) be a dynamical system with ergodic invariant probability measure \(\nu\), and let \(C : X \to M_d\) be a \(\log\)-integrable linear cocycle for \(T\); here \(\log\)-integrable means that \(\int_X \log \max(1, \|C(x)\|) \, d\nu(x) < \infty\). Then the Lyapunov exponents \(\vartheta_k(C)\) of \(C\) exist and are given for \(k \in \{1, \ldots, d\}\) by \((\wedge^k \text{ denotes the } k\text{-fold exterior product})\)

\[
\vartheta_1(C) + \cdots + \vartheta_k(C) = \lim_{n \to \infty} \frac{1}{n} \log \| \wedge^k C(T^{n-1}x) \cdots C(Tx)C(x) \| \quad \text{for } \nu\text{-almost all } x \in X.
\]

We say that \((X, T, C, \nu)\) satisfies the **Pisot condition** if \(\vartheta_1(C) > 0 > \vartheta_2(C)\).

We always assume that the continued fraction algorithm \((\Delta, T, A)\) is endowed with an ergodic \(T\)-invariant probability measure \(\nu\) such that the map \(A\) is \(\nu\)-measurable; here \(GL(d, \mathbb{Z})\) carries the discrete topology. Then the Pisot condition together with the Oseledets theorem implies that there is a constant \(\delta < 0\) such that, for \(\nu\)-almost all \(x \in \Delta\), there is a hyperplane \(V\) of \(\mathbb{R}^d\) with

\[
\lim_{n \to \infty} \frac{1}{n} \log \| A^{(n)}(x) v \| \leq \delta \quad \text{for all } v \in V.
\]

This implies a.e. strong convergence of \((\Delta, T, A)\).

2.2. **Substitutive and \(S\)-adic dynamical systems, shifts of directive sequences.** Substitutions will be one of the main objects in our constructions. Let \(A = \{1, 2, \ldots, d\}\) be a finite ordered alphabet and let \(\sigma : A^* \to A^*\) be an endomorphism of the free monoid \(A^*\) of words over \(A\), which is equipped with the operation of concatenation. If \(\sigma\) is non-erasing, i.e., if \(\sigma\) does not map a non-empty word to the empty word, then we call \(\sigma\) a substitution over the alphabet \(A\). A word \(w\) is a factor of a word \(v\) if there exist words \(p, s\) such that \(v = pws\). Moreover, if \(p\) is the empty word, then \(w\) is a prefix of \(v\), which will often be denoted by \(w \leq v\). On the space \(A^\mathbb{N}\) of one-sided infinite sequences over \(A\) (equipped with the product topology of the discrete topology on \(A\)), the notions of factor and prefix are defined in analogous manner. With the substitution \(\sigma\) we associate the language

\[
L_\sigma = \{ w \in A^* : w \text{ is a factor of } \sigma^n(i) \text{ for some } i \in A, \ n \in \mathbb{N} \},
\]

i.e., \(L_\sigma\) is the set of words that occur as subwords in iterations of \(\sigma\) on a letter of \(A\). Using the language \(L_\sigma\), the **substitutive dynamical system** \((X_\sigma, \Sigma)\) is defined by

\[
X_\sigma = \{ \omega \in A^\mathbb{N} : \text{each factor of } \omega \text{ is contained in } L_\sigma \},
\]

with \(\Sigma\) being the **shift map** \((\omega_n)_{n \in \mathbb{N}} \mapsto (\omega_{n+1})_{n \in \mathbb{N}}\). \(X_\sigma\) is obviously \(\Sigma\)-invariant. The nature of a substitution \(\sigma\) very much depends on its abelianized counterpart, its so-called **incidence matrix**

\[
M_\sigma = ([\sigma^i(j)]_{1 \leq i,j \leq d},
\]

where \(|w|_i\) denotes the number of occurrences of a letter \(i \in A\) in the word \(w \in A^*\). We assume that the incidence matrix of \(\sigma\) is unimodular, i.e., we consider the set of substitutions

\[
S_d = \{ \sigma : \sigma \text{ is a substitution over } A = \{1, \ldots, d\}, \ M_\sigma \in M_d \}.
\]

The abelianization of a word \(w \in A^*\) is \(I(w) = [\{w|_1, \ldots, w|_d\}^t]\), so that \(I(\sigma(w)) = M_\sigma I(w)\).

Substitutive dynamical systems (and related tiling flows) have been studied extensively in the literature; see for instance [BG19, BST18, Fog02, Que10]. So-called unit Pisot substitutions received particular interest; a unit Pisot substitution is a substitution \(\sigma\) whose incidence matrix \(M_\sigma\) has a characteristic polynomial which is the minimal polynomial of a Pisot unit. Recall that a Pisot number is an algebraic integer greater than 1 whose Galois conjugates are all contained in the open unit disk. This class of substitutions is of importance for several reasons; one of them is its relation to strongly convergent multidimensional continued fraction algorithms, a relation that will be important in the present paper. Note also that a unit Pisot substitution \(\sigma\) is primitive in the sense that its incidence matrix admits a positive power. This implies that the associated

\[1\text{We denote the shift map on any space of sequences by } \Sigma; \text{ this should not cause any confusion.}\]
symbolic dynamical system \((X_\sigma, \Sigma)\) is minimal (i.e., \(X_\sigma\) has no nontrivial closed shift-invariant subset); see e.g. [Que10]. The main conjecture in this context, the so-called Pisot substitution conjecture, claims that, for each unit Pisot substitution \(\sigma\), the substitutive dynamical system \((X_\sigma, \Sigma)\) is measurably conjugate to a minimal translation on the torus \(\mathbb{T}^{d-1}\), and, hence, has purely discrete spectrum. Although there are many partial results (see e.g. [ABB+15, Bar16, Bar18, HS03, MAIS]), this conjecture is still open. However, given a single unit Pisot substitution \(\sigma\), there are many algorithms that can be used to verify that \((X_\sigma, \Sigma)\) has purely discrete spectrum; see [AL11, BST10, MAIS, SS02]. Thus, for each single unit Pisot substitution \(\sigma\), this property is easy to check, which is important for us. Indeed, in the present paper we show that wide classes of symbolic dynamical systems of Pisot type are measurably conjugate to minimal translations on the torus, provided that the same is true for a particular substitutive element of the class (see Theorem 3.5).

The concept of \(S\)-adic dynamical system constitutes a generalization of substitutive dynamical systems; see for instance [AMS14, ABM+20, BD14, BST19, Thu19], where \(S\)-adic dynamical systems are studied in a similar context as in the present paper. An \(S\)-adic dynamical system is defined in terms of a sequence \(\sigma = (\sigma_n)_{n \in \mathbb{N}}\) of substitutions over a given alphabet \(A\) in a way that is analogous to the definition of a substitutive dynamical system. In particular, let

\[ L_\sigma = \{ w \in A^\omega : w \text{ is a factor of } \sigma_{(0,n)}(i) \text{ for some } i \in A, n \in \mathbb{N} \}, \]

be the language associated with \(\sigma\), with

\[ \sigma_{(k,n)} = \sigma_k \circ \sigma_{k+1} \circ \cdots \circ \sigma_{n-1} \quad (0 \leq k \leq n). \]

Then the \(S\)-adic dynamical system \((X_\sigma, \Sigma)\) is defined by setting

\[ X_\sigma = \{ \omega \in A^\mathbb{N} : \text{ each factor of } \omega \text{ is contained in } L_\sigma \}. \]

The sequence \(\sigma\) is called a directive sequence of \((X_\sigma, \Sigma)\). Note that the \(S\)-adic dynamical system of a periodic directive sequence \((\sigma_0, \ldots, \sigma_{n-1})^\infty\) is equal to the substitutive dynamical system \((X_{\sigma_{(0,n)}}, \Sigma)\).

We say that a directive sequence \(\sigma\) has purely discrete spectrum if the system \((X_\sigma, \Sigma)\) is uniquely ergodic (i.e., it has a unique shift-invariant measure \(\mu\), minimal, and has purely discrete measure-theoretic spectrum (i.e., the measurable eigenfunctions of the Koopman operator \(U_\tau : L^2(X_\sigma, \Sigma, \mu) \to L^2(X_\sigma, \Sigma, \mu), f \mapsto f \circ \tau\), span \(L^2(X_\sigma, \Sigma, \mu)\)).

There is a tight link between \(S\)-adic dynamical systems and continued fraction algorithms. For the classical continued fraction algorithm, this is worked out in detail in [AF01, AF05], for multidimensional continued fractions algorithms, see for instance [BST19, Thu19]. Indeed, for each given vector, a continued fraction algorithm creates a sequence of partial quotient matrices. If these matrices are nonnegative and integral (i.e., if the algorithm is positive), they can be regarded as incidence matrices of a directive sequence of substitutions over a given alphabet \(A\) in a way that is analogous to the definition of a substitutive dynamical system. In particular, let

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Assume throughout the paper that the space \(S_0^d\) of sequences over the substitutions \(S^d\) carries the product topology of the discrete topology on \(S\). Let \(D \subset S_0^d\) be a shift-invariant set of directive sequences (which is not to be confused with the \(S\)-adic shift \((X_\sigma, \Sigma)\) of a single directive sequence \(\sigma \in D\)); note that we do not require \(D\) to be closed. We define the linear cocycle \(Z\) over \((D, \Sigma)\)
by
\[ Z : D \to M_d, \quad (\sigma_n)_{n \in \mathbb{N}} \mapsto \Sigma_{\sigma_0} \]
(recall that \( M_\sigma \) is the incidence matrix of \( \sigma \)). Analogously to the linear cocycle \( A \), we define
\[ Z^{(n)}(\sigma) = Z(\Sigma^{n-1} \sigma) \cdots Z(\Sigma \sigma) Z(\sigma), \]
so that \( Z^{(n)}(\sigma) = \Sigma_{\sigma_{n-1}} \cdots \Sigma_{\sigma_1} \Sigma_{\sigma_0} = \Sigma_{\sigma_{(0,n)}} \).

Like in the substitutive case, also in the \( S \)-adic case properties of the incidence matrices of the substitutions \( \sigma_n \) will be decisive for the behavior of the \( S \)-adic dynamical system \((X_\sigma, \Sigma)\). We have under mild conditions (see Section 4.1) that, for some vector \( u \in \mathbb{R}_+^d \), which is called a \textit{generalized right eigenvector of} \( \sigma \) (or of \( (\Sigma_{\sigma_n})_{n \in \mathbb{N}} \)) and can be seen as the generalization of the Perron–Frobenius eigenvector of a primitive matrix. When normalized by \( \| u \|_1 = 1 \), the vector \( u \) is called the \textit{normalized generalized right eigenvector of} \( \sigma \) (or of \( (\Sigma_{\sigma_n})_{n \in \mathbb{N}} \)).

Moreover, we wish to carry over the Pisot property of the substitutive case to this more general setting. This will be done by imposing the Pisot condition in Definition 2.1 to the Lyapunov exponents of the cocycle \((D, \Sigma, Z, \nu)\) for a convenient \( \Sigma \)-invariant Borel measure \( \nu \). Thus we do not consider a single sequence \( \sigma \) but the behavior of \( \nu \)-almost all sequences in \( D \).

Finally, recall that in general a shift (or equivalently, a \textit{symbolic dynamical system}) is a closed and shift-invariant set \( Y \) of sequences \( \omega \in \mathcal{A}^\mathbb{N} \) over some alphabet \( \mathcal{A} \). The \textit{language} of \( Y \) is the set of all factors of the sequences in \( Y \).

2.3. \( S \)-adic shifts given by continued fraction algorithms. Our goal is to set up symbolic realizations of positive continued fraction algorithms, which in turn will provide symbolic models of toral translations, in a way that is described in Section 2.4 below. To this end, we associate with each \( \mathbf{x} \in \Delta \) a sequence of substitutions \( \sigma = (\sigma_n)_{n \in \mathbb{N}} \in \mathcal{S}_d^\mathbb{N} \) with generalized right eigenvector \( \mathbf{x} \).

In particular, given \( \mathbf{x} \in \Delta \) we regard the partial quotient matrices \( \Sigma_{\sigma_n} = \Sigma(A(T^n \mathbf{x})) \) as incidence matrices of substitutions, i.e., for each \( n \in \mathbb{N} \) we choose \( \sigma_n \) with incidence matrix \( M_{\sigma_n} = \Sigma_{\sigma_n} \). This obviously implies that \( M_{\sigma_{(0,n)}} = \Sigma(A^{(n)}(\mathbf{x})) \).

\textbf{Definition 2.2 (\( S \)-adic realizations).} We call a map \( \varphi : \Delta \to \mathcal{S}_d \) a substitution selection for a positive \((d-1)\)-dimensional continued fraction algorithm \((\Delta, T, A)\) if the incidence matrix of \( \varphi(\mathbf{x}) \) is equal to \( \Sigma_{\sigma_n} \mathbf{x} \) for all \( \mathbf{x} \in \Delta \). The corresponding \textit{substitutive realization} of \((\Delta, T, A)\) is the map \( \varphi : \Delta \to \mathcal{S}_d^\mathbb{N}, \quad \mathbf{x} \mapsto (\varphi(T^n \mathbf{x}))_{n \in \mathbb{N}}, \)

\( \Sigma \)

\begin{align*}
\varphi(\Delta) & \xrightarrow{\Sigma} \varphi(\Delta)
\end{align*}

\begin{align*}
\Delta & \xrightarrow{T} \Delta
\end{align*}

\begin{align*}
\varphi & \xrightarrow{\Sigma} \varphi
\end{align*}

\text{together with the shift \((\varphi(\Delta), \Sigma)\). For any \( \mathbf{x} \in \Delta \), the sequence \( \varphi(\mathbf{x}) \) is called an \textit{\( S \)-adic expansion} of \( \mathbf{x} \). and \((X_{\varphi(\mathbf{x})}, \Sigma)\) is called the \textit{\( S \)-adic dynamical system of \( \mathbf{x} \) w.r.t. \((\Delta, T, A, \varphi)\). If \( \varphi(\mathbf{x}) = \varphi(\mathbf{y}) \) for all \( \mathbf{x}, \mathbf{y} \in \Delta \) with \( A(\mathbf{x}) = A(\mathbf{y}) \), then \( \varphi \) is called a faithful substitution selection and \( \varphi \) is a faithful substitutive realization.}

Note that the diagram
\begin{align*}
(2.4) \quad \Delta & \xrightarrow{T} \Delta \\
\varphi(\Delta) & \xrightarrow{\varphi} \varphi(\Delta)
\end{align*}

\text{commutes. If \( T \) converges weakly at \( \mathbf{x} \) for almost all \( \mathbf{x} \in \Delta \), then \((\Delta, T, \nu)\) is measure-theoretically isomorphic to its substitutive realization, which we write as}
\begin{align*}
(2.5) \quad (\Delta, T, \nu) \cong (\varphi(\Delta), \Sigma, \nu \circ \varphi^{-1}).
\end{align*}

The following definition will play a crucial role in the sequel. A \textit{Pisot matrix} is an integer matrix with characteristic polynomial equal to the minimal polynomial of a Pisot number, a \textit{unit Pisot}
matrix is a unimodular Pisot matrix, and a Pisot substitution is a substitution whose incidence matrix is a Pisot matrix.

**Definition 2.3** (Pisot sequences and points). A periodic sequence \((M_0, M_1, \ldots, M_{n-1})^\infty \in \mathbb{M}_d^\ast\) \([\sigma_0, \sigma_1, \ldots, \sigma_{n-1}]^\infty \in \mathcal{S}_d^\infty\) is called a periodic Pisot sequence if \(M_0M_1 \cdots M_{n-1}\) is a Pisot matrix \([\sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{n-1} = 1\) is a Pisot substitution.

For a multidimensional continued fraction algorithm \((\Delta, T, \sigma, A)\), we say that \(x_0 \in \Delta\) is a periodic Pisot point if there is an \(n \geq 1\) such that \(T^n(x_0) = x_0\) and \(A^{(n)}(x_0)\) is a Pisot matrix.

We also need to recall the notion of properness. A substitution \(\sigma\) over \(A\) is left [right] proper if there exists \(j \in A\) such that \(\sigma(i) = j\) for all \(i \in A\). A sequence of substitutions \(\sigma = (\sigma_n)\) is left [right] proper if for each \(k \in \mathbb{N}\) there exists \(n > k\) such that \(\sigma_{(k,n)}\) is left [right] proper. It is proper if it is both left and right proper. This is a natural assumption introduced in [DHS99] in order to relate Bratteli–Vershik systems associated with stationary, properly ordered Bratteli diagrams with substitutive dynamical systems.

### 2.4. Natural codings, bounded remainder sets, and Rauzy fractals

In this section we introduce some terminology related to symbolic codings of toral translations with respect to finite partitions (see [Che09] for more details). For \(t \in \mathbb{R}^d\) we consider the translation

\[ R_t : \mathbb{T}^d \to \mathbb{T}^d, \quad x \mapsto x + t \pmod{\mathbb{Z}^d} \]

on \(\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d\). We assume that \(t = (t_1, \ldots, t_d)\) is totally irrational in the sense that \(1, t_1, \ldots, t_d\) are rationally independent. This implies that \(R_t\) is minimal and uniquely ergodic.

We want to provide symbolic codings of \(R_t\) with respect to a given finite partition. There are many possible codings, and the simplest partitions, using polytopes for example, do not give the best results in terms of multiscale bounded remainder sets. We rather consider partitions of a fundamental domain of \(\mathbb{T}^d\) which is chosen in such a way that on each atom the map \(R_t\) is a translation by a vector. This induces an exchange of domains on this fundamental domain. This yields the notion of natural partition and natural coding which we describe now.

**Definition 2.4** (Natural partition). A measurable fundamental domain of \(\mathbb{T}^d\) is a set \(\mathcal{F} \subset \mathbb{R}^d\) with Lebesgue measure 1 that satisfies \(\mathcal{F} + \mathbb{Z}^d = \mathbb{R}^d\). A collection \(\{\mathcal{F}_1, \ldots, \mathcal{F}_h\}\) is said to be a natural partition of \(\mathcal{F}\) with respect to \(R_t\) if

- \(\bigcup_{i=1}^h \mathcal{F}_i = \mathcal{F}\);
- the (Lebesgue) measure of \(\mathcal{F}_i \cap \mathcal{F}_j\) is zero for all \(i \neq j\), \(1 \leq i, j \leq h\);
- each set \(\hat{F}_i\), \(1 \leq i \leq h\), is the closure of its interior and has boundary of measure zero;
- there exist vectors \(t_1, \ldots, t_h\) in \(\mathbb{R}^d\) such that \(t_i + \mathcal{F}_i \subset \mathcal{F}\) with \(t_i \equiv 0 \pmod{\mathbb{Z}^d}\), \(1 \leq i \leq h\).

A natural partition is called bounded if the set \(\mathcal{F}\) is bounded.

A natural partition \(\{\mathcal{F}_1, \ldots, \mathcal{F}_h\}\) of a measurable fundamental domain \(\mathcal{F}\) of \(\mathbb{T}^d\) allows to define a map \(\tilde{R}_t : \mathcal{F} \to \mathcal{F}\) (which depends on the partition) as an exchange of domains defined a.e. on \(\mathcal{F}\) as \(\tilde{R}_t(x) = x + t_i\) whenever \(x \in \mathcal{F}_i\). The map \(\tilde{R}_t : \mathcal{F} \to \mathcal{F}\) is defined on \(\mathcal{F} \setminus \bigcup_{i=1}^h \partial \mathcal{F}_i\), hence, it is defined almost everywhere. The dynamical system \((\mathcal{F}, \tilde{R}_t, \lambda|_{\mathcal{F}})\), where \(\lambda\) denotes the Lebesgue measure, is minimal, uniquely ergodic, and measurably isomorphic to \((\mathbb{T}^d, R_t)\) (endowed with the Haar measure). One has for a.e. \(x \in \mathcal{F}\), \(\tilde{R}_t(x) \equiv R_t(x) \pmod{\mathbb{Z}^d}\). The collection \(\{\mathcal{F}_1 + t_1, \ldots, \mathcal{F}_h + t_h\}\) also forms a natural partition of \(\mathcal{F}\), hence the terminology exchange of domains (see Figure 2 for an illustration). The language associated with the partition \(\{\mathcal{F}_1, \ldots, \mathcal{F}_h\}\) is the set of words \(i_0 \cdots i_n \in \{1, \ldots, h\}^*\) such that \(\bigcap_{k=0}^{n} \tilde{R}_t^{-k} \mathcal{F}_{i_k} \neq \emptyset\).

**Definition 2.5** (Natural coding). A shift \((X, \Sigma)\) is a natural coding of \((\mathbb{T}^d, R_t)\) if its language is the language of a natural partition \(\{\mathcal{F}_1, \ldots, \mathcal{F}_h\}\) and \(\bigcap_{n \geq 1} \bigcap_{k=0}^{n} \tilde{R}_t^{-k} \mathcal{F}_{i_k}\) is reduced to one point for any \((i_n)_{n \in \mathbb{N}} \in X\), where \(\tilde{R}_t\) stands for the associated exchange of domains.

---

2We stress the fact that in this paper we mainly work with unit Pisot substitutions and matrices.

3This is a partition up to zero measure sets.
A sequence \((i_n)_{n \in \mathbb{N}} \in \{1, \ldots, h\}^\mathbb{N}\) is said to be a \textit{natural coding} of \((\mathbb{T}^d, R^d)\) w.r.t. the natural partition \(\{F_1, \ldots, F_h\}\) if there exists \(x \in \mathcal{F}\) such that \((i_n)_{n \in \mathbb{N}} \in \{1, \ldots, h\}^\mathbb{N}\) codes the orbit of \(x\) under the action of \(R^d\), i.e., \(R^n_d(x) = x + \sum_{k=0}^{n-1} t_{i_k} \in F_{i_n}\) for all \(n \in \mathbb{N}\).

If \((X, \Sigma)\) is a natural coding of \((\mathbb{T}^d, R^d)\) w.r.t. a bounded natural partition \(\{F_1, \ldots, F_h\}\) we call it a \textit{bounded natural coding}. In this case by [Che09] Theorem A and the remark after Theorem B, one can define a continuous surjective map \(\chi : X \to \mathcal{F}\). Moreover, by the same result there exists a a one-to-one coding map \(\Phi\) defined a.e. on \(\mathcal{F}\) that satisfies \(\chi \circ \Phi(x) = x\) for a.e. \(x\), and that associates with \(x\) the natural coding of its orbit under the action \(R^d\) w.r.t. the partition \(\{F_1, \ldots, F_h\}\). Furthermore, by [Che09] Theorems A and B, and the remark after Theorem B the subshift \((X, \Sigma)\) is minimal and uniquely ergodic, \((\mathbb{T}^d, R^d)\) is a topological factor of \((X, \Sigma)\), and \((\mathbb{T}^d, R^d)\) is measure-theoretically isomorphic to \((X, \Sigma)\). This implies that \((X, \Sigma)\) has purely discrete measure-theoretic spectrum.

Here we just give an example. Consider the translation \(R^d\) on \(\mathbb{T}^1\) with \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\). The partition \(\{F_1, F_2\}\) of \(\mathcal{F} = [0, 1]\) given by \(F_1 = [0, 1-\alpha)\) and \(F_2 = [1-\alpha, 1)\) is a natural partition (which corresponds to a Sturmian dynamical system [MH40]) because \(R^d_\alpha(x) = x + \alpha\) for \(x \in F_1\) and \(R^d_\alpha(x) = x + \alpha - 1\) for \(x \in F_2\). The natural coding of a point \(x \in \mathbb{T}^1\) is the (Sturmian) sequence \((i_n)_{n \in \mathbb{N}}\) given by \(R^d_n(x) \in F_{i_n}\) \((n \in \mathbb{N})\). On the contrary, it is easy to see that the partition of \([0, 1]\) by the intervals \([0, \frac{1}{2})\) and \([\frac{1}{2}, 1)\) is not a natural partition for \(R^d\).

**Definition 2.6** (Bounded remainder set). A \textit{bounded remainder set} of a dynamical system \((X, T, \mu)\) with invariant probability measure \(\mu\) is a measurable set \(Y \subseteq X\) such that there exists \(C > 0\) with the property

\[
\left| \# \{0 \leq n < N : T^n(x) \in Y \} - N \mu(Y) \right| \leq C \quad \text{for all } N \in \mathbb{N} \text{ and a.e. } x \in X.
\]

Bounded natural codings and bounded remainder sets are closely related (see for instance [Rau84] and Theorem 3.8 below). We will define bounded natural partitions using \textit{Rauzy fractals}. To define Rauzy fractals we denote by

\[
\pi_u : \mathbb{R}^d \to 1^\perp \quad \text{the projection along } u \text{ on } 1^\perp,
\]

where \(1^\perp\) is the hyperplane orthogonal to \(1 = (1, 1, \ldots, 1)\).

**Definition 2.7** (Rauzy fractal and subtile). Let \((X_\sigma, \Sigma)\) be an \(S\)-adic dynamical system with \(\sigma \in S_\mathbb{Q}^+\) having the generalized right eigenvector \(u\). The \textit{Rauzy fractal} associated with \(\sigma = (\sigma_n)_{n \in \mathbb{N}}\) is defined as

\[
\mathcal{R}_\sigma = \{ \pi_u 1(p) : p \preceq \sigma_{[0,n]}(j) \text{ for infinitely many } n \in \mathbb{N}, j \in \mathcal{A} \},
\]

and for each word \(w \in \mathcal{A}^*\) a \textit{subtile} of \(\mathcal{R}_\sigma\) is defined by

\[
\mathcal{R}_\sigma(w) = \{ \pi_u 1(p) : p w \preceq \sigma_{[0,n]}(j) \text{ for infinitely many } n \in \mathbb{N}, j \in \mathcal{A} \}.
\]

We clearly have

\[
\mathcal{R}_\sigma = \bigcup_{w \in \mathcal{A}^* \mathcal{R}_\sigma(w) \quad (n \in \mathbb{N}),
\]
and in particular \( R_\sigma = \bigcup_{i \in A} R_\sigma(i) \). In Figure 3, we illustrate the definition of Rauzy fractals for the periodic directive sequence \( \sigma = (\gamma_1, \gamma_2)\infty \), with \( \gamma_1, \gamma_2 \) being the Cassaigne–Selmer Substitutions defined in (6.1) below. Rauzy fractals associated with periodic sequences \( \sigma \) (and therefore related to substitutive dynamical systems) go back to [Rau82] and have been studied extensively; see for instance [AI01, BS05, BST10, CS01, Fog02, IR06, ST09, Thu19]. Our definition of \( R_\sigma \) is equivalent to the one in [BST19 Section 2.9] which uses limit sequences of \( \sigma \), i.e., infinite sequences that are images of \( \sigma_{[0,n]} \) for all \( n \in \mathbb{N} \).

For convenience we define a further “projection” that will provide translations on \( T^{d-1} \) in the main results given in Section 3. We set

\[
\pi' : \mathbb{R}^d \to \mathbb{R}^{d-1}, \quad (x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_{d-1}),
\]

i.e., we omit the last coordinate of a vector. (In doing so, we make an arbitrary choice; it would also be possible to omit any other coordinate.) Sometimes we will just write \( x' \) instead of \( \pi'(x) \). Accordingly for the subtiles embedded in \( \mathbb{R}^{d-1} \) via \( \pi' \) we will write

\[
R'_\sigma(w) = \pi'(R_\sigma(w)) \quad (w \in A^*).
\]

2.5. **Cylinders and positive range.** To state our theorems, we need a few more definitions on partitions associated with continued fraction algorithms.

**Definition 2.8 (Cylinder and follower sets, positive range).** Let \((D, \Sigma, \nu)\) be a dynamical system with \( D \subset S^N \) and a shift invariant Borel measure \( \nu \). The cylinder set of \((\omega_0, \ldots, \omega_{n-1})\) is defined as

\[
[\omega_0, \ldots, \omega_{n-1}] = \{(v_k)_{k \in \mathbb{N}} \in D : (v_0, \ldots, v_{n-1}) = (\omega_0, \ldots, \omega_{n-1})\}
\]

and \( \Sigma^n[\omega_0, \ldots, \omega_{n-1}] \) is the follower set of \((\omega_0, \ldots, \omega_{n-1})\). Moreover, we say that \((\omega_n)_{n \in \mathbb{N}}\) has positive range in \((D, \Sigma, \nu)\) if

\[
\inf_{n \in \mathbb{N}} \nu(\Sigma^n[\omega_0, \ldots, \omega_{n-1}]) > 0.
\]

Similarly, the cylinder sets of a multidimensional continued fraction algorithm \((\Delta, T, A, \nu)\) are given by

\[
\Delta^{(n)}(x) = \{y \in \Delta : A(y) = A(x), A(Ty) = A(Tx), \ldots, A(T^{n-1}y) = A(T^{n-1}x)\},
\]

with \( \Delta^{(0)}(x) = \Delta; \) for convenience, we set \( \Delta(x) = \Delta^{(1)}(x) \). In this context the follower sets are the sets of the form \( T^n\Delta^{(n)}(x) \). Then \( x \in \Delta \) is said to have positive range in \((\Delta, T, A, \nu)\) if

\[
\inf_{n \in \mathbb{N}} \nu(T^n\Delta^{(n)}(x)) > 0.
\]

Figure 3. Illustration of the definition of the Rauzy fractal \( R_\sigma \) corresponding to the periodic directive sequence \( \sigma = (\gamma_1, \gamma_2)\infty \), where \( \gamma_1, \gamma_2 \) are the Cassaigne–Selmer Substitutions defined in (6.1). The abelianizations \( I(p) \) of the prefixes define a broken line. Its vertices are projected along \( u \) to \( 1^\perp \) in order to define the Rauzy fractal \( R_\sigma \), where \( u \) is the normalized generalized right eigenvector of \( \sigma \). Its subtiles \( R_\sigma(1), R_\sigma(2), \) and \( R_\sigma(3) \) are indicated by different shades of grey.
Cylinder sets of $(D, \Sigma, \nu)$ are measurable because all cylinders are open sets in the subspace topology on $D$. This is the reason why we assumed $\nu$ to be a Borel measure. We also recall that $D$ is not assumed to be closed. Note that measurability of the cylinder sets of $(\Delta, T, A, \nu)$ holds because $A$ is measurable by assumption.

We note that all the classical algorithms we are aware of satisfy even the (global) finite range property (cf. [IY87]) stating that the collection of follower sets

$$D = \{ T^n \Delta^{(n)}(x) : x \in \Delta, n \in \mathbb{N} \}$$

is finite, where sets differing only on a set of $\nu$-measure zero are identified. For instance, although the Jacobi–Perron algorithm is multiplicative, $D$ consists of only two elements; see also Section 6.3. By the $T$-invariance of $\nu$, the finite range property obviously implies positive range for $x \in \Delta$ if we suppose that all cylinders satisfy $\nu(\Delta^{(n)}(x)) > 0$; this will be the case for the algorithms considered in Section 6.

If $(\Delta, T, A, \nu)$ has the finite range property and $\bigcap_{n \in \mathbb{N}} \Delta^{(n)}(x) = \{ x \}$ for almost all $x \in \Delta$, i.e., the collection of cylinders $\{ \Delta(x) : x \in \Delta \}$ is a generating partition, then $\{ U \cap \Delta(x) : U \in D, x \in \Delta \}$ forms a (measurable countable) generating Markov partition of $(\Delta, T)$; see e.g. [Yur95, Theorem 10.1]. Most of the classical continued fraction algorithms (like Brun, Selmer, and Jacobi–Perron) are designed in a way that this Markov partition property holds.

We need that any set $B \subset \Delta$ with $\nu(B) > 0$ included in the follower set $T^n \Delta^{(n)}(x)$ leads to an intersection $T^{-n} B \cap \Delta^{(n)}(x)$ with positive measure. To this end, we always assume the stronger property that

$$\nu(E) = 0 \implies \nu \circ T(E) = 0 \quad \text{for all measurable sets } E.$$

Although $\nu \circ T$ is usually not a measure, we use the notation $\nu \circ T \ll \nu$ because (2.11) is reminiscent of absolute continuity.

The notation $\nu \circ \Sigma \ll \nu$ has the analogous meaning in the context of a shift $(D, \Sigma, \nu)$.

3. Main results

We present two types of results, the first type is stated in the framework of multidimensional continued fraction algorithms in Section 3.1, the second one is stated in terms of $S$-adic dynamical systems and directive sequences in Section 3.2. For both frameworks two theorems are given. The first one requires the existence of a single substitutive dynamical system with purely discrete spectrum which corresponds to a periodic sequence in the set of $S$-adic sequences under consideration. The existence of this single system already implies purely discrete spectrum for a whole shift of $S$-adic dynamical systems. It is stated in Theorem 3.1 for multidimensional continued fraction algorithms and in Theorem 3.5 for shifts of directive sequences. The second one yields unconditional purely discrete spectrum results for accelerations and is contained in Theorem 3.3 for multidimensional continued fraction algorithms and in Theorem 3.6 for shifts of directive sequences. All these results are then made more explicit in terms of bounded remainder sets with Theorem 3.3.

3.1. Main results on multidimensional continued fraction algorithms. In this section we provide our main results for multidimensional continued fraction algorithms. We recall that we use the abbreviation $x' = \pi'(x)$ for the projection $\pi'$ defined in (2.8), in particular, following (2.9) we write $\mathcal{R}'_\varphi(i) = \pi'(\mathcal{R}_\varphi(i))$. The notation $\ll$ is defined at the end of Section 2.4.

**Theorem 3.1.** Let $(\Delta, T, A, \nu)$ be a positive $(d-1)$-dimensional continued fraction algorithm satisfying the Pisot condition and $\nu \circ T \ll \nu$. Let $\varphi$ be a faithful substitutive realization of $(\Delta, T, A, \nu)$. Assume that there is a periodic Pisot point $x_0 \in \Delta$ with positive range in $(\Delta, T, A, \nu)$ such that $\varphi(x_0)$ has purely discrete spectrum. Then, for $\nu$-almost all $x \in \Delta$, the $S$-adic dynamical system $(X_{\varphi(x)}, \Sigma)$ is a bounded natural coding of the minimal translation by $\pi'(x)$ on $T^{d-1}$ w.r.t. the partition $\{ -\mathcal{R}'_{\varphi(x)}(i) : i \in A \}$; in particular, its measure-theoretic spectrum is purely discrete.

It will follow from Theorem 3.8 that the sets $-\mathcal{R}'_{\varphi(x)}(i), i \in A$, are bounded remainder sets. If the directive sequence $\varphi(x)$ is assumed to be (left) proper (as defined in Section 2.3), Theorem 3.8...
shows that we can even refine these bounded remainder sets from letters to factors. In particular, in this case the Rauzy fractals $-R'_{\varphi(x)}(w)$, $w \in A^n$, associated with factors of length $n$ are bounded remainder sets for each $n \in \mathbb{N}$.

**Remark 3.2.**

(i) We note that $(X_{\varphi(x)}, \Sigma)$ is a substitutive dynamical system since $\varphi(x)$ is a periodic sequence of substitutions. For such systems, some combinatorial coincidence conditions (as for instance the ones used in [ABB+15, BK06, BST10, IR06]) can be used to establish purely discrete measure-theoretic spectrum (see also Section 4.2 for precise statements). We could therefore replace the purely discrete spectrum condition in Theorem 3.1 by 

\[ \varphi(x_0) \circ \varphi(Tx_0) \circ \cdots \circ \varphi(T^{n-1}x_0) \text{ satisfies the super coincidence condition from IR06 Definition 4.2}. \]

However, since coincidence conditions require quite some notation we decided to introduce them later in this paper in order to make our main results easier to read. The Pisot substitution conjecture implies that all Pisot substitutions satisfy the super coincidence condition. To get an impression of the techniques used in the substitutive case for proving purely discrete spectrum, see also Lemma 6.1 where we use the balanced pair algorithm to prove purely discrete spectrum of a substitutive dynamical system.

(ii) In Theorem 3.3 we can omit the requirement that $\varphi$ is faithful if we replace $\Delta$ by $\varphi$ in the definition of the cylinder sets $\Delta^{(n)}(x)$ in (2.10), if we assume that $\varphi$ is measurable, and if we assume positive range with respect to this new definition of cylinder.

Since the Pisot substitution conjecture is not proved, we cannot omit the requirement of a periodic Pisot point with purely discrete spectrum in Theorem 3.1 and we do not even know whether there always exists a substitutive realization $\varphi$ that admits such a point. However, we are able to establish the following unconditional theorem that guarantees the existence of accelerations $(\Delta, T^k)$ for which there exists a faithful substitutive realization $\varphi$ with a periodic Pisot point $x_0$ such that $\varphi(x_0)$ has purely discrete spectrum.

**Theorem 3.3.** Let $(\Delta, T, A, \nu)$ be a positive ($d-1$)-dimensional continued fraction algorithm satisfying the Pisot condition and $\nu \ll T$ and assume that there exists a periodic Pisot point with positive range. Then, there exist a positive integer $k$ and a (faithful) substitutive realization $\varphi$ of $(\Delta, T^k, A, \nu)$ such that for $\nu$-almost all $x \in \Delta$ the $S$-adic dynamical system $(X_{\varphi(x)}, \Sigma)$ is a bounded natural coding of the minimal translation by $\pi(x)$ on $\mathbb{T}^{d-1}$ w.r.t. the partition $\{-R_{\varphi(x)}(i) : i \in A\}$; in particular, its measure-theoretic spectrum is purely discrete. Moreover, we have $(\Delta, T^k, \nu, \varphi) \equiv (\varphi(\Delta), \Sigma, \nu \circ \varphi^{-1}).$

**Remark 3.4.** The set of translations in Theorems 3.1 and 3.3 does not cover $\mathbb{T}^{d-1}$ since the translations are of the form $R_t$ with $t \in [0,1]^{d-1}$ and $||t|| \leq 1$. However, $R_t$ is conjugate to all translations $R_s$ with $s \in \text{GL}(d-1, \mathbb{Z}) \cdot t$, and $\{t \in [0,1]^{d-1} : ||t|| \leq 1\}$ is mapped by

\[(t_1, \ldots, t_{d-1}) \mapsto (t_1t_1 + t_2, \ldots, t_1 + t_2 + \cdots + t_{d-1})\]

to $\{t \in [0,1]^{d-1} : 0 \leq t_1 \leq t_2 \leq \cdots \leq t_{d-1} \leq 1\}$. Then, taking permutations of the coordinates of the latter set gives the whole torus $\mathbb{T}^{d-1}$.

Verifying purely discrete spectrum for some concrete substitutive dynamical systems will allow us to use Theorem 3.1 in Section 6 in order to prove a.e. purely discrete spectrum for many continued fraction algorithms like for instance the Jacobi–Perron, Brun, Cassaigne–Selmer and Arnoux–Rauzy–Poincaré algorithms. Indeed, it is well known that these algorithms have the finite range property, and the Pisot condition holds for all these algorithms when $d = 3$. In the case of Brun, the Pisot property also holds for $d = 4$. Applying Theorem 3.1 to these algorithms, according to Remark 3.4 we are able to realize almost all translations in $\mathbb{T}^2$ and $\mathbb{T}^3$ via systems of the form $(X_{\varphi(x)}, \Sigma)$, $x \in \Delta$. Since the Cassaigne–Selmer algorithm (for $d = 3$) gives rise to languages $L_{\varphi(x)}$ of factor complexity $p(n) = 2n + 1$, this entails that there exist natural codings for almost all translations of $\mathbb{T}^2$ with factor complexity $2n + 1$, see Corollary 6.3. Recall that the factor complexity $p(n)$ of a language counts the number of elements of a given length $n$ contained in this language. Looking at [BCBD19, BST19], we also see many other consequences...
for these algorithms and their associated shifts of directive sequences. We will come back to these consequences in Theorem 3.8 and Section 6.

3.2. Main results on shifts of directive sequences. We now give variants of the results of the previous section in terms of directive sequences.

**Theorem 3.5.** Let $D \subset S^N_d$ be a shift-invariant set of directive sequences equipped with an ergodic $\Sigma$-invariant Borel probability measure $\nu$ satisfying $\nu \circ \Sigma \ll \nu$. Assume that the linear cocycle $(D, \Sigma, Z, \nu)$ defined by $Z((\sigma_n)_{n \in \mathbb{N}}) = \Sigma_0$ satisfies the Pisot condition, and that there is a periodic Pisot sequence in $D$ having positive range in $(D, \Sigma, \nu)$ and purely discrete spectrum. Then for $\nu$-almost all $\sigma \in D$ the $S$-adic dynamical system $(X_\sigma, \Sigma)$ is a bounded natural coding of the minimal translation by $\pi'(u)$ on $\mathbb{T}^{d-1}$ w.r.t. the partition $\{\mathcal{R}_\sigma^i : i \in \mathcal{A}\}$. Here $u$ is the normalized generalized right eigenvector of $\sigma$. In particular, the measure-theoretic spectrum of $(X_\sigma, \Sigma)$ is purely discrete.

To get an analogue of Theorem 3.3 for directive sequences, we do not start with a shift of directive sequences but rather with its abelianization, i.e., a shift of sequences of matrices $(\mathcal{D}, \Sigma)$, for which we would like to find a map $s : \mathcal{M}_d \to S^N_d$ such that almost all $\sigma \in \mathcal{D}$ with $s((M_n)_{n \in \mathbb{N}}) = (s(M_n))_{n \in \mathbb{N}}$, have purely discrete spectrum. Again, we have to consider the accelerated shift $(\mathcal{D}, \Sigma^k)$ for a suitable power $\Sigma^k$ to gain such a result. The main issue is the construction of a substitution with purely discrete spectrum associated with a given unit Pisot matrix which is done in Proposition 5.8.

**Theorem 3.6.** Let $\mathcal{D} \subset \mathcal{M}_d^N$ be a shift-invariant set of sequences of unimodular matrices equipped with an ergodic $\Sigma$-invariant Borel probability measure $\nu$ satisfying $\nu \circ \Sigma \ll \nu$. Assume that the linear cocycle $(\mathcal{D}, \Sigma, Z, \nu)$ defined by $Z((M_n)_{n \in \mathbb{N}}) = \Sigma_0$ satisfies the Pisot condition, and that there is a periodic Pisot sequence in $\mathcal{D}$ having positive range in $(\mathcal{D}, \Sigma, \nu)$. Then there exists a positive integer $k$ and a map $\psi : \mathcal{D} \to S^N_d$ satisfying $\psi \circ \Sigma^k = \Sigma \circ \psi$ such that for $\nu$-almost all $M \in \mathcal{D}$ the $S$-adic dynamical system $(X_{\psi(M)}, \Sigma)$ is a bounded natural coding of the minimal translation by $\pi'(u)$ on $\mathbb{T}^{d-1}$ w.r.t. the partition $\{\mathcal{R}_{\psi(M)}^i : i \in \mathcal{A}\}$. Here $u$ is the normalized generalized right eigenvector of $M$. In particular, the measure-theoretic spectrum of $(X_{\psi(M)}, \Sigma)$ is purely discrete.

**Remark 3.7.** Let $M = (M_n)$ and $\psi(M) = (\sigma_n)$. According to (5.7) the map $\psi$ in Theorem 3.6 can be chosen in a way that $M_{nk,(nk+1)k}$ is the incidence matrix of $\sigma_n$. This choice is needed to derive Theorem 3.3 from Theorem 3.6.

The main difference between the results in Section 3.1 and the ones in Section 3.2 is that in the latter case there can be several directive sequences in $D$ with the same normalized generalized right eigenvector.

3.3. Main results on natural codings and bounded remainder sets. We now prove that natural codings with respect to bounded fundamental domains (see Definition 2.5) provide bounded remainder sets and that, moreover, Rauzy fractals can be considered as canonical bounded remainder sets, up to some affine map. In the following theorem we need the fundamental domain $\mathcal{F}$ to be bounded and the partition of $\mathcal{F}$ to have $d$ atoms for a translation on $\mathbb{T}^{d-1}$. Recall that we set $x' = \pi'(x)$ for the projection $\pi'$ defined in (2.8).

**Theorem 3.8.** Assume that $(X, \Sigma)$ is the natural coding of a minimal translation $R_\nu$ on $\mathbb{T}^{d-1}$ w.r.t. a natural partition $\{\mathcal{F}_1, \ldots, \mathcal{F}_d\}$ of a bounded fundamental domain $\mathcal{F}$. Then the atoms $\mathcal{F}_1, \ldots, \mathcal{F}_d$ are bounded remainder sets of $R_\nu$. Their Lebesgue measures are rationally independent.

If, moreover, $(X, \Sigma)$ is an $S$-adic dynamical system with $X = X_\sigma$ for some $\sigma \in S^N_d$, then

- $\sigma$ admits a normalized generalized eigenvector $u$,
- the $i$-th coordinate of $u$ is given by the measure of $\mathcal{F}_i$ for $1 \leq i \leq d$,
- there is an affine map $H : \mathbb{R}^d \to \mathbb{R}^{d-1}$ such that $\mathcal{F}_i = H(\mathcal{R}_{\pi(i)}^i)$ for $1 \leq i \leq d$,
- $(X_\sigma, \Sigma)$ is a natural coding of $R_{\pi'}$ w.r.t. the natural partition $\{-\mathcal{R}_{\nu}^i : 1 \leq i \leq d\}$.
Furthermore, if the directive sequence $\sigma$ is (left) proper, then for each word $i_0i_1\cdots i_n \in L_{\sigma}$, the “cylinder set” $F_{i_0} \cap R_1^{-1}F_{i_1} \cap \cdots \cap R_1^{-n}F_{i_n}$ is also a bounded remainder set of $R_1$; in particular, $R_\sigma^{-1}(i_0i_1\cdots i_n)$ is a bounded remainder set of $R_\sigma$.

A similar result holds if one replaces left properness by right properness. Theorem 3.8 leads us to state the following conjecture stating, roughly speaking, that a bounded remainder set that “extends to factors” must have fractal boundary.

**Conjecture 3.9.** Let $\{F_1, \ldots, F_h\}$ be a natural partition of a minimal translation $R_h$ such that all sets $F_{i_0} \cap R_1^{-1}F_{i_1} \cap \cdots \cap R_1^{-n}F_{i_n}$, $i_0i_1\cdots i_n \in \{1, \ldots, h\}^*$, are bounded remainder sets for $R_h$. Then $F_i$ cannot have piecewise smooth boundaries ($1 \leq i \leq h$).

One argument supporting this conjecture is that such natural codings provide Markov partitions for automorphisms of the torus, and Markov partitions cannot have smooth boundaries for hyperbolic automorphisms of the torus in dimension 3 or more [Bow78].

In [Lin87] it is shown that only “trivial” axis-parallel boxes can be bounded remainder sets for Kronecker sequences and toral translations. The bounded remainder sets constructed in [GL15] are based on polytopes. In all these cases the bounded remainder sets do not “extend to factors” like in the last part of Theorem 3.8 or in Conjecture 3.9.

After some preparations in Section 4 the proofs of all main results will be contained in Section 5. The proof of the $S$-adic results in Theorem 3.5 and Theorem 3.6 will be given in Section 5.1 and Section 5.2, respectively. The results of multidimensional continued fraction statements, namely Theorems 3.1 and 3.3, will then be deduced from the according $S$-adic result in Section 5.3. Finally, Theorem 3.8 is proved in Section 5.4.

### 4. Preparations for the proofs of the main theorems

Throughout the proofs of our main results we will need notation, definitions, and results that are recalled in this section.

#### 4.1. Properties of sequences of substitutions

In our main theorems, we put certain assumptions, most notably, the Pisot condition from Definition 2.1. We will now discuss combinatorial properties that will be satisfied by almost all directive sequences $\sigma$ under these assumptions. We need these combinatorial properties because they occur in some results from [BST19] that will be important for us. Accordingly, most of the definitions stated in the present subsection are taken from [BST19] Section 2.

Let $\sigma = (\sigma_n) \in S^\infty_0$ be a sequence of substitutions over a given alphabet $A = \{1, \ldots, d\}$. We say that $\sigma$ is primitive, if for each $k \in \mathbb{N}$ there exists $n > k$ such that $M_{\sigma(k,n)}$ is a positive matrix. If each factor $(\sigma_0, \ldots, \sigma_m)$, $m \in \mathbb{N}$, occurs infinitely often in $\sigma$, then $\sigma$ is recurrent. As observed in [Fur60] p. 91–95, primitivity and recurrence of $\sigma$ allow for an analog of the Perron–Frobenius theorem for the associated sequence $(M_{\sigma_n})$ of incidence matrices. In particular, if $\sigma$ is primitive and recurrent, then the generalized right eigenvector $u$ defined in (2.3) exists.

A sequence of substitutions $\sigma$ is said to be unimodular if it takes its values in the set of unimodular matrices.

Another important property is algebraic irreducibility. A sequence of substitutions $\sigma = (\sigma_n)$ over the alphabet $A$ is called algebraically irreducible if for each $k \in \mathbb{N}$ the matrix $M_{\sigma(k,n)}$ has irreducible characteristic polynomial provided that $n \in \mathbb{N}$ is large enough. Note that for $S$-adic dynamical systems that arise from multidimensional continued fraction algorithms which are almost everywhere exponentially convergent, we can even prove that, for almost every sequence of substitutions, for each $k \in \mathbb{N}$ the characteristic polynomial of $M_{\sigma(k,n)}$ is the minimal polynomial of a Pisot unit for $n$ large enough; see [BST19] Lemma 8.7. This is true in particular if we assume the Pisot condition.

Finally, we need a balance property for the language related to a sequence of substitutions. Let $L$ be a language over a finite alphabet $A = \{1, \ldots, d\}$. We say that $L$ is $C$-balanced if for each two words $w, w' \in L$ with $|w| = |w'|$ we have $||w|i - |w'|i| \leq C$ for each $i \in A$. It is said balanced...
if it is C-balanced for some C. We define

\[ B_C = \{ \sigma \in S^N_{d^2} : L_\sigma \text{ is C-balanced} \}. \]

Balance is a classical notion in word combinatorics and symbolic dynamics. We recall that the minimal shifts having 1-balanced language over a two-letter alphabet are the Sturmian ones [MH40]. This natural notion has been widely studied from many viewpoints, for instance in ergodic theory and word combinatorics (see e.g. [CFZ00]), in number theory in connection with Fraenkel’s conjecture [Fran73] [TiJo00], and in operations research, for optimal routing and scheduling; see e.g. [AGH00] [BC04] [BJ08].

Balance can be generalized to factors. We say that L is balanced on the factor v ∈ L if there exists some C_v ≥ 1 such that, for any two words w, w' ∈ L with |w| = |w'|, we have |w|_v - |w'|_v ≤ C_v, and L is balanced on factors if it is balanced on each v ∈ L. Here, |w|_v denotes the number of occurrences of the factor v in w. Without further precision, balance refers a priori to letters hereafter.

In the sequel, we will strongly rely on the relation between balance and bounded remainder sets. The quantity C occurring in the definition of bounded remainder sets (i.e., in Definition 2.6) can be considered as a notion of local symbolic discrepancy; see e.g. [Ada04b]. To illustrate this we characterize balance by the following geometric version of [Ada03] Proposition 7. We recall that π_n is defined in 2.6.

Proposition 4.1. Let (X, Σ) be a uniquely ergodic minimal shift over the alphabet A = {1, . . . , d}. Let u = (u_1, . . . , u_d) be the vector whose entry u_i equals the measure of the “letter cylinder” [i] = \{ (v_i) ∈ X : v_0 = i \} for each i ∈ A. Then the language of X is balanced on letters if and only if sup\{||π_n(I(w))|| : w \text{ in the language of X}\} is bounded. Moreover, (X, Σ) is balanced on the factor v if and only if the cylinder [v] is a bounded remainder set.

This result will be essential for the proof of Lemma 5.6 which relates a given unit Pisot matrix to substitutions Σ having balanced language L_Σ; see also [Ada04b] Theorem 1.

We will need results from [BST19] which require a set of technical conditions which goes under the name property PRICE. A directive sequence Σ = (σ_n) ∈ S^N_{d^2} has Property PRICE if the following conditions hold for some strictly increasing sequences (n_k)_{k∈N} and (ℓ_k)_{k∈N} and a vector v ∈ R_{d^2} \backslash \{0\}.

(P) There exists h ∈ N and a positive matrix B such that M_{σ_{(t_k + h - t_k)}} = B for all k ∈ N.
(R) We have (σ_{n_k}, σ_{n_k + 1}, . . . , σ_{n_k + ℓ_k - 1}) = (σ_0, σ_1, . . . , σ_{ℓ_k - 1}) for all k ∈ N.
(I) The directive sequence Σ is algebraically irreducible.
(C) There is C > 0 such that L^{(n_k + ℓ_k)}(Σ) is C-balanced for all k ∈ N.
(E) We have \lim_{k→∞} v(n_k) = v.

We note that if Σ satisfies PRICE, then the same is true for Σσ by [BST19] Lemma 5.10.

4.2. Tilings by Rauzy fractals and coincidence conditions. As mentioned before, the Rauzy fractals defined in Section 2.4 play a crucial role in proving that the S-adic dynamical system (X_Σ, Σ) has purely discrete spectrum. The importance of Rauzy fractals is due to the fact that one can “see” on them the toral translation, to which we want to conjugate (in the measure-theoretic sense) an S-adic dynamical systems (X_Σ, Σ); this is worked out in [BST19] Section 8.

In the substitutive case, the proof of this conjugacy strongly relies on a certain self-affinity property of the subtiles R_Σ(i), i ∈ A (see e.g. [SpW02]). In the S-adic case these subtiles are no longer self-affine. However, they still satisfy a certain set equation that allows to express them as unions of shrunk copies of subtiles R_Σ(Σ)(i) corresponding to a shift of the original directive sequence Σ. In particular, in [BST19] Proposition 5.6 it is shown that each R_Σ(i), i ∈ A, can be written as

\[ R_Σ(i) = \bigcup_{p \in A^*} \left( \bigcup_{j \in A : p \leq \sigma_0(n)j} \pi_u(I(p) + M_{σ_{\sigma_0(n)}} R_{Σ^u} σ(j)) \right) \quad (n \in \mathbb{N}). \]

An S-adic Rauzy fractal R_Σ has thus two different kinds of natural subsets: the subtiles R_Σ(w) defined in (2.7) and the (level n) subdivision tiles π_u(I(p) + M_{σ_{\sigma_0(n)}} R_{Σ^u} σ(j)) occurring on the right hand side of (4.2) for some i ∈ A. In this section we will mostly use the subdivision tiles.
We will need the collection
\[ C_\sigma = \{ x + R_\sigma(i) : x \in \mathbb{Z}^d \cap \mathbf{1}^+, i \in \mathcal{A} \} \]
consisting of the translations of (the subtile of) the Rauzy fractal $R_\sigma$ by vectors in the lattice $\mathbb{Z}^d \cap \mathbf{1}^+$. As shown e.g. in [BST19], the fact that $C_\sigma$ forms a tiling of $\mathbf{1}^+$ implies that $(X_\sigma, \Sigma)$ has purely discrete spectrum. Here, a tiling of $\mathbb{R}^d$ is a collection of sets that covers $\mathbb{R}^d$ in a way that the intersection of any two distinct sets has Lebesgue measure 0. Related results for the substitutive case are contained in [AI01, Theorem 2] and [CS01, Theorem 3.8]; for the classical example that initiated the whole theory we refer to [Rau82].

It is proved in [BST19, Proposition 7.5] that, if PRICE holds, $C_\sigma$ is a locally finite multiple tiling of $\mathbf{1}^+$ by compact tiles (in the sense that a.e. point of $\mathbf{1}^+$ is contained in exactly $p$ elements of $C_\sigma$ for some given $p \geq 1$). It is a priori not clear how to decide for a given directive sequence $\sigma$ if this multiple tiling is actually a tiling. However, as shown in [BST19, Section 7] the following coincidence conditions can be used to get checkable criteria for this tiling property.

A directive sequence $\sigma = (\sigma_n)_{n \in \mathbb{N}}$ satisfies the geometric coincidence condition if for each $R > 0$, there is $k \in \mathbb{N}$ such that, for all $n \geq k$, there exist $z_n \in \mathbf{1}^+, i_n \in \mathcal{A}$, such that
\[
\{(y, j) \in \mathbb{Z}^d \times \mathcal{A} : \| \pi_{\alpha}^{-1}(\sigma_n)(y - z_n) \| \leq R, 0 \leq (1, y) < |\sigma_n(j)|\}
\subset \{(1(p), j) : p \in \mathcal{A}^*, j \in \mathcal{A}, p i_n \leq |\sigma_n(j)|\}
\]
(recall that $w \preceq v$ means that $w$ is a prefix of $v$). This geometric coincidence condition is a rephrasing of the one defined in [BST19, Section 2.11]: since we do not want to define discrete hyperplanes and dual substitutions here, we use equivalent statements with usual substitutions and abelianizations of words.

It turns out that the effective version of the geometric coincidence condition taken from [BST19, Proposition 7.9 (iv)], which states that there are $n \in \mathbb{N}$, $z \in \mathbf{1}^+$, $i \in \mathcal{A}$, $C > 0$, such that $L_{\Sigma^* \sigma}$ is $C$-balanced and
\[
\{(y, j) \in \mathbb{Z}^d \times \mathcal{A} : \| \pi_{\alpha}^{-1}(\sigma_n)(y - z) \|_\infty \leq C, 0 \leq (1, y) < |\sigma_n(j)|\}
\subset \{(1(p), j) : p \in \mathcal{A}^*, j \in \mathcal{A}, p i_n \leq |\sigma_n(j)|\},
\]
with $u^{(n)} = \pi_{\alpha}^{-1}(\sigma_n) u$, is more useful for our purposes.

These conditions guarantee that $C_\sigma$ contains an exclusive point, i.e., a point contained in only one tile of $C_\sigma$. The fact that $C_\sigma$ is a locally finite multiple tiling by compact tiles then leads to the conclusion that $C_\sigma$ is actually a tiling; see Proposition 4.2 below.

The geometric coincidence condition can be seen as an $S$-adic analog of the geometric coincidence condition (or super-coincidence condition) in [BK06, IR06, BST10], which provides a tiling criterion in the substitutive case. This criterion is a coincidence type condition in the same vein as the various coincidence conditions introduced in the usual Pisot framework; see e.g. [Sol97, AL11].

Results from [BST19] that are central for our proofs are contained in the following proposition.

**Proposition 4.2.** Let $\sigma \in S_0^T$ be a directive sequence satisfying PRICE. Then the following assertions are equivalent.

(i) The collection $C_\sigma$ forms a tiling.
(ii) The collection $C_{\Sigma^* \sigma}$ forms a tiling for some $n \in \mathbb{N}$.
(iii) The collection $C_{\Sigma^n \sigma}$ forms a tiling for all $n \in \mathbb{N}$.
(iv) The sequence $\sigma$ satisfies the geometric coincidence condition.
(v) The sequence $\sigma$ satisfies the effective version of the geometric coincidence condition.

**Proof.** This result is proved in [BST19, Lemma 7.1, Proposition 7.5, and Proposition 7.9]. However, the proof of the implication (i) $\Rightarrow$ (v) in [BST19, Proposition 7.9] is somewhat sketchy. Since this implication will be important in the sequel and in order to explain the (effective version of the) geometric coincidence condition, we give a more detailed proof which is illustrated in Figure 4.

Let $u$ be the normalized generalized right eigenvector of $\sigma$ which exists because PRICE implies that $\sigma$ is primitive and recurrent. Assume that there are $n \in \mathbb{N}$, $z \in \mathbf{1}^+$, $i \in \mathcal{A}$, $C > 0$, such that $\Sigma^n \sigma \in B_C$ and (4.4) holds. We show that $\pi_u M_{\Sigma^n \sigma} z$ is an exclusive point of the collection $C_\sigma$. 
R this will already imply that \( \mathcal{C} \). Since \([\text{BST}19, \text{Proposition 7.5}]\) states that \( \mathcal{C} \) of (4.6) this implies that \( \mathcal{C} \). More precisely, the given point \( \pi u^{-1} M_{\sigma_{(0,n)}} z \) can only be contained in a level \( n \) subdivision tile whose translation point is contained in the (shaded) parallelepiped \( \pi u^{-1} M_{\sigma_{(0,n)}} (z + [-C, C]^3) \). As can be seen, all translation points inside the shaded parallelepiped belong to level \( n \) subdivision tiles of the same tile of \( \mathcal{C} \) (namely, \( \mathcal{R}_\sigma (i) \); this fact is proved using the effective version of the geometric coincidence condition). Therefore, \( \pi u^{-1} M_{\sigma_{(0,n)}} z \) can belong only to level \( n \) subdivision tiles of a single tile of \( \mathcal{C} \). Thus it is an exclusive point of \( \mathcal{C} \).

Since \([\text{BST}19, \text{Proposition 7.5}]\) states that \( \mathcal{C} \) is a locally finite multiple tiling by compact tiles this will already imply that \( \mathcal{C} \) is in fact a tiling.

By the set equation (4.2) for each \( i' \in A \) the subset \( \mathcal{R}_\sigma (i') \) can be written as

\[
\mathcal{R}_\sigma (i') = \bigcup_{p' \in A^*} \pi u^{-1} (l(p') + M_{\sigma_{(0,n)}} R_{\Sigma^n} (j')).
\]

The tiles of \( \mathcal{C} \) form a multiple tiling and, hence, a covering of \( 1^+ \). Thus we can choose \( (x, i') \in (\mathbb{Z}^d \cap 1^+) \times A \) with \( \pi u^{-1} M_{\sigma_{(0,n)}} z \in x + R_{\sigma} (i') \). To prove exclusivity we have to show that this choice of \( (x, i') \) is unique. By (4.5) there exist \( p' \in A^* \), \( j' \in A \) with \( p' \leq \sigma_{(0,n)} (j') \) such that

\[
\pi u^{-1} M_{\sigma_{(0,n)}} z \in \pi u^{-1} (x + l(p') + M_{\sigma_{(0,n)}} R_{\Sigma^n} (j')).
\]

Using \( \pi u^{-1} M_{\sigma_{(0,n)}} = \pi u^{-1} M_{\sigma_{(0,n)}} \pi u^{-1} \), this implies that

\[
\pi u^{-1} M_{\sigma_{(0,n)}} z \in \pi u^{-1} M_{\sigma_{(0,n)}} \left( \pi u^{(n)} M_{\sigma_{(0,n)}}^{-1} (x + l(p')) + R_{\Sigma^n} (j') \right).
\]

Since \( z \in 1^+ \), \( \pi u^{(n)} y \in 1^+ \) for all \( y \in R^d \), and \( R_{\Sigma^n} (j') \subset 1^+ \), this is equivalent to (4.6)

\[
z \in \pi u^{(n)} M_{\sigma_{(0,n)}}^{-1} (x + l(p')) + R_{\Sigma^n} (j').
\]

By the definition of the Rauzy fractal, balance of the language \( L_{\Sigma^n} \) is related to bounds on the size of \( R_{\Sigma^n} \). In particular, \( \Sigma^n \in B_C \) yields \( \| y \|_\infty \leq C \) for all \( y \in R_{\Sigma^n} \) (see also \([\text{BST}19, \text{Lemma 4.1}]\)). By (4.6) this implies that

\[
\| \pi u^{(n)} M_{\sigma_{(0,n)}}^{-1} (x + l(p')) - z \| \leq C.
\]
We may now apply (4.4) to conclude that \((x + l(p'), j') = (l(p), j)\) for some \(p \in \mathcal{A}^*\), \(j \in \mathcal{A}\) with \(p \leq \sigma_{[0,n]}(j)\). Since \(j = j'\), both words, \(p\) and \(p'\) are prefixes of the same word \(\sigma_{[0,n]}(j)\).

Thus \(x = l(p - p') \in \mathbb{N}^d \cup (-\mathbb{N})^d\). But because \(x \in 1^d\), this implies \(x = 0\) and, hence, \(p = p'\), thus \(i' = i\). (The set \(\{\pi_u M_{\sigma_{[0,n]}, y} : \|z - y\|_\infty \leq C\}\) is the shaded cube in Figure 4 in view of the set equation (4.5), the effective version of geometric coincidence in (4.4) guarantees that this cube contains only translation points of tiles that occur as subdivision tiles of \(\mathcal{R}_\sigma(i)\).) Therefore, \((x, i') = (0, i)\) is the only choice for \((x, i')\) and, hence, \(\mathcal{R}_\sigma(i) = \mathcal{R}_\sigma(i) + 0\) is the only tile of \(\mathcal{C}_\sigma\) containing \(\pi_u M_{\sigma_{[0,n]}, z}\).

\[\square\]

4.3. Purely discrete spectrum implies geometric coincidence. In our main theorems, substitutive dynamical systems with purely discrete spectrum play a key role. The following lemma shows that in the substitutive case purely discrete spectrum is equivalent to the geometric coincidence condition, and thus, by Proposition 4.2 also to its effective version. This will be crucial in the proofs of Theorems 3.5 and 3.6; see also the discussion before Lemma 5.3. Indeed, let \(\tau\) be a unit Pisot substitution that satisfies the geometric coincidence condition. We will show that the existence of occurrences of long blocks of \(\tau\) in a given directive sequence \(\sigma\) allows to “transfer” the effective version of the coincidence condition from \(\tau\) to \(\sigma\). Using the following lemma, this “transfer” works for purely discrete spectrum property as well.

**Lemma 4.3.** Let \(\sigma\) be a unit Pisot substitution. Then, \((X_\sigma, \Sigma)\) has purely discrete spectrum if and only if \(\sigma\) satisfies the geometric coincidence condition.

**Proof.** In Section 4.2, we recalled the well-known fact that if \(\sigma\) satisfies the geometric coincidence condition, then \((X_\sigma, \Sigma)\) has purely discrete spectrum; see [IR06]. Consider now the reverse implication. As mentioned in the introduction of [BK06], it follows from [CS03] Theorem 3.1 that \((X_\sigma, \Sigma)\) has purely discrete spectrum if and only if the associated tiling flow \(T\) has purely discrete spectrum (as e.g. in [Sad10], just note that if all the tiles in the self-similar tiling space \(T\) have length 1, the spectrum of \(T\) on \(T\) is (up to a multiplicative constant) the logarithm of the spectrum of the shift operator \(\Sigma\) on \(X_\sigma\)). According to [BK06] Corollary 9.4 and Remark 18.5], the flow \(T\) has purely discrete spectrum if and only if only if the collection \(C_\sigma\) of Rauzy fractals associated with \(\sigma\) forms a tiling. Thus, Proposition 4.2 implies that the substitution \(\sigma\) satisfies the geometric coincidence condition. \(\square\)

5. PROOFS OF THE MAIN RESULTS

This section contains the proofs of all our main results. In Sections 5.1 and 5.2 we prove the results stated in Section 4.2 on shifts of directive sequences. In Section 5.3 we will use these results to derive the theorems on multidimensional continued fraction algorithms formulated in Section 3.1. Section 5.4 is devoted to the proof of Theorem 3.8 on natural codings and bounded remainder sets.

5.1. **Proof of Theorem 3.5.** For convenience, we recall the assumptions of Theorem 3.5. Let \(D \subset S^N_d\) be a shift-invariant set of directive sequences equipped with an ergodic \(\Sigma\)-invariant Borel probability measure \(\nu\) satisfying \(\nu \circ \Sigma \ll \nu\). Assume that

- the linear cocycle \((D, \Sigma, Z, \nu)\) defined by \(Z((\sigma_n)_{n \in \mathbb{N}}) = \nu M_{\sigma_n}\) satisfies the Pisot condition;
- there is a periodic Pisot sequence with purely discrete spectrum and positive range in \((D, \Sigma, \nu)\).

We first show that under these assumptions \(\nu\)-almost all \(\sigma \in D\) satisfy property PRICE. To this end we need the following auxiliary result (recall that \(B_C\) is defined in (4.1) and denotes the set of sequences in \(S^N_d\) with \(C\)-balanced language).

**Lemma 5.1.** Under the assumptions of Theorem 3.5 we have \(\lim_{C \to \infty} \nu(D \cap B_C) = 1\), in particular \(D \cap B_C\) is \(\nu\)-measurable for all \(C > 0\).

**Proof.** The Pisot condition yields that \(\nu\left(\bigcup_{C \in \mathbb{N}} (D \cap B_C)\right) = 1\) by [BD14] Theorem 6.4. Since \(B_C \subset B_{C'}\) for all \(C < C'\), it only remains to show that \(D \cap B_C\) is \(\nu\)-measurable for all \(C > 0\). Let
$C > 0$ be arbitrary but fixed and set
$$B_C' = \bigcap_{n \in \mathbb{N}} \bigcup_{(\sigma_0, \ldots, \sigma_{n-1}) \in \mathcal{S}_k : [\sigma_0, \ldots, \sigma_{n-1}] \cap B_C \neq \emptyset} [\sigma_0, \ldots, \sigma_{n-1}]$$
(recall that the cylinders $[\sigma_0, \ldots, \sigma_{n-1}]$ are subsets of $D$ according to Definition 2.8). Then we clearly have $D \cap B_C \subseteq B_C'$. If $\sigma \in B_C'$, then we have $\sigma \in D$ and the finite languages
$$L_\nu^{(n)} = \{ w \in A^* : w \text{ is a factor of } [\sigma_{[m,n]}(i) \text{ for some } i \in A, m \leq n \}$$
are $C$-balanced for all $n \in \mathbb{N}$. Since $L_\nu^{(0)} \subseteq L_\nu^{(1)} \subseteq \cdots$, also $L_\nu = \bigcup_{n \in \mathbb{N}} L_\nu^{(n)}$ is $C$-balanced, i.e., $\sigma \in B_C$. Hence, we have $D \cap B_C = B_C'$. Since cylinders are measurable (they are open sets and $\nu$ is a Borel measure on $D$) and countable unions and intersections of measurable sets are measurable, we obtain that $B_C' = D \cap B_C$ is $\nu$-measurable.

**Proposition 5.2.** _Under the assumptions of Theorem 3.5, $\nu$-almost every $\sigma \in D$ satisfies PRICE._

**Proof.** By the assumptions of Theorem 3.5, $D$ contains a periodic Pisot sequence with positive range. In particular, there exists a sequence $\tau = (\tau_n) \in D$ with the following properties:

(a) There is $j \geq 1$ such that $\Sigma \tau = \tau$ and $\tau_{[0,j)}$ is a Pisot substitution;

(b) $\inf_{n \in \mathbb{N}} \nu(\Sigma^n[\tau_0, \ldots, \tau_{n-1}]) > 0$.

Since $\tau_{[0,j)}$ is a Pisot substitution by (a), [CS01, Proposition 1.3] implies that it is primitive as well and, hence, there is $k \in \mathbb{N}$ such that $\tau_{[0,k]}$ has positive incidence matrix. Set $h = kj$. Because $\nu \circ \Sigma^h \ll \nu$, (b) implies that $\nu([\tau_0, \ldots, \tau_{n-1}]) > 0$. Thus by Lemma 5.1 there is $C \in \mathbb{N}$ such that $\nu(\Sigma^{-h}(D \cap B_C)) \geq 1 - \nu([\tau_0, \ldots, \tau_{n-1}])$, hence, $\nu([\tau_0, \ldots, \tau_{n-1}]) \cap \Sigma^{-h}B_C > 0$.

By the Poincaré Recurrence Theorem, we have for almost all $\sigma = (\sigma_n)_{n \in \mathbb{N}} \in D$ some $\ell_0(\sigma) \geq h$ such that $\Sigma_{\ell_0(\sigma)-h} \sigma \in [\tau_0, \ldots, \tau_{n-1}] \cap \Sigma^{-h}B_C$, i.e., $(\sigma_0, \ldots, \sigma_{\ell_0(\sigma)-1})$ ends with $(\tau_0, \ldots, \tau_{n-1})$ and $\Sigma_{\ell_0(\sigma)} \sigma \in B_C$. We extend $\ell_0(\sigma)$ for almost all $\sigma \in D$ to a sequence $(\ell_k(\sigma))_{k \in \mathbb{N}}$ such that

- $(\sigma_0, \ldots, \sigma_{\ell_k(\sigma)-1})$ ends with $(\sigma_0, \ldots, \sigma_{\ell_k(\sigma)-1})$ (and, a fortiori, with $(\tau_0, \ldots, \tau_{n-1})$),
- $\Sigma_{\ell_k+1(\sigma)} \sigma \in B_C$,
- $\ell_{k+1}(\sigma) \geq 2\ell_k(\sigma)$,

for all $k \in \mathbb{N}$. To this end, assume that $\ell_0(\sigma), \ldots, \ell_k(\sigma)$ are already defined for almost all $\sigma \in D$. Consider the set of all $\sigma$ having a given value $\ell_k = \ell_k(\sigma)$ and a given prefix $(\sigma_0, \ldots, \sigma_{\ell_k})$. Assume that this set has positive measure, which implies that $\nu([\sigma_0, \ldots, \sigma_{\ell_k-1}] \cap \Sigma^{-\ell_k}B_C) > 0$. Then, for almost all $\sigma$ in this set, we obtain (by the Poincaré Recurrence Theorem) some $\ell_{k+1}(\sigma)$ with the required properties. Applying this for all choices of $\ell_k$ and $(\sigma_0, \ldots, \sigma_{\ell_k})$, we get some $\ell_{k+1}(\sigma)$ for almost all $\sigma \in D$. Therefore, such a sequence $(\ell_k(\sigma))_{k \in \mathbb{N}}$ exists for almost all $\sigma \in D$.

Setting $n_k(\sigma) = \ell_{k+1}(\sigma) - \ell_k(\sigma)$, we obtain that conditions (P), (R) and (C) of property PRICE hold for almost all $\sigma \in E_G$. By [BST19, Lemma 5.7], we can replace $(n_k)$ and $(\ell_k)$ by subsequences such that condition (E) holds. These subsequences also satisfy (P), (R), and (C). From the Pisot condition and [BST19, Lemma 8.7], we obtain that almost all $\sigma \in D$ are algebraically irreducible, i.e., also (I) holds a.a. and we are done. \[\square\]

With Proposition 5.2 at our disposal we can use a slight variation of [BST19, Theorem 3.1] to show without much effort that under the conditions of Theorem 3.5 the following is true: For almost all $\sigma \in D$ the dynamical system $(X_\sigma, \Sigma)$ has an $m$-to-1 factor which is a minimal translation on $\mathbb{T}^{d-1}$ for some $m \in \mathbb{N}$. However, in order to prove Theorem 3.5, we have to show that $m = 1$, i.e., that $(X_\sigma, \Sigma)$ is measurably conjugate to a minimal translation on $\mathbb{T}^{d-1}$, which is way more difficult. According to [BST19, Theorem 3.1], in order to achieve this, we have to prove that almost all $\sigma \in D$ satisfy the geometric coincidence condition. The idea of the proof of this is as follows. The geometric coincidence condition 4.3 is satisfied for a given directive sequence $\sigma \in D$ if certain sets defined in terms of balls of arbitrarily large radius $R$ are contained in sets that are defined in terms of the directive sequence $\sigma$. According to the effective version of the geometric coincidence condition 4.4, it is even sufficient to consider balls whose radius is chosen in terms of certain balance properties of languages related to $\sigma$. By the assumptions of Theorem 3.5, there exists a substitutive system $(X_\tau, \Sigma)$, with $\tau^\infty \in D$, which has purely discrete
spectrum and, hence, satisfies the geometric coincidence condition (4.3) for balls of arbitrarily large radii $R$. In the following two lemmas we show that this has the following consequence: Each $S$-adic dynamical system whose directive sequence $\sigma = (\sigma_n)$ has PRICE and contains a sufficiently long block $(\sigma_n, \ldots, \sigma_{n+\ell-1})$ with $\sigma_{[n,n+\ell]} = \tau^m$ (i.e., $m$ is sufficiently large) followed by some tail $\Sigma^{n+\ell} \sigma$ that has $C$-balanced language $L_{\Sigma^{n+\ell}}$, satisfies the effective version (4.4) of geometric coincidence condition. Informally speaking, in $\sigma$ we need a sufficiently long block of substitutions which satisfies the coincidence condition (4.3), which is followed by a tail that is “balanced enough” to guarantee the coincidence condition for the whole sequence $\sigma$. Using the Poincaré Recurrence Theorem we are able to show that almost all directive sequences contain such a block.

This will finally imply Theorem 3.5.

Lemma 5.3. Let $\tau$ be a unit Pisot substitution with geometric coincidence. Then for each $C > 0$ there are $m = m_\tau(C) \in \mathbb{N}$, $z \in 1^+$, and $i \in A$ such that for each $t \in \mathbb{R}_+^d \setminus \{0\}$ we have

$$
\{(y, j) \in \mathbb{Z}^d \times A : \|\pi_t M^-m y - z\|_\infty \leq C, 0 \leq (1, y) < |\tau^m(j)| \}
\subseteq \{(l(p), j) : p \in A^*, j \in A, p i \leq |\tau^m(j)|\}.
$$

Remark 5.4. If we look at the definition of geometric coincidence in (4.3) the lemma states that the inclusion in the definition of geometric coincidence still holds if we add the projection $\pi_t$ for some nonnegative vector $t$. Indeed, because the elements $M^-m y$ that are projected are close to a hyperplane that is “sufficiently orthogonal” to $t$ and $1$, this projection does not change these vectors too much.

Proof. Since $\tau$ satisfies the geometric coincidence condition, there exist, for each $R > 0$ and sufficiently large $m \in \mathbb{N}$, some $i \in A$ and $z' \in M^-m 1^\perp = (M^{-m}1)^\perp$, such that

$$
\{(y, j) \in \mathbb{Z}^d \times A : \|M^-m y - z'\|_\infty \leq R, 0 \leq (1, y) < |\tau^m(j)| \}
\subseteq \{(l(p), j) : p \in A^*, j \in A, p i \leq |\tau^m(j)|\}.
$$

Since $1 M^{-m} 1 / |1 M^{-m} 1|$ converges to a dominant eigenvector of $1 M$, which is positive, there exists a constant $c_1 > 0$ such that $\|x\|_\infty \leq c_1 \|\pi_t x\|_\infty$ for all $t \in \mathbb{R}_+^d \setminus \{0\}, x \in (1 M^{-m} 1)^\perp, m \in \mathbb{N}$. Let $\pi_{t,n}$ denote the projection along $t$ on $(1 M^{-m} 1)^\perp$. There is another constant $c_2 > 0$ such that $\|x - \pi_{t,n} x\|_\infty \leq c_2$ for all $x \in \mathbb{R}_+^d$ with $0 \leq (1 M^{-m} 1, x) < \max_{j \in A} (1 M^{-m} 1, e_j)$.

Therefore, we have

$$
\|M^-m y - z'\|_\infty \leq \|\pi_t M^-m y - z'\|_\infty + c_2 \leq c_1 \|\pi_t (M^-m y - z')\|_\infty + c_2
$$

for all $y \in \mathbb{Z}^d, z' \in (1 M^{-m} 1)^\perp$ with $0 \leq (1, y) < \max_{j \in A} |\tau^m(j)|$.

Choosing $m = m_\tau(C)$ such that $c_2 R + c_2$ holds for $R = c_1 C + c_2$ and some $z' \in 1^\perp$, $i \in A$, we obtain that (5.1) holds with $z = \pi_t z'$.

We now prove geometric coincidence for directive sequences $\sigma = (\sigma_n)$ containing a long block $(\sigma_n, \ldots, \sigma_{n+\ell-1})$ satisfying $\sigma_{[n,n+\ell]} = \tau^m$ followed by a tail $\Sigma^{n+\ell} \sigma \in B_C$. Indeed, this constellation will allow us to apply Lemma 5.3 in order to fulfill the effective version of the geometric coincidence condition for $\Sigma^{n+\ell} \sigma$. Thus $\Sigma^{n+\ell} \sigma$ gives rise to tilings which will lead to the desired conclusion.

Lemma 5.5. Let $\tau$ be a unit Pisot substitution that satisfies geometric coincidence. Let $\sigma = (\sigma_n)$ be a sequence satisfying PRICE and $C > 0$ such that there are $\ell, n \in \mathbb{N}$ such that for $m = m_\tau(C)$ as in Lemma 5.3 we have $\sigma_{[n,n+\ell]} = \tau^m$ and $\Sigma^{n+\ell} \sigma \in B_C$. Then $C_\sigma$ forms a tiling of $1^\perp$.

Proof. Let $\tau$, $\sigma$, and as in the statement of the lemma, let $u$ be a generalized right eigenvector of $\sigma$, and write $u^{(k)} = M^{-1} u$ as in (4.4). Obviously, $u^{(n)}$ is a generalized right eigenvalue of $\Sigma^n \sigma$.

Since $\sigma$ satisfies PRICE, $\Sigma^n \sigma$ also satisfies PRICE by [BST19, Lemma 5.10]. We want to prove that $\Sigma^n \sigma$ satisfies (4.4). To this end we apply Lemma 5.3 to $\tau$ and $t = u^{(n+\ell)}$. Since $\sigma_{[n,n+\ell]} = \tau^m$ this yields that

$$
\{(y, j) \in \mathbb{Z}^d \times A : \|\pi_t u^{(n)} M^{-1} \sigma_{[n,n+\ell]} y - z\| \leq C, 0 \leq (1, y) < |\sigma_{[n,n+\ell]}(j)| \}
\subseteq \{(l(p), j) : p \in A^*, j \in A, p i \leq |\sigma_{[n,n+\ell]}(j)|\}.
$$
Thus all conditions of Proposition 4.2, in particular (4.4), are satisfied by $\Sigma^n \sigma$, hence, by Proposition 4.2 both $C_{\Sigma^n \sigma}$ and $C_{\sigma}$ form a tiling of $1^\perp$. □

We are now in a position to prove Theorem 3.5. Indeed, we use the Poincaré Recurrence Theorem in order to show that under the conditions of Theorem 3.5, Lemma 5.5 can be applied to almost all directive sequences $\sigma \in D$.

**Conclusion of the proof of Theorem 3.5** According to the assumptions of Theorem 3.5 there is a periodic sequence $(\tau_0, \ldots, \tau_{k-1})^\infty \in D$ with positive range such that $\tau = \tau_{\{0, k\}}$ is a Pisot substitution and the substitutive dynamical system $(X, \Sigma)$ has purely discrete spectrum and thus Lemma 4.3 implies that $\tau$ satisfies the geometric coincidence condition. By Lemma 5.1 and the positive range of $(\tau_0, \ldots, \tau_{k-1})^\infty$ there is $C \in \mathbb{N}$ such that

$$
\nu(D \cap B_C) > 1 - \inf_{n \in \mathbb{N}} \nu(\Sigma^n[\tau_0, \ldots, \tau_{n-1}]).
$$

This yields that $\nu(\Sigma^n[\tau_0, \ldots, \tau_{n-1}] \cap B_C) > 0$ and, since $\nu \circ \Sigma^n \ll \nu$, $\nu(\tau_0, \ldots, \tau_{n-1}) \cap \Sigma^{-n}B_C) > 0$ for all $n \in \mathbb{N}$. Choose $m = m_\tau(C)$ as in Lemma 5.3. By the Poincaré Recurrence Theorem, for almost all sequences $\sigma \in D$, there exists $n$ such that $\Sigma^n \sigma \in [\tau_0, \ldots, \tau_{k-1}] \cap \Sigma^{-n}B_C$, which is equivalent to the conditions $\sigma_{[n,n+i]} = \tau_{\sigma}$ and $\Sigma^n \sigma \in B_C$ in the formulation of Lemma 5.5. Thus, since PRICE holds for a.a. $\sigma \in D$ by Proposition 5.2, Lemma 5.5 yields geometric coincidence for almost all $\sigma \in D$. This implies that $C_{\sigma}$ forms a tiling of $1^\perp$. We may thus apply [BST19, Proposition 8.5] to conclude that $(X, \Sigma, \mu)$ is conjugate to the translation by $\pi_x = \pi_x - u$ on $1^\perp/\mathbb{Z}^d$ for all $i \in \{1, \ldots, d\}$, where $u$ is the normalized generalized right eigenvector of $\sigma$. Taking $i = d$ and omitting the $d$-th coordinate, we obtain that $(X, \Sigma, \mu)$ is conjugate to the translation by $-\pi(u) = \pi_x$ on $T^{d-1}$, thus also to the translation by $u$. In particular, $(X, \Sigma, \mu)$ has purely discrete measure-theoretic spectrum. Moreover, the shift $(X, \Sigma)$ is a natural coding of $R_{\sigma}$, w.r.t. the natural partition $\{R_{\sigma}(1), \ldots, R_{\sigma}(d)\}$. Indeed, the required topological properties of the atoms of the natural partition are established in [BST19, Theorem 3.1]. We then consider the action of the domain exchange from [BST19, Proposition 8.4] on the pieces of the Rauzy fractal which gives $R_{\sigma}(i) + e_i - u \subset R_{\sigma}$, for $1 \leq i \leq d$. This yields after applying $\pi'$ that $-R_{\sigma}'(i) - e_i + u' \subset -R_{\sigma}'$ for $1 \leq i < d$ and $-R_{\sigma}'(d) + u' \subset -R_{\sigma}'$. Lastly, the fact that the intersection of cylinders from Definition 2.5 always contain a single point holds by [BST19, Lemma 8.3]. □

5.2. **Proof of Theorem 3.6** To prove Theorem 3.6 we need to get rid of the condition on the existence of a periodic Pisot sequence with purely discrete spectrum present in Theorem 3.5. In other words, under the conditions of Theorem 3.6 we have to provide an “accelerated” substitution with purely discrete spectrum (i.e., satisfying the geometric coincidence condition by Lemma 4.3). This is the objective of Proposition 5.8 which, for any given unit Pisot matrix $M$, provides a substitution with incidence $M^n$ (for some $n \geq 1$) having purely discrete spectrum.

We start with two technical lemmas. Lemma 5.6 recalls the classical connection between the Pisot property and balance (see e.g. [Ada04b, Theorem 1], which states that a substitution whose incidence matrix is a unit Pisot matrix is balanced). Its proof relies on the geometric characterization of balance given in Proposition 4.1. Moreover, Lemma 5.7, recalls that for any given integer vector $x$ with nonnegative entries, there exists a word $w$ with uniformly bounded balance (w.r.t. the direction of $x$), whose abelianization satisfies $s(w) = x$.

**Lemma 5.6.** Let $M$ be a unit Pisot matrix with dominant right eigenvector $u$. There exists a constant $C > 0$ such that each substitution $\sigma$ satisfying $M\sigma = M^k$ for some $k \in \mathbb{N}$ and

$$
(5.3) \max_{p \in A^* : p \preceq \pi(i), i \in A} ||\pi_u l(p)|| < 2
$$

has $C$-balanced language $L_\sigma$.

**Proof.** Let $\sigma$ be a substitution satisfying the conditions indicated in the statement of the lemma. Let $n \in \mathbb{N}$ be arbitrary but fixed and choose a prefix $p$ of $\sigma^n(i)$ for some $i \in A$. Then we have $p = \sigma^{n-1}(p_{n-1}) \cdots \sigma(p_1)p_0$ for some prefixes $p_j$ of $\sigma(i_j), i_j \in A, (\text{with } \sigma(i_j) \in p_j i_{j-1} A^*)$, thus

$$
l(p) = M^{k(n-1)}l(p_{n-1}) + \cdots + M^k l(p_1) + l(p_0).
$$
Let \( v \) be a dominant left eigenvector of \( M \), \( q < 1 \) the maximal absolute value of the non-dominant eigenvalues of \( M \) and \( \tilde{\pi}_u \) the projection along \( u \) on \( v^\perp \). Then we have a constant \( c_1 > 0 \) such that \( \| M^\ell x \| \leq c_1 q^\ell \| x \| \) for all \( \ell \in \mathbb{N}, \ x \in v^\perp \). Thus we have \( \| \tilde{\pi}_u M^\ell x \| = \| M^\ell \tilde{\pi}_u x \| \leq c_1 q^\ell \| \tilde{\pi}_u x \| \) for all \( x \in \mathbb{R}^d \), hence

\[
(5.4) \quad \| \tilde{\pi}_u (p) \| \leq \frac{c_1}{1 - q^k} \max_{q \in A^*: q \leq \sigma(i), i \in A} \| \tilde{\pi}_u (q) \|.
\]

There is a constant \( c_2 > 0 \) such that \( \| \pi_u x \| \leq c_2 \| x \| \) for all \( x \in v^\perp \) and \( \| \tilde{\pi}_u x \| \leq c_2 \| x \| \) for all \( x \in v^\perp \). Thus \([5.4]\) and \([5.3]\) yield

\[
\| \pi_u (p) \| = \| \pi_u \tilde{\pi}_u (p) \| \leq c_2 \| \tilde{\pi}_u (p) \| \leq \frac{c_1 c_2}{1 - q^k} \max_{q \in A^*: q \leq \sigma(i), i \in A} \| \tilde{\pi}_u (q) \| \leq \frac{2 c_1 c_2}{1 - q^k}.
\]

If \( v \in v^\perp \), then \( v \) is a factor of \( \sigma^n(i) \) for some \( n \in \mathbb{N}, \ i \in A \). Thus there are two prefixes \( p_1, p_2 \) of \( \sigma^n(i) \) such that \( p_1 v = p_2 \) and, hence, \( \| \pi_u (v) \| \leq \| \pi_u (p_1) \| + \| \pi_u (p_2) \| < \frac{4 c_1 c_2}{1 - q^k} \). Moreover, for two factors \( v_1, v_2 \) with \( I(v_1) \cap I(v_2) \neq \emptyset \), we have \( \| I(v_1) \cap I(v_2) \| \leq \| I(v_1) \| + \| I(v_2) \| \leq 2 \| I(v_1) \| \), and thus \( L_\sigma \) is \( C \)-balanced with \( C = \frac{8 c_1 c_2}{1 - q^k} \), according to Proposition 4.1. \( \square \)

**Lemma 5.7.** Let \( x \in \mathbb{R}^d \). Then there exists a word \( w \in A^* \) such that \( I(w) = x \) and \( \| \pi_x (p) \| \leq 1 - \frac{1}{2d-2} \) for all \( p \leq w \). Moreover, \( w \) starts with the letter corresponding to the largest coordinate of \( x \).

**Proof.** This is proved in [Mel73, Tij80]. \( \square \)

The construction of the desired substitution is contained in the following proposition.

**Proposition 5.8.** Let \( M \) be a nonnegative unit Pisot matrix. Then there exists a substitution \( \sigma \) with incidence matrix \( M_\sigma \) satisfying \( M_\sigma = M^n \) for some \( n \in \mathbb{N} \) such that the geometric coincidence condition holds. Moreover, we can choose \( \sigma \) in a way that \( \sigma(i) \leq \sigma(j) \) or \( \sigma(j) \leq \sigma(i) \) for all \( i, j \in A \).

**Proof.** Let \( u \) be a dominant right eigenvector of \( M \) and

\[
P = \{ y \in \mathbb{Z}^d : 0 \leq \langle 1, M^n y \rangle \leq \max_{i \in A} \langle 1, M^n e_i \rangle \text{ for some } n \in \mathbb{N}, \ |\pi_u y| \leq C \},
\]

with \( C \) as in Lemma 5.6. Note that \( P \) is a finite set since \( \langle 1, M^n y \rangle = \langle M^n, 1, y \rangle \) and \( u \in \mathbb{R}^d \).

Write \( P = \{ y_\ell : 0 \leq \ell \leq L \} \) such that \( 0 = (u, y_0) < (u, y_1) < \cdots < (u, y_L) \); this is possible since \( u \) has rationally independent coordinates. Then for \( n \in \mathbb{N} \) large enough we have

\[
(5.5) \quad \| \pi_u M^n y \| \leq \frac{1}{3}
\]

for all \( y \in P \) and \( M^n (y_{\ell+1} - y_\ell) \in \mathbb{N}^d \) for all \( 0 \leq \ell < L \). Let the words \( w_\ell \) be given by Lemma 5.7 with \( x = x_\ell = M^n (y_{\ell+1} - y_\ell) \) for \( 0 \leq \ell < L \), and set \( \sigma(j) = w_0 w_1 \cdots w_{L_j-1} \) for all \( j \in A \), with \( L_j \) such that \( y_{L_j} = e_j \). Let \( p \leq \sigma(j) \) for some \( j \in A \). Then there is \( \ell \in \{ 0, \ldots, L - 1 \} \) such that \( p = w_0 \cdots w_{\ell-1} p' \) with \( p' \leq w_\ell \) (here \( w_0 \cdots w_{\ell-1} \) is the empty word for \( \ell = 0 \)). This yields

\[
\| \pi_u (p) \| = \| \pi_u y_\ell + \pi_u (p') \| \leq \| \pi_u y_\ell \| + \| \pi_u (p') \|
\]

\[
\leq \| \pi_u y_\ell \| + \| \pi_u M^n (y_{\ell+1} - y_\ell) \| + \| \pi_M^n (y_{\ell+1} - y_\ell) \| < 2,
\]

where the last inequality follows from \([5.5]\) and \([5.7]\). We can now apply Lemma 5.6 to conclude that \( L_\sigma \) is \( C \)-balanced. By the construction of \( \sigma \), we have \( M_\sigma = M^n \) and

\[
\{(y, j) \in \mathbb{Z}^d \times A : \| \pi_u M_\sigma^n y \| \leq C, 0 \leq \langle 1, y \rangle < |\sigma(j)| \} = \bigcup_{j \in A} \{ (I(w_0 \cdots w_\ell), j) : 0 \leq \ell < L_j - 1 \} \cup \bigcup_{i \in A} \{ (I(p), j) : p \in A^*, j \in A, p i \leq \sigma(j) \}.
\]
Moreover, $\sigma(i)$ is a prefix of $\sigma(j)$ if and only if $\langle u, e_i \rangle = \langle u, e_j \rangle$. Let $i_0 \in A$ be chosen in a way that $\langle u, e_{i_0} \rangle = \max_{j \in A} \langle u, e_j \rangle$. Then the $i_0$-th coordinate of $x_\ell$ is the largest one for each $1 \leq \ell < L$ if $n$ is chosen large enough. In the construction of [Bar16], the word $w = w_\ell$ in Lemma 5.7 starts with the letter $i_0$ for each $x = x_\ell$ ($1 \leq \ell < L$). Therefore, we can choose $n$ large enough such that all words $w_\ell$, $0 \leq \ell < L$, start with $i_0$. This means that we can sharpen the inclusion in (5.6) to
\[
\bigcup_{j \in A} \{ (l(w_0 \cdots w_\ell), j) : 0 \leq \ell < L_j - 1 \} \subset \{ (l(p), j) : p \in A^*, j \in A, p i_0 \preceq \sigma(j) \}.
\]
Together with (5.6) this yields
\[
\{(y, j) \in \mathbb{Z}^d \times A : \|\pi u M_\sigma^{-1} y\| \leq C, 0 \leq \langle 1, y \rangle < |\sigma(j)|\}
\subset \{ (l(p), j) : p \in A^*, j \in A, p i_0 \preceq \sigma(j) \}
\]
and $\sigma$ satisfies the effective geometric coincidence condition (4.4), and, hence, the geometric coincidence by Proposition 4.2.

Remark 5.9. To prove Proposition 5.8 we could also have used the condition from [Bar16 Corollary 2] to check geometric coincidence. This condition requires that the last letter of $\sigma(i)$ is equal for all $i \in A$ and the first letter of $\sigma(i)$ is different from the first letter of $\sigma(j)$ if $i \neq j$. If $M$ is a unit Pisot matrix it is also primitive and thus there is $n \in \mathbb{N}$ such that $M^n$ is a positive matrix. By this positivity there is clearly a substitution $\sigma$ with incidence matrix $M^n$ having this property. However, since [Bar16] deals with an $\mathbb{R}$-action which is a suspension of the shift $S$, a some more detailed discussion (like the one contained in [Bor06]) would be needed to adapt the results of [Bar16] to our setting.

We can now finish the proof of Theorem 3.6. The idea is to provide a suitable substitutive realization in the same flavor as the substitutive realizations associated with multidimensional continued fraction algorithms from Section 2.3. Analogously to compositions of substitutions, for products of matrices $C_k, \ldots, C_n$ ($k \leq n$) we will use the notation $C_k \cdots C_n = C_{[k,n]}$ in the sequel.

Proof of Theorem 3.6. Let $(\mathcal{D}, \Sigma, Z, \nu)$ be as in the statement of Theorem 3.6. Then there is a periodic sequence $(M_0, \ldots, M_{k-1}) \in \mathcal{D}$ with positive range such that $M_{[0,k]}$ is a Pisot matrix. Since $M_{[0,k]}$ and $M_{[i,k]}$ are similar matrices, also $M_{[i,k]} M_{[0,i]}$ is a Pisot matrix for each $i \in \{0, \ldots, k-1\}$. By Proposition 5.8 we know that there is a substitution $\tau_i$ with incidence matrix $M_{\tau_i} = M_{[i,k]} M_{[0,i]}$ satisfying the geometric coincidence condition (replace $k$ by $km$ for some $m \in \mathbb{N}$ if necessary). We choose $\tau_i$ in a way that $\tau_i = \tau_j$ if $M_{[i,k]} M_{[0,i]} = M_{[j,k]} M_{[0,j]}$ ($0 \leq i, j < k$).

Choose a map $s : M^k_d \to S_d$ with the properties that
\begin{itemize}
  \item the incidence matrix of $s(M_0, \ldots, M_{k-1})$ is $M_{[0,k]}$ for all $(M_0, \ldots, M_{k-1}) \in M^k_d$,
  \item $s(M_0, \ldots, M_{k-1}) = s'(M'_0, \ldots, M'_{k-1})$ if $M_{[0,k]} = M'_{[0,k]}$,
  \item $s(M_0, \ldots, M_{k-1}) = \tau_i$ if $M_{[0,k]} = M_{[i,k]} M_{[0,i]}$ for some $0 \leq i < k$.
\end{itemize}

Then the map
\[
\psi : \mathcal{D} \to D, \quad (M_n)_{n \in \mathbb{N}} \mapsto (s(M_{kn}, \ldots, M_{kn+k-1}))_{n \in \mathbb{N}}
\]
is well defined, and we have the commutative diagram
\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{s} & \mathcal{D} \\
\downarrow{\psi} & & \downarrow{\psi} \\
D & \xrightarrow{\Sigma} & D
\end{array}
\]
The acceleration $\Sigma^k$ of $\Sigma$ may no longer be ergodic with respect to $\nu$. Thus the system $(D, \Sigma, \nu')$ may be nonergodic, with $\nu' = \nu \circ \psi^{-1}$. However, we will now show that $(D, \Sigma, \nu')$ can be partitioned into ergodic systems that satisfy the conditions of Theorem 3.5. Since all cylinders in

\footnote{These systems correspond to sets of directive sequences that may be not closed in $D$. This is why we chose not to confine ourselves to closed sets of directive sequences.}
$D$ are measurable, $\nu'$ is a Borel probability measure on $D$. Suppose that $(D, \Sigma, \nu')$ is not ergodic. Then there exists a $\Sigma$-invariant (up to measure zero) subset $\hat{D} \subseteq D$ with $0 < \nu'(\hat{D}) < 1$. Then $\psi^{-1}(\hat{D}) \subseteq \mathfrak{D}$ is $\Sigma^k$-invariant, hence $\bigcup_{i=0}^{k-1} \Sigma^{-i}\psi^{-1}(\hat{D})$ is $\Sigma$-invariant and, by ergodicity of $\nu$, equal to $\mathfrak{D}$ up to measure zero. Therefore, we have $\nu'(\hat{D}) = \nu(\psi^{-1}(\hat{D})) \geq 1/k$. Since $D \setminus \hat{D}$ is also $\Sigma$-invariant, we also have $\nu'(\hat{D}) \leq 1 - 1/k$. We repeat the argument until we have a measurable partition $\{D_1, \ldots, D_k\}$ of $D$, with $1 \leq \ell \leq k$, such that $(D_j, \Sigma, \nu'|_{D_j})$ is ergodic for all $1 \leq j \leq \ell$.

Let now $j$ be fixed. We need to prove that, for some $1 \leq i \leq k$, the periodic Pisot sequence $(\tau_i)_{n \in \mathbb{N}}$ has positive range in $D_j$. For all $1 \leq i \leq k$, we have

$$
\nu'(\Sigma^n[(\tau_i)^n] \cap D_j) \geq \nu(\Sigma^k[(\hat{M}_i, \ldots, \hat{M}_{k-1}, \hat{M}_0, \ldots, \hat{M}_{i-1})^n] \cap \psi^{-1}(D_j))
= \nu(\Sigma^{-i}\Sigma^k[(\hat{M}_i, \ldots, \hat{M}_{k-1}, \hat{M}_0, \ldots, \hat{M}_{i-1})^n] \cap \Sigma^{-i}\psi^{-1}(D_j))
\geq \nu(\Sigma^k[(\hat{M}_0, \ldots, \hat{M}_{k-1})^n(\hat{M}_0, \ldots, \hat{M}_{i-1})] \cap \Sigma^{-i}\psi^{-1}(D_j)).
$$

Since

$$
\Sigma^j \bigcap_{n \in \mathbb{N}} \Sigma^k[(\hat{M}_0, \ldots, \hat{M}_{k-1})^n(\hat{M}_0, \ldots, \hat{M}_{i-1})] = \bigcap_{n \in \mathbb{N}} \Sigma^{kn+i}[(\hat{M}_0, \ldots, \hat{M}_{k-1})^n(\hat{M}_0, \ldots, \hat{M}_{i-1})]
$$

and

$$
\nu\left( \bigcap_{n \in \mathbb{N}} \Sigma^{kn+i}[(\hat{M}_0, \ldots, \hat{M}_{k-1})^n(\hat{M}_0, \ldots, \hat{M}_{i-1})] \right) = \inf_{n \in \mathbb{N}} \nu(\Sigma^{kn+i}[(\hat{M}_0, \ldots, \hat{M}_{k-1})^n(\hat{M}_0, \ldots, \hat{M}_{i-1})]) > 0
$$

by the positive range of $(\hat{M}_0, \ldots, \hat{M}_{k-1})^\infty$, $\nu \cdot \Sigma^i \ll \nu$ gives that

$$
\nu\left( \bigcap_{n \in \mathbb{N}} \Sigma^k[(\hat{M}_0, \ldots, \hat{M}_{k-1})^n(\hat{M}_0, \ldots, \hat{M}_{i-1})] \right) > 0.
$$

Therefore, there is some $1 \leq i \leq k$ such that $\inf_{n \in \mathbb{N}} \nu'(\Sigma^n[(\tau_i)^n] \cap D_j) > 0$. Note that the constant sequence $(\tau_i)_{n \in \mathbb{N}}$ may not be in $D_j$, but the proof of Theorem 3.5 also goes through for Pisot directive sequences with positive range that are not contained in $D_j$ (but in the closure of $D_j$ in $D$). Thus $(\tau_i)_{n \in \mathbb{N}} \in D_j$ is a periodic Pisot sequence having positive range in $(D_j, \Sigma, \nu'|_{D_j})$ and purely discrete spectrum. Since the cocycle $Z$ satisfies the Pisot condition, the same is true for the cocycle $Z_j : D_j \to M_d$, $(\sigma_n) \mapsto {}^iM_{\sigma_n}$. Summing up we can apply Theorem 3.5 to $(D_j, \Sigma, Z_j, (\sigma_n) \mapsto {}^iM_{\sigma_n})$. This proves the result.

5.3. Proofs of Theorems 3.1 and 3.3 We now prove Theorems 3.1 and 3.3 by reducing them to Theorems 3.5 and 3.6 (see also Remark 3.7), respectively.

Proof of Theorem 3.1 Recall that $(\Delta, T, A, \nu)$ is a positive $(d-1)$-dimensional continued fraction algorithm satisfying the Pisot condition and $\nu \circ T \ll \nu$, that $\varphi$ is a faithful substitutive realization of $(\Delta, T, A, \nu)$, and that there is a periodic Pisot point $x_0$ such that $\varphi(x_0)$ has purely discrete spectrum and positive range in $(\Delta, T, A, \nu)$. Then we have $(\Delta, T, \nu) \equiv (\varphi(\Delta), \Sigma, \nu \circ \varphi^{-1})$, hence $\nu \circ \varphi^{-1}$ is an ergodic $\Sigma$-invariant Borel probability measure satisfying $\nu \ll \nu \circ \varphi^{-1}$, the linear cocycle $(\varphi(\Delta), \Sigma, Z, \nu \circ \varphi^{-1})$ defined by $Z((\sigma_n)_{n \in \mathbb{N}}) = {}^iM_{\sigma_n}$ satisfies the Pisot condition, and $\varphi(x_0)$ is a periodic Pisot sequence with purely discrete spectrum having positive range in $(\varphi(\Delta), \Sigma, \nu \circ \varphi^{-1})$. Therefore, by Theorem 3.5, for $\nu$-almost all $x \in \Delta$ the $S$-adic dynamical system $(X_{\varphi(x)}, \Sigma)$ is a natural coding of the minimal translation by $\varphi'(u)$ on $\mathbb{T}^{d-1}$ with respect to the partition $\{-R'\varphi(x)(i) : i \in A\}$ of the bounded fundamental domain $-R'\varphi(x)$, where $u$ is the normalized generalized right eigenvector of $\varphi(x)$. Since $x$ is the normalized generalized right eigenvector of $\varphi(x)$ we have $x = u$, which proves Theorem 3.1.

Theorem 3.3 follows from Theorem 3.6 in the following way.
Proof of Theorem 3.6. Recall that \((\Delta, T, A, \nu)\) is a positive \((d-1)\)-dimensional continued fraction algorithm satisfying the Pisot condition and \(\nu \circ T \ll \nu\), and that there is a periodic Pisot point \(x_0 \in \Delta\) having positive range in \((\Delta, T, A, \nu)\). Define \(\eta : \Delta \to M^d_{\mathbb{Z}}\) by \(x \mapsto (A(T^n x))_{n \in \mathbb{N}}\). Then we have \((\Delta, T, \nu) \cong (\eta(\Delta), \Sigma, \nu \circ \eta^{-1})\), hence, \(\nu \circ \eta^{-1}\) is an ergodic \(\Sigma\)-invariant Borel probability measure satisfying \(\nu \circ \eta^{-1} \circ \Sigma \ll \nu \circ \eta^{-1}\), the linear cocycle \((\eta(\Delta), \Sigma, Z, \nu \circ \eta^{-1})\) defined by \(Z(M_n)_{n \in \mathbb{N}} = M_0\) satisfies the Pisot condition, and \(\eta(x_0)\) has positive range in \((\eta(\Delta), \Sigma, \nu \circ \eta^{-1})\). Therefore, by Theorem 3.6 there exists a positive integer \(Z\) and a map \(\psi : \eta(\Delta) \to S^d_0\) (which we choose as in (5.7)) satisfying \(\psi \circ \Sigma^k = \Sigma \circ \psi\) such that for \(\nu\)-almost all \(x \in \Delta\) the \(S\)-adic dynamical system \((X_{\psi(\eta(x))}, \Sigma)\) is a natural coding of the minimal translation by \(\psi' (x)\) on \(\mathbb{T}^{d-1}\) with respect to a partition of a bounded fundamental domain. Setting \(\varphi = \psi \circ \eta\), we obtain that the diagram

\[
\begin{array}{ccc}
\Delta & \xrightarrow{T^k} & \Delta \\
\downarrow \eta & & \downarrow \eta \\
\eta(\Delta) & \xrightarrow{\Sigma^k} & \eta(\Delta) \\
\downarrow \psi & & \downarrow \psi \\
\varphi(\Delta) & \xrightarrow{\Sigma} & \varphi(\Delta)
\end{array}
\]

commutes. Because we chose \(\psi\) as in (5.7), \(\varphi\) is a substitutive realization of \((\Delta, T^k, A, \nu)\) such that for \(\nu\)-almost all \(x \in \Delta\) the \(S\)-adic dynamical system \((X_{\varphi(x)}, \Sigma)\) is a natural coding of the minimal translation by \(\varphi'(x)\) on \(\mathbb{T}^{d-1}\) with respect to the partition \(\{-R_{\varphi'(x)}(i) : i \in A\}\) of the bounded fundamental domain \(-R_{\varphi'(x)}\). This implies that \((X_{\varphi(x)}, \Sigma)\) has purely discrete spectrum. Since by construction, \(x\) is a generalized right eigenvector of \(\varphi(x)\), the map \(\varphi\) is injective, thus \((\Delta, T^k, \nu) \cong (\varphi(\Delta), \Sigma, \nu \circ \varphi^{-1})\).

5.4. Proof of Theorem 3.8. We now establish the relation between a natural coding with \(d\) atoms, bounded remainder sets, and Rauzy fractals asserted in Theorem 3.8. To this end, we need Lemma 5.10 that states in a nutshell that balance implies strong convergence. Like in Section 2.1, strong convergence refers to the convergence of the column vectors \(\lim_{n \to \infty} \pi_n M_{\sigma(n), n} e_i = 0\) for all \(i \in A\) and

\[
(5.8) \quad \lim_{n \to \infty} \sup \{ \| \pi_n M_{\sigma(n), n} I(w) \| : w \in L_{\Sigma^n \sigma} \} = 0.
\]

Proof. Assume that \(\sigma = (\sigma_n)_{n \in \mathbb{N}} \in S^d_0\) has balanced language \(L_\sigma\) and a generalized right eigenvector \(u\) with rationally independent coordinates. We first show that \(\sigma\) is a primitive sequence of substitutions. Suppose that there exists \(k \in \mathbb{N}\) such that \(M_{\sigma_{(k,n)}}\) is not positive for all \(n > k\). Then there exist coordinates \(i, j\) such that the \((i, j)\)-element of \(M_{\sigma_{(k,n)}}\) is 0 for infinitely many \(n\), i.e., \(M_{\sigma_{(k,n)}} e_j \in e_i^\perp\). Since the cones \(M_{\sigma_{(k,n)}} \mathbb{R}_{\geq 0}^d\) form a nested sequence of nonempty compact sets, their intersection is nonempty, and we obtain that \(e_i^\perp \cap \bigcap_{n \in \mathbb{N}} M_{\sigma_{(k,n)}} \mathbb{R}_{\geq 0}^d \neq \{0\}\), thus \(M_{\sigma_{(k,n)}} (e_i^\perp) \cap \bigcap_{n \in \mathbb{N}} M_{\sigma_{(k,n)}} \mathbb{R}_{\geq 0}^d \neq \{0\}\), which implies that \(u \in M_{\sigma_{(k,n)}} (e_i^\perp)\), contradicting that \(u\) has rationally independent coordinates. Therefore, \(\sigma\) is primitive.

Choose a sequence \((i_n)_{n \in \mathbb{N}} \in \mathbb{A}^\mathbb{N}\) such that \(i_n \preceq \sigma_n(i_{n+1})\) for all \(n \in \mathbb{N}\), and let \(\omega(n)\) be such that \(\sigma_{[n,t]} (i_t) \prec \omega(n)\) for all \(t > n\), i.e., \(\omega(n)\) is a so-called limit sequence of \(\Sigma^n \sigma\). Set

\[
P = \{ w \in \mathbb{A}^* : w \prec \omega(0) \} \quad \text{and} \quad P_{(j)} = \{ w \in \mathbb{A}^* : w \prec \sigma_{[0,n]}(j) \} \quad (j \in \mathbb{A}, n \in \mathbb{N}).
\]

Since \(\sigma\) is balanced, the set \(\pi_n I(P)\) is bounded by Proposition 4.1. From \(P_{(i_n)} \subseteq P_{(i_{n+1})} \subseteq \cdots \subseteq \bigcup_{n \in \mathbb{N}} P_{(i_n)} = P\), we obtain that there is a sequence of positive numbers \((\varepsilon_n)_{n \in \mathbb{N}}\) with \(\lim_{n \to \infty} \varepsilon_n = 0\).
0 such that
\[ \|x\| \leq \varepsilon_n \quad \text{for all } x \in \mathbb{1}^\perp \text{ satisfying } x + \pi_u l(P^{(i_n)}_n) \subseteq \pi_u l(P). \]

We can now show that \( \pi_u M_{\sigma [0,n]} Q_n \) is small, where
\[ Q_n = \{ w \in \mathcal{A}^*: \gamma j \prec \omega(n) \text{ and } \rho j \prec \omega(n) \text{ for some } p \in \mathcal{A}^*, j \in \mathcal{A} \} \]
is the set of return words in \( \omega(n) \) to some letter. More precisely, we have
\[ ||\pi_u M_{\sigma [0,n]} l(w)|| \leq 2\varepsilon_k \quad \text{for all } w \in Q_n, \text{ provided that } M_{\sigma [k,n]} \text{ is a positive matrix.} \]

To prove this, let \( w \in Q_n \) arbitrary. If \( M_{\sigma [k,n]} \) is a positive matrix and \( j \in \mathcal{A} \), then there exists \( v \in \mathcal{A}^* \) with \( v i_k \leq \sigma [k,n] (j) \). Because \( w \in Q_n \), we can find some \( p \in \mathcal{A}^* \) such that
\[ \pi_u l(\sigma [0,n] (p) \sigma [0,k] (v)) + \pi_u l(P^{(i_k)}_k) \subseteq \pi_u l(P) \] and \( \pi_u l(\sigma [0,n] (p w) \sigma [0,k] (v)) + \pi_u l(P^{(i_k)}_k) \subseteq \pi_u l(P) \),
which implies that
\[ ||\pi_u M_{\sigma [0,n]} l(w)|| \leq ||\pi_u l(\sigma [0,n] (p) \sigma [0,k] (v))|| + ||\pi_u l(\sigma [0,n] (p w) \sigma [0,k] (v))|| \leq 2\varepsilon_k. \]

Since \( \sigma \) is primitive, \( M_{\sigma [k,n]} \) is positive for all \( k \in \mathbb{N} \) and sufficiently large \( n \) (depending on \( k \)). Thus for each \( k \in \mathbb{N} \) there is \( n \geq k \) such that \( ||\pi_u M_{\sigma [0,n]} l(w)|| \leq 2\varepsilon_k \) holds for all \( w \in Q_n \).

Next we show that, for each \( n \in \mathbb{N} \), the Minkowski sum
\[ (5.10) \quad I(Q_n) - \sum_{j=1}^d I(Q_n) \text{ contains a basis of } \mathbb{R}^d \text{ with vectors in } \{0,1\}^d. \]

First note that \( I(Q_n) \) contains a basis of \( \mathbb{R}^d \) by the rational indepence of \( u \) and the balance of \( L_\sigma \). If this was not the case then, since \( I(Q_n) \subset \mathbb{Z}^d \), there would exist \( v^+ \in \mathbb{Z}^d \) with \( I(Q_n) \subset \mathbb{Z}^d \), contradicting that \( u \) is rationally independent. Thus we may choose words \( w_i \in Q_n \) such that \( \{ I(w_i) : 1 \leq i \leq d \} \) forms a basis of \( \mathbb{R}^d \). If \( I(w_i) \in \{0,1\}^d \) for all \( i \), then we have found a basis of the required form because \( 0 \in I(Q_n) \). Otherwise note that each non-empty factor \( w \in \omega(n) \) can be written as
\[ (5.11) \quad w = v_1 a_1 v_2 a_2 \ldots v_\ell a_\ell \quad \text{with } 1 \leq \ell \leq d, v_j \in Q_n, a_j \in \mathcal{A} \text{ for all } 1 \leq j \leq \ell, a_j \neq a_k \text{ if } j \neq k. \]

Indeed, let \( a_1 \) be the first letter of \( w \) and \( v_1 \) the longest (possibly empty) word such that \( v_1 a_1 \leq w \); then \( v_1 \in Q_n \) and \( (v_1 a_1)^{-1} w \) has no occurrence of \( a_1 \); if \( w \neq v_1 a_1 \), then let \( a_2 \in \mathcal{A} \) be the first letter of \( (v_1 a_1)^{-1} w \) and \( v_2 \) the longest word such that \( v_2 a_2 \leq (v_1 a_1)^{-1} w \); repeat this procedure until \( (v_1 \ldots v_\ell a_\ell)^{-1} w \) (which has no occurrences of \( a_1, \ldots, a_\ell \)) is the empty word. Now, if \( w_i \notin \{0,1\}^d \) and \( w_i = v_1 a_1 v_2 a_2 \ldots v_\ell a_\ell \), then we can replace \( w_i \) by the shorter word \( v_j \) for some \( j \), or, when all \( I(v_j) \) are in the span of the other basis vectors, we replace \( I(w_i) \) by \( I(w_i) - \sum_{j=1}^{\ell} I(v_j) \) without losing the basis property. Since \( I(w_i) - \sum_{j=1}^{\ell} I(v_j) = I(a_1 \ldots a_\ell) \in \{0,1\}^d \) and the replacement by a shorter word can happen only finitely many times, this proves (5.10).

From (5.9) and (5.10) we see that, for each \( n \in \mathbb{N} \), there is a basis of \( \mathbb{R}^d \) with vectors \( x \in \{0,1\}^d \) satisfying \( ||\pi_u M_{\sigma [0,n]} x|| \leq 2(d+1)\varepsilon_k \) for all \( k < n \) such that \( M_{\sigma [k,n]} \) is positive. In particular, we have the same basis for infinitely many \( n \), and obtain that \( \lim_{n \to \infty} \pi_u M_{\sigma [0,n]} e_i = 0 \) for all \( i \in \mathcal{A} \).

Finally, let \( w \in L_{\Sigma \in \mathbb{R}} \). By primitivity, \( w \) is a factor of \( \omega(n) \). Writing \( w \) as in (5.11), we obtain that \( ||\pi_u M_{\sigma [0,n]} l(w)|| \leq 2d\varepsilon_k + \sum_{i=1}^d ||\pi_u M_{\sigma [0,n]} e_i|| \) for all \( k < n \) such that \( M_{\sigma [k,n]} \) is positive. This proves the lemma. \( \square \)

\textbf{Proof of Theorem 5.8}. Let \( (X, \Sigma) \) be the natural coding of the minimal translation \( R_k \) on \( \mathbb{T}^{d-1} \) w.r.t. the natural partition \( \{ F_1, \ldots, F_d \} \) of the bounded fundamental domain \( F \). We consider the associated exchange of domains \( \hat{R}_k \) defined on \( \hat{F} \) (see Section 2.4). Let \( t_k \) be such that \( \hat{R}_k (x) = x + t_k \) on \( F_k \) (note that \( t_k - t \in \mathbb{Z}^d \)), and let \( u = (u_1, \ldots, u_d) \) with \( u_i = \lambda(F_i) \), where \( \lambda \)
denotes the Lebesgue measure. Then we have $\sum_{i=1}^{d} u_i = 1$. Since $F$ is bounded and $(F, \hat{R}_t, \lambda|_F)$ is ergodic, we have for almost all $x \in F$, by the Birkhoff Ergodic Theorem,

$$\sum_{i=1}^{d} u_i t_i = \sum_{i=1}^{d} t_i \int_{F} d\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (\hat{R}_t^{k+1}(x) - \hat{R}_t^k(x)) = \lim_{n \to \infty} \frac{1}{n} (\hat{R}_t^n(x) - x) = 0.$$

Define the matrix $N \in \mathbb{R}^{(d-1)\times d}$ by $Ne_i = t_i$, i.e., the columns of $N$ are given by the vectors $t_i$. Then we have $Nu = 0$ and, by the minimality of $\hat{R}_t$, the vectors $t_i$ span $\mathbb{R}^{d-1}$, thus the kernel of $N$ is $\mathbb{R}u$. Hence there exists $c > 0$ such that $|x|_\infty \leq c \|Nx\|_\infty$ for all $x \in 1^d$. If $w$ is in the language of $X$, then $NI(w) = \sum_{i=1}^{d} |w_i| t_i = \hat{R}_t[w](x) - x$ for some $x \in F$, thus $\|NI(w)\|_\infty \leq \text{diam}(F)$, where $\text{diam}(F)$ denotes the diameter of $F$. Hence, we have

$$|w_i| - |w| u_i \leq \|I(w) - |w| u\|_\infty \leq c \|N(I(w) - |w| u)\|_\infty = \|NI(w)\|_\infty \leq c \text{diam}(F).$$

Therefore, $F_i$ is a bounded remainder set for all $1 \leq i \leq d$, and the language of $X$ is $(2c \text{diam}(F))$-balanced.

Minimality of $R_k$ implies total irrationality of $t$. We will show that this in turn implies that the vector $u = (\lambda(F_1), \ldots, \lambda(F_d))$ has rationally independent coordinates. Indeed, suppose on the contrary, that $(z, u) = 0$ for some $z \in \mathbb{Z}^d \setminus \{0\}$. Consider the $d \times d$ matrix $\tilde{N}$ which is obtained from $N$ by subtracting $t$ from each column and adding the row $(1, \ldots, 1)$ at the bottom. Because $t_i - t \in \mathbb{Z}^d$ the matrix $\tilde{N}$ is an integer matrix. Moreover, since $Nu = 0$ and $\|u\|_1 = 1$, we have $\tilde{N}u = (1)$. If det $\tilde{N} \neq 0$, then we have $\tilde{N}y = z$ for some $y \in \mathbb{Q}^d \setminus \{0\}$; if det $\tilde{N} = 0$, then we have $\tilde{N}y = 0$ for some $y \in \mathbb{Z}^d \setminus \{0\}$. In both cases, we have $0 = \tilde{N}y, u = \langle y, (1) \rangle$, contradicting the total irrationality of $t$.

Assume now that $X = X_\sigma$ for some sequence of substitutions $\sigma \in \Sigma_\sigma^d$. Because $F_1, \ldots, F_d$ are bounded remainder sets with measures $u_1, \ldots, u_d$, $X_\sigma$ has uniform letter frequencies. Thus [BD14, Theorem 5.7] implies that $u = (u_1, \ldots, u_d)$ is the (rationally independent) normalized generalized right eigenvector of $\sigma$ (moreover, $\sigma$ is primitive; see the first part of the proof of Lemma 5.10). Let $\omega(0) \in \sigma^0$ be as in the proof of Lemma 5.10 and write $\omega(0) = \omega_0 \omega_1 \cdots$ with $\omega_n \in \Sigma$. Then there is some $z \in F$ such that $R_\sigma^0(z) \in F_{\omega_n}$ for all $n \in \mathbb{N}$. Define the affine map $H : \mathbb{R}^d \to \mathbb{R}^{d-1}$ by $H(x) = x + N x$. Then, because $\mathbb{R}u$ is in the kernel of $N$, we have $H(x) = H(\pi_u x)$, in particular $H(\pi_u e_i) = z + t_i$. By minimality, we have

$$f_i = \{z + NI(p) : p \in A^*, p \prec \omega(0)\} \subseteq H(R_\sigma(i)) \quad \text{for all } i \in A.$$

On the other hand, if $p \prec x_0(j)$ for infinitely many $(n, j) \in \mathbb{N} \times A$, then there are words $w_n \in L_{\Sigma \sigma, \sigma}$ such that $\sigma_{(n, 0)}(w_n) p \prec \omega(0)$ for all $n \in \mathbb{N}$, hence $H(M_{\sigma_{(n, 0)}} I(w_n) + I(p)) \in F_i$, which implies that $H(I(p)) \in F_i$, written Lemma 5.10. Hence, we have $H(R_\sigma(i)) \subseteq F_i$, thus $H(R_\sigma(i)) = F_i$. This means that $(F, R_t)$ is the domain exchange $H(R_\sigma(i)) \to H(R_\sigma(i)) + H(\pi_u e_i)$. Therefore, $(F, R_t)$ is conjugate to the domain exchange $R_\sigma'(i) \to R_\sigma'(i) + e_i' - u'$, and $(X_\sigma, \Sigma)$ is a natural coding of $R_\sigma'$ w.r.t. the natural partition $\{[-R_\sigma'(i) : 1 \leq i \leq d]\}$, by the same arguments as in the proof of Theorem 3.3.

Assume now that the directive sequence $\sigma$ is left proper. Then by [BCBD+19, Lemma 3.2] the shift $(X_\sigma, \Sigma)$ can be represented as $(X_\sigma, \Sigma)$, where $\sigma$ is proper (and still unit). Like $\sigma$, also $\sigma'$ is primitive (see again the first part of the proof of Lemma 5.10). From [BCBD+19 Corollary 5.5], we gain that if a primitive unit proper $S$-adic shift $(X_\sigma, \Sigma)$ is balanced on letters, then it is also balanced on all its factors. Hence, cylinders associated to factors are also bounded remainder sets, by Proposition 4.4.

6. Examples

In this section we show that our theory can easily be applied to many well-known multidimensional continued fraction algorithms. According results for the case of the Brun and the Arnoux–Rauzy algorithm for $d = 3$ are treated in [BST19], and for the Cassaigne–Selmer algorithm ($d = 3$) in [FN20]. Using our new theory, the conditions we need to check are easier to verify than the ones in [BST19, FN20]. This even allows us to treat the Arnoux–Rauzy algorithm in arbitrary dimension $d \geq 3$ (see Section 6.2), the (multiplicative) Jacobi–Perron algorithm ($d = 3$)
in Section 6.3} and the Brun algorithm for \( d = 4 \) in Section 6.4. We first start with the Cassaigne–Selmer algorithm in Section 6.1 for which we can also prove more general results than the ones in [FY20].

Save for the Arnoux–Rauzy algorithm, we focus on algorithms with dimension \( d \in \{3, 4\} \). This is due to recent somewhat surprising numerical experiments from [BST20] which indicate that the second Lyapunov exponent is positive for most of the classical continued fraction algorithms if the dimension is beyond a certain threshold. In other words, the Pisot condition (see Definition 2.1) seems to be violated in these cases. For instance, the Brun and Jacobi–Perron algorithms seem to have positive second Lyapunov exponent in dimension \( d \geq 10 \), contrarily to what was expected e.g. in [LAG93] [HK00]. For the Selmer algorithm the Pisot condition seems to be violated already in dimension \( d \geq 5 \).

6.1. The Cassaigne–Selmer algorithm. In 2015 Cassaigne announced a 2-dimensional continued fraction algorithm that was first studied in [CLL17]. This algorithm is called Cassaigne–Selmer algorithm because it is measurably conjugate to the 2-dimensional Selmer algorithm (see [CLL17]; Selmer’s algorithm goes back to [Sel61]). Cassaigne’s representation of this algorithm is remarkable because it admits a set of substitutions that is particularly relevant from a symbolic point of view. As shown in [CLL17], the \( S \)-adic symbolic dynamical systems defined in terms of these substitutions have factor complexity \( 2n+1 \) and, as underlined in [BCBD19], belong to the family of so-called dendric subshifts. We recall that the factor complexity of a symbolic dynamical system counts the number of factors of a given length of its associated language, i.e., of the factors that occur in the elements of this symbolic dynamical system.

Let \( \Delta = \{(x_1, x_2, x_3) \in [0, 1]^3 : x_1 + x_2 + x_3 = 1\} \). Using the matrices

\[
C_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},
\]

we define the matrix valued function

\[
A_c : \Delta \to \text{GL}(3, \mathbb{Z}), \quad x \mapsto \begin{cases} C_1 & \text{if } x \in \Delta(1) := \{(x_1, x_2, x_3) \in \Delta : x_1 \geq x_3\}, \\ C_2 & \text{if } x \in \Delta(2) := \{(x_1, x_2, x_3) \in \Delta : x_1 < x_3\}
\end{cases}
\]

(here, we slightly abuse notation by writing \( \Delta(j) \) for \( \Delta(x) \) with \( A_c(x) = C_j \)). Then \( T_c \) is given by

\[
T_c : \Delta \to \Delta, \quad x \mapsto \begin{cases} \left( \frac{x_1 - x_3}{x_1 + x_2}, \frac{x_3}{x_1 + x_2}, \frac{x_2}{x_1 + x_2} \right) & \text{if } x_1 \geq x_3, \\ \left( \frac{x_3}{x_1 + x_2}, \frac{x_1 - x_3}{x_1 + x_2}, \frac{x_2}{x_1 + x_2} \right) & \text{if } x_1 < x_3,
\end{cases}
\]

and \((\Delta, T_c, A_c)\) is called Cassaigne–Selmer algorithm. In [AL18] Proposition 22 it is proved that the density of the invariant probability measure \( \nu_c \) of \( T_c \) equals \( \frac{12}{\pi^2(1-x_1)(1-x_3)} \). Following [CLL17] we define the Cassaigne–Selmer substitutions

\[
\gamma_1 : \{1 \mapsto 1, 2 \mapsto 13, 3 \mapsto 2\} \quad \text{and} \quad \gamma_2 : \{1 \mapsto 2, 2 \mapsto 13, 3 \mapsto 3\}
\]

The corresponding faithful substitution selection is defined by \( \varphi(x) = \gamma_j \) if \( x \in \Delta(j) \). By Definition 2.2 the map

\[
\varphi : \Delta \to \{\gamma_1, \gamma_2\}^\mathbb{N} \quad \text{with} \quad \varphi(x) = (\varphi(T^n x))_{n \in \mathbb{N}}
\]

is a faithful substitutive realization of \((\Delta, T_c, A_c)\). We have \( T_c(\Delta(1)) = T_c(\Delta(2)) = \Delta \), thus the algorithm satisfies the finite range property and each \( x \in \Delta \) has positive range (in the sense of Definition 2.8). Moreover, \( \varphi(\Delta) = \{\gamma_1, \gamma_2\}^\mathbb{N} \) (up to a set of measure zero). According to [LAG93] Section 6] and [Sch00] Chapter 7, \( T_c \) is \( \nu_c \)-almost everywhere weakly convergent, \((\Delta(1), \Delta(2))\) is a generating (Markov) partition for \( T_c \), and, hence, one has \( (\Delta, T_c, \nu_c) \cong (\{\gamma_1, \gamma_2\}^\mathbb{N}, \Sigma, \nu_c \circ \varphi^{-1}) \). The linear cocycle \( A_c \) is log-integrable since the Cassaigne–Selmer algorithm is additive with \( A_c \), taking only 2 values. By [Sch04] [BST20], we know that \((\Delta, T_c, A_c, \nu_c)\) satisfies the Pisot condition.
Moreover, since \( \nu_c \) is a Borel probability measure which is equivalent to the Lebesgue measure and \( T_c \) maps open sets to open sets, we have \( \nu_c \circ T \ll \nu_c \).

To apply Theorem 3.1 we have to find a periodic Pisot point \( x \in \Delta \) (see Definition 2.3) such that \( \varphi(x) \) has purely discrete spectrum. To this end, consider

\[
\tau = \gamma_1 \circ \gamma_2 : \begin{align*}
1 & \mapsto 13 \\
2 & \mapsto 12 \\
3 & \mapsto 2
\end{align*}
\]

and let \( x \in \Delta \) be the dominant right eigenvector of \( M_\tau \). Then we have \( \varphi(x) = (\gamma_1, \gamma_2)^\infty \). Since \( M_\tau \) is a Pisot matrix, we conclude that \( x \) is a periodic Pisot point which has positive range by the above considerations. Therefore, it only remains to prove the following lemma to be able to apply Theorem 3.1

**Lemma 6.1.** Let \( \tau = \gamma_1 \circ \gamma_2 \). Then \( \tau \) is a unit Pisot substitution and the substitutive dynamical system \((X_\tau, \Sigma)\) has purely discrete spectrum.

**Proof.** The characteristic polynomial \( X^3 - 2X^2 + X - 1 \) of \( M_\tau = t_1 C_1 t_2 \) is the minimal polynomial of a Pisot unit. We have to prove that substitutive dynamical system \((X_\tau, \Sigma)\) has purely discrete spectrum.

Let \( \sigma \) be a unit Pisot substitution over the alphabet \( A = \{1, 2, 3\} \). To check if the substitutive dynamical system \((X_\tau, \Sigma)\) is measurably conjugate to a minimal translation on \( T^2 \) one can apply the balanced pair algorithm (see e.g. [SS02, Section 3], [BK06, Section 17] or [BST10, Section 5.8]). A balanced pair is a pair \((v_1, v_2) \in A^* \times A^*\) with \( \mathbf{I}(v_1) = \mathbf{I}(v_2) \). It is called irreducible if no proper prefixes of \( v_1 \) and \( v_2 \) give rise to a balanced pair. Each balanced pair can be decomposed into irreducible balanced pairs in an obvious way. We recall the balanced pair algorithm for \( \sigma \). It starts with \( I_0 = \{(12, 21), (13, 31), (23, 32)\} \). Given \( I_k \), for some \( k \in \mathbb{N} \), the set \( I_{k+1} \) is defined recursively by the set of all irreducible balanced pairs occurring in a decomposition of a balanced pair \((\sigma(v_1), \sigma(v_2))\) with \((v_1, v_2) \in I_k \). We say that the balanced pair algorithm terminates if for some \( k \in \mathbb{N} \) the set \( I_k \setminus (I_0 \cup \cdots \cup I_{k-1}) = \emptyset \) and if each \((v_1, v_2) \in \bigcup_{j=0}^{k} I_j \) eventually contains a coincidence, i.e., there is a pair of the form \((i, i) \in A \times A\) that occurs in \((\sigma^j(v_1), \sigma^j(v_2))\) for some \( j \in \mathbb{N} \). According to [SS02, Section 3] the balanced pair algorithm terminates if and only if \((X_\sigma, \Sigma)\) has purely discrete spectrum.

In our case, we get \((12, 21) \xrightarrow{\tau} (1312, 1213)\) which splits into the irreducible pairs \((1, 1)\), a coincidence, and \((312, 213)\). Moreover, \((13, 31) \xrightarrow{\tau} (1321, 213)\) does not split and \((23, 32) \xrightarrow{\tau} (122, 212)\) splits into \((12, 21)\), and the coincidence \((2, 2)\). Thus

\[
I_1 = \{(1, 1), (2, 2), (12, 21), (312, 213), (132, 213)\}.
\]

We have to go on with the new pairs \((1, 1), (2, 2), (312, 213), (132, 213)\) occurring in \( I_1 \). While coincidences yield only coincidences again, we get the pairs \((321, 213) \xrightarrow{\tau} (21312, 12132)\) and \((132, 213) \xrightarrow{\tau} (13212, 12132)\). Splitting these yields the new pairs \((321, 213), (321, 213)\). Summing up the set \( I_2 \) contains the new pairs \((3, 3)\) and \((321, 213)\). We only have to check the one which is not a coincidence, getting \((321, 213) \xrightarrow{\tau} (21213, 21132)\). This gives (up to switching the order of the pairs) no new pairs in \( I_3 \). Since all occurring pairs eventually end up in coincidences, the balanced pair algorithm terminates for \( \tau \) and, hence, \((X_\tau, \Sigma)\) has purely discrete spectrum. \(\square\)

Note that the periodic directive sequence \((\gamma_1, \gamma_2)^\infty\) is obviously proper. Hence, combining Theorem 3.1 with Theorem 3.8 we obtain the following result. Recall that \( x' = \pi'(x) \) for the projection \( \pi' \) defined in (2.8). The according projections of the subtiles, \( R'_{\sigma'}(w), w \in A^* \), are defined in (2.9).

**Theorem 6.2.** Let \((\Delta, T_c, A_c, \nu_c)\) be the Cassaigne–Selmer algorithm, and let \( \varphi \) be the substitutive realization defined in (6.2). Then \((\Delta, T_c, \nu_c) \xrightarrow{\varphi} \{x \in \Delta \} \) and for \( \nu_c \)-almost all \( x \in \Delta \) the following assertions hold.

(i) The shift \( X_\varphi(x) \) is a natural coding of the toral translation \( R_x \) w.r.t. the natural partition \( \{R_x(i) : i \in A\} \).
The $S$-adic dynamical system $(X_\varphi(x), \Sigma) \cong (T^2, R_{x})$ has purely discrete spectrum.

The set $-R'_{\varphi}(w)$ is a bounded remainder set for $R_{x}$ for each $w \in A^*$.

According to [CLL17], each sequence in $X_\varphi(x)$ has factor complexity $2n + 1$. Thus Theorem 6.2 has the following consequence (observe Remark 3.4 and the fact that $\nu_c$ is equivalent to the Lebesgue measure).

Corollary 6.3. For (Lebesgue) almost all $t \in T^2$, there exists a minimal subshift $X \subset \{1, 2, 3\}^N$ with factor complexity $2n + 1$ and language balanced on factors such that $(X, \Sigma)$ is a natural coding of the toral translation $R_{t}$.

This result is optimal in the sense that according to [BB13] we cannot reach a smaller factor complexity for a natural coding of a two-dimensional translation. The asserted balance on factors means that all $F_{i0} \cap R_{t}^{-1}F_{i1} \cap \cdots \cap R_{t}^{-n}F_{in}$ are bounded remainder sets of $R_{t}$, with the notation of Theorem 6.2. We mention that the dimension group of $X$ can be completely described: it is isomorphic to $\left(\mathbb{Z}^3, \{x \in \mathbb{Z}^3 : \langle x, u \rangle > 0\} \cup \{0\}, 1\right)$, where $u$ stands for the associated normalized generalized right eigenvector; see [BCBD+19] for more on this topic. All this extends many properties of Sturmian sequences to sequences on 3-letter alphabets.

The Selmer algorithm also exists in higher dimensions (see e.g. [BFK15, BFK19]). However, to be able to extend the previous results to higher dimensions, two problems occur: firstly, one has to find a suitable substitutive realization leading to sequences of factor complexity $(d-1)n + 1$; secondly, as mentioned above the second Lyapunov exponent seems to be negative only for $d \leq 4$ [BST20].

6.2. The Arnoux–Rauzy algorithm. In this section we apply our results to the Arnoux–Rauzy algorithm in arbitrary dimension $d \geq 3$. As for the Cassaigne–Selmer algorithm (with $d = 3$), the Arnoux–Rauzy algorithm generates symbolic dynamical systems that have factor complexity $(d-1)n + 1$ and belong to the family of dendric subshifts.

Define the set of Arnoux–Rauzy substitutions over the alphabet $A = \{1, \ldots, d\}$ by

$$(6.4) \quad \alpha_i : i \mapsto i, \quad j \mapsto ij \quad \text{for} \quad j \in A \setminus \{i\} \quad (i \in A).$$

Let

$$\Delta(i) = \left\{ (x_1, \ldots, x_d) : x_i \geq \sum_{j \neq i} x_j \right\}.$$ 

Using the transposed incidence matrices of $\alpha_i$, we define the matrix valued function

$$A_{AR} : \bigcup_{i \in A} \Delta(i) \to GL(3, \mathbb{Z}), \quad x \mapsto ^tM_{\alpha_i}, \quad \text{if} \quad x \in \Delta(i),$$

which gives that

$$T_{AR}(x_1, \ldots, x_d) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_i - \sum_{j \neq i} x_j}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_d}{x_i}\right) \quad \text{if} \quad x \in \Delta(i).$$

We have $T_{AR}(\Delta(i)) = \{x \in [0, 1]^d : \|x\|_1 = 1\}$ for all $i \in A$, thus the image of $T_{AR}$ need not be contained in $\bigcup_{i \in A} \Delta(i)$. For this reason we have to restrict the domain of $T_{AR}$ to the $d$-dimensional Rauzy simplex, which is defined by

$$\Delta_{AR} = \left\{ x \in [0, 1]^d : \|x\|_1 = 1 \text{ and } T_{AR}^n(x) \in \bigcup_{i \in A} \Delta(i) \text{ for all } n \in \mathbb{N} \right\}.$$ 

The Rauzy simplex is defined in a way that $T_{AR}(\Delta_{AR}) = \Delta_{AR}$. The algorithm $(\Delta_{AR}, T_{AR}, A_{AR})$ is called Arnoux–Rauzy algorithm and goes back to [AR91]. The Rauzy simplex has zero Lebesgue measure by [AS13, Section 7]. We consider $T_{AR}$-invariant probability measures $\nu$ of $(\Delta_{AR}, T_{AR})$ satisfying $\nu \ll T \ll \nu$ (the latter condition is satisfied for instance for Borel probability measures $\nu$ w.r.t. the subspace topology on $\Delta_{AR}$; see e.g. [AHST16]). Clearly, the map $\varphi$ defined by $\varphi(x) = \alpha_j$ when $x \in \Delta(j)$ is a faithful substitution selection. We have $T_{AR}(\Delta_{AR}(i)) = \Delta_{AR}$, thus the algorithm satisfies the finite range property and each $x \in \Delta$ has positive range (in the sense of Definition 2.3).

The associated substitutive realization $\varphi$ thus satisfies $\varphi(\Delta_{AR}) = \{\alpha_1, \ldots, \alpha_d\}^N$ (up to a set of
measure zero). By [AD19], we know that the Lyapunov exponents of the Arnoux–Rauzy algorithm satisfy $\vartheta_1(A_{\AR}) > 0 > \vartheta_2(A_{\AR})$ for any ergodic invariant measure $\nu$ with support $\Delta_{\AR}$.

By induction on $d$, we can show that

$$\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_d = \tilde{\alpha}^d, \quad \text{with } \tilde{\alpha}(i) = 1(i+1) \text{ for } 1 \leq i < d, \quad \tilde{\alpha}(d) = 1.$$ 

The substitution $\tilde{\alpha}$ is the $d$-bonacci substitution; the characteristic polynomial of the incidence matrix $M_{\tilde{\alpha}}$ of $\tilde{\alpha}$ is $x^d - x^{d-1} - \cdots - x - 1$. Thus the dominant right eigenvector $x \in \Delta_{\AR}$ of $M_{\tilde{\alpha}}$ is a periodic Pisot point. It has, like all points of $\Delta_{\AR}$, positive range. Also, it is well known that $(X_{\tilde{\alpha}}, \Sigma)$ has purely discrete spectrum; see [FS02, BS05, Bar10]. Moreover, $\tilde{\alpha}$ is clearly proper. Thus, combining Theorem 3.1 with Theorem 3.8 and using the result on factor complexity from [AR91, RZ00] we obtain the following result (parts of which were proved for $d = 3$ in [BST19]).

Recall that $X' = \pi'(x)$ for the projection $\pi'$ defined in (2.8). The according projections of the subtiles, $R'_{\sigma}(w), w \in A^\ast$, are defined in (2.9).

**Theorem 6.4.** Let $(\Delta_{\AR}, T_{\AR}, A_{\AR}, \nu)$ be the Arnoux–Rauzy algorithm for $d \geq 2$, where $\nu$ is an ergodic invariant probability measure with support $\Delta_{\AR}$, and let $\varphi$ be as above. Then we have $(\Delta_{\AR}, T_{\AR}, \nu) \overset{\mathcal{E}}{\cong} ((\alpha_1, \ldots, \alpha_d)^N, \Sigma, \nu \circ \varphi^{-1})$ and for $\nu$-almost all $x \in \Delta_{\AR}$ the following assertions hold:

(i) The shift $X_{\varphi(x)}$ is a natural coding of the toral translation $R_{X'}$, w.r.t. the natural partition $\{ -R'_{\sigma}(i) : i \in A \}$.

(ii) The $S$-adic dynamical system $(X_{\varphi(x)}, \Sigma) \cong (T^2, R_{X'})$ has purely discrete spectrum.

(iii) The set $-R'_{\sigma}(w)$ is a bounded remainder set for $R_{X'}$ for each $w \in A^\ast$.

(iv) The shift $X_{\varphi(x)}$ has factor complexity $(d - 1)n + 1$ and is balanced on factors.

Note that Arnoux–Rauzy shifts are also dendric and their dimension group has a similar description as the one given in the previous section for the Cassaigne–Selmer shifts (see [BCBD+19]).

### 6.3. The Jacobi–Perron algorithm

One of the most famous multidimensional continued fraction algorithms is the Jacobi–Perron algorithm; see e.g. [Sch00, Chapter 4] or [Lag93, Section 2]. We want to apply our theory to the case $d = 3$. In this case the Jacobi–Perron algorithm is defined on the set $\Delta = \{(x_1, x_2, x_3) \in \mathbb{R}_+^3 : x_1 + x_2 + x_3 = 1, x_1 \leq x_3, x_2 \leq x_3\}. Let $L = \{(a, b) \in \mathbb{N}^2 : 0 \leq a \leq b, b \neq 0\}$ and for $(a, b) \in L$ define the matrices

$$J_{a,b} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & a & b \end{pmatrix}$$

and the sets $\Delta_{a,b} = \{(x_1, x_2, x_3) \in \Delta : ax_1 \leq x_2 < (a + 1)x_1 	ext{ and } bx_1 \leq x_3 < (b + 1)x_1\}$. Then $U_{\varphi} = \{\Delta_{a,b} : (a, b) \in L\}$ forms a partition of $\Delta$. We can thus define the matrix valued function

$$A_{\varphi} : \Delta \rightarrow GL(3, \mathbb{Z}), \quad x \mapsto J_{a,b} \quad \text{if } x \in \Delta_{a,b}.$$ 

This function is used to define the piecewise linear function $T_{\varphi}$ according to (2.1) which yields

$$T_{\varphi}(x_1, x_2, x_3) = \begin{pmatrix} x_2 - ax_1 \\ 1 - (a+b)x_1 \\ 1 - (a+b)x_1 \\ 1 - (a+b)x_1 \end{pmatrix} \quad \text{if } x \in \Delta(a,b).$$

The algorithm $(\Delta, T_{\varphi}, A_{\varphi})$ is called (2-dimensional) Jacobi–Perron algorithm and goes back to Jacobi’s posthumously published work [LJ68]. Note that, contrary to the Cassaigne–Selmer algorithm, this algorithm is multiplicative (its linear cocycle $A_{\varphi}$ produces infinitely many different matrices). It is known from [Sch90] that the invariant measure $\nu_{\varphi}$ of $T_{\varphi}$ is equivalent to the Lebesgue measure on $\Delta$ and, hence, has full support and satisfies $\nu_{\varphi} \circ T \ll \nu_{\varphi}$ (however, there is no known simple expression for the density of $\nu_{\varphi}$; for more on this subject, see [Bro96]). Let

$$B((a_0, b_0), \ldots, (a_{n-1}, b_{n-1}))$$

$$= \{x \in \Delta : (A_{\varphi}(x), A_{\varphi}(Tx), \ldots, A_{\varphi}(T^{n-1}x)) = (J_{a_0,b_0}, \ldots, J_{a_{n-1},b_{n-1}}) \}$$

$$= \bigcap_{k=0}^{n-1} T_{\varphi}^{-k}(\Delta_{a_k,b_k})$$


for \((a_0, b_0), \ldots, (a_{n-1}, b_{n-1}) \in L\). The cylinder \(B((a_0, b_0), \ldots, (a_{n-1}, b_{n-1}))\) is nonempty if and only if the pairs \((a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\) satisfy the following admissibility condition (see [Sch00 Section 4.1]):

\[
\Delta = \{ a_n \leq b_n, b_n \neq 0, \text{ and if } a_n = b_n \text{ then } a_{n+1} = 0, \}
\]

which implies that the Jacobi–Perron algorithm satisfies the finite range property. In other words, this admissibility condition is a sofic condition that can be recognized by a finite graph. It is proved in [Lag93 p. 322] that the cocycle \(A_{2p}\) is log-integrable (this is nontrivial in this case because \(A_{2p}\) has infinite range). Thus, because \(\nu_{2p}\) has full support, each \(x \in \Delta\) has positive range. The fact that \(A_{2p}\) satisfies the Pisot condition is proved in [Sch00 Chapter 16]. Following [Ber16] we define the \textit{Jacobi–Perron substitutions}

\[
\iota_{a,b} : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 12^a3^b \\
\end{cases} \quad (a,b) \in L
\]

on the alphabet \(A = \{1, 2, 3\}\) and set \(S = \{\iota_{a,b} : (a,b) \in L\}\). It is easy to see that \(\iota_{a,b}\) is the incidence matrix of \(\iota_{a,b}\) for each pair \((a,b) \in L\). Define the substitution selection \(\varphi\) on \(\Delta\) by setting \(\varphi(x) = \iota_{a,b}\) if \(x \in \Delta_{a,b}\). The associated faithful substitutive realization \(\varphi\) yields \((\Delta_{2p}, T_{2p}, \nu_{2p}) \cong (D_{2p}, \Sigma_{2p}, \nu_{2p} \circ \varphi^{-1})\), where \(D_{2p}\) is the set of all directive sequences whose sequence of incidence matrices \((M_{(a_n,b_n)})\) satisfies the admissibility condition (6.5). This isomorphy is due to the fact that the set \(\{\Delta_{a,b} : (a,b) \in L\}\) is a generating (Markov) partition for \(T_{2p}\), which yields weak convergence (see [Lag93 Section 5]).

To apply Theorem 3.1 it remains to establish that there exists a periodic Pisot point \(x \in \Delta\) for which \(\varphi(x)\) has purely discrete spectrum. This assertion is easily checked. Indeed, \(\sigma = \iota_{0,1}\) is a unit Pisot substitution (see also [DFPLR04] for relations between the Jacobi–Perron algorithm and Pisot numbers) for which \((\sigma)^\infty \in D_{2p}\) is admissible. Moreover, using for instance the balanced pair algorithm (as we did in Lemma 6.1 for another substitution) one easily checks that \((X, \Sigma, \sigma)\) has purely discrete spectrum. This implies that the right eigenvector \(x \in \Delta\) of the incidence matrix of \(\sigma\) is a periodic Pisot point with \(\varphi(x)\) having purely discrete spectrum. Thus, all the conditions of Theorem 3.1 are satisfied and, because of right properness of all directive sequences, we arrive at the following result. Recall again that \(x' = \pi'(x)\) for the projection \(\pi'\) defined in (2.8). The according projections of the subtiles, \(R'_w(x), w \in A^*\), are defined in (2.9).

**Theorem 6.5.** Let \((\Delta, T_{2p}, A_{2p}, \nu_{2p})\) be the 2-dimensional Jacobi–Perron algorithm. Then we have \((\Delta, T_{2p}, \nu_{2p}) \cong (D_{2p}, \Sigma, \nu_{2p} \circ \varphi^{-1})\) and for \(\nu_{2p}\)-a.a. \(x \in \Delta\) the following assertions hold.

(i) The shift \(X_{\varphi(x)}\) is a natural coding of the toral translation \(R_x\) w.r.t. the natural partition \(\{-R'_{\varphi}(i) : i \in A\}\).

(ii) The \(S\)-adic dynamical system \((X_{\varphi(x)}, \Sigma) \cong (T^2, R_x)\) has purely discrete spectrum.

(iii) The set \(-R'_{\varphi}(w)\) is a bounded remainder set for \(R_w\) for each \(w \in A^*\).

(iv) The shift \(X_{\varphi(x)}\) is balanced on factors.

6.4. The Brun algorithm. The case \(d = 3\) of the Brun algorithm is treated in [BST19]. Here we consider the unordered version of the Brun algorithm, as defined in [DHS13], with special emphasis on the case \(d = 4\). We start with the definition of the algorithm for arbitrary \(d \geq 3\). For this algorithm, we have \(\Delta = \{x \in [0,1]^d : \|x\|_1 = 1\}\), and the set of \textit{Brun substitutions} over \(A\) is defined by

\[
\beta_{ij} : j \mapsto ij, \quad k \mapsto k \text{ for } k \in A \setminus \{j\}
\]

(we emphasize that in [BF11] the authors deal with other substitutions related to this algorithm). Let

\[
\Delta(i,j) = \{(x_1, \ldots, x_d) : x_i \geq x_j \geq x_k \text{ for all } k \in A \setminus \{i,j\}\}
\]

Using the transposed incidence matrices of \(\beta_{ij}\), we define the matrix valued function

\[
A_b : \Delta \to \text{GL}(d, \mathbb{Z}), \quad x \mapsto 'M_{\beta_{ij}} \text{ if } x \in \Delta(i,j),
\]
which yields
\[ T_n(x_1, \ldots, x_d) = \left( \frac{x_1}{1 - x_j}, \ldots, \frac{x_{i-1}}{1 - x_j}, \frac{x_i - x_j}{1 - x_j}, \frac{x_{i+1}}{1 - x_j}, \ldots, \frac{x_d}{1 - x_j} \right) \] if \( x \in \Delta(i, j). \)

The algorithm \((\Delta, T_n, A_n)\) is called (unordered) Brun algorithm. It goes back to Bru74, Bru95. The faithful substitution selection corresponding to the substitutions in (6.7) is defined by \( \varphi(x) = \beta_{ij} \) if \( x \in \Delta(i, j) \), and by Definition 2.2 the map

\[ \varphi : \Delta \to \{(\beta_{ij}, i, j) : i \neq j \}^\mathbb{N} \]

with \( \varphi(x) = (\varphi(T^n x))_{n \in \mathbb{N}} \)
is a faithful substitutive realization of \((\Delta, T_n, A_n)\). As indicated in DHS13, the directive sequences \( \sigma = (\sigma_n) \) that are generated by this algorithm are characterized by the admissibility condition

\[ (\sigma_n, \sigma_{n+1}) \in \{(\beta_{ij}, \beta_{jk}) : i \in A, j \in A \setminus \{i\} \} \cup \{(\beta_{ij}, \beta_{jk}) : i \in A, j \in A \setminus \{i\}, k \in A \setminus \{j\} \} \]

for all \( n \in \mathbb{N} \).

This is again a sofic condition that can be recognized by a finite graph. Thus \( \varphi(\Delta) = D_n \) for a sofic shift \( D_n \) and the algorithm satisfies the finite range property. Moreover, since the invariant measure of the Brun algorithm \( \nu_n \) is a Borel probability measure equivalent to the Lebesgue measure (see e.g. AL18, Proposition 28), each \( x \in \Delta \) has positive range (see Definition 2.8). Moreover, as \( T_n \) maps open sets to open sets, we have \( \nu_n \circ T \ll \nu_n \).

Now we confine ourselves to the case \( d = 4 \). The linear cocycle \( A_n \) is log-integrable since \( A_n \) takes only 12 values. By [HK00, Har02], we know that \((\Delta, T_n, A_n, \nu_n)\) satisfies the Pisot condition (in HK00, Har02) an acceleration of Brun’s algorithm is considered; however, because this acceleration, which is in turn equivalent to the modified Jacobi–Perron algorithm, see Podl77, is a return map to a set of positive measure, the Pisot property is invariant under this acceleration). This implies that \( \{(\Delta(i, j) : i \neq j \} \) is a generating partition for \( D_n \) and that \( T_n \) is weakly convergent, hence, \((\Delta, T_n, \nu_n) \cong (D_n, \Sigma, \nu_n \circ \varphi^{-1}) \).

To apply Theorem 3.1 we have to find a periodic Pisot point \( x \in \Delta \) (see Definition 2.3) such that \( \varphi(x) \) has purely discrete spectrum. To this end, consider

\[ \tau = \beta_{12} \circ \beta_{23} \circ \beta_{34} \circ \beta_{41} : \begin{cases} 1 &\mapsto 12341 \\ 2 &\mapsto 12 \\ 3 &\mapsto 123 \\ 4 &\mapsto 1234 \end{cases} \]

and let \( x \in \Delta \) be the dominant right eigenvector of \( M_\tau \). Then \( \varphi(x) = (\beta_{12}, \beta_{23}, \beta_{34}, \beta_{41})^\infty \in D_n \) is an admissible sequence. Since \( M_\tau \) is a Pisot matrix we conclude that \( x \) is a periodic Pisot point which has positive range by the above considerations. Along the same lines as in Lemma 6.1 one can show that \( \varphi(x) \) has purely discrete spectrum. Combining Theorem 3.1 with Theorem 6.8 (and noting that the substitutions in (6.7) give rise to proper directive sequences with probability one (w.r.t. \( \nu_n \circ \varphi^{-1} \) under the admissibility condition (6.9) because \( \tau \) is left proper) we thus obtain the following result. Once again, recall that \( x' = \pi'(x) \) for the projection \( \pi' \) defined in (2.8). The according projections of the subtiles, \( R_\varphi(w), w \in A^* \), are defined in (2.9).

**Theorem 6.6.** Let \((\Delta, T_n, A_n, \nu_n)\) be the Brun algorithm with \( d = 4 \), and let \( \varphi \) be the substitutive realization defined in (6.8). Then \((\Delta, T_n, \nu_n) \cong (D_n, \Sigma, \nu_n \circ \varphi^{-1}) \) and for \( \nu_n \)-almost all \( x \in \Delta \) the following assertions hold.

(i) The shift \( X_{\varphi(x)} \) is a natural coding of the toral translation \( R_\varphi \) w.r.t. the natural partition \( \{ -R_\varphi(i) : i \in A \} \).

(ii) The \( S \)-adic dynamical system \((X_{\varphi(x)}, \Sigma) \cong (T^2, R_\varphi)\) has purely discrete spectrum.

(iii) The set \( -R_\varphi(w) \) is a bounded remainder set for \( R_\varphi \) for each \( w \in A^* \).

(iv) The shift \( X_{\varphi(x)} \) is balanced on factors.

Note that this result gives a natural coding for (Lebesgue) a.a. points of \( T^3 \) in terms of “Brun \( S \)-adic sequences” by Remark 3.4 and by recalling that the ergodic invariant measure \( \nu_n \) of the Brun algorithm is equivalent to Lebesgue measure.
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