The topology of sums in powers of an algebraic number

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(joint with Boris Solomyak)

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Background

Let $1 < \theta < 2$ be our parameter. Put

$$\Lambda_n(\theta) = \left\{ \sum_{k=0}^{n} a_k \theta^k \mid a_k \in \{-1, 0, 1\} \right\}$$

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Theorem (folklore)

If θ is transcendental, then 0 is a limit point of $\Lambda(\theta)$.

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$$D_n(\theta) = \left\{ \sum_{k=0}^{n} a_k \theta^k \mid a_k \in \{0, 1\} \right\}.$$

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Since $x - y \in \Lambda_n(\theta)$, we are done.

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Theorem (Erdős-Komornik, 1998)

If $\theta < \frac{1+\sqrt{5}}{2}$ and not Pisot, then $\Lambda(\theta)$ has a finite accumulation point.

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Theorem (S+Solomyak, 2009)

If θ is not Perron, then $\Lambda(\theta)$ is dense in $\mathbb{R}$.

Digression: $\ell(\theta), L(\theta)$

Put $D(\theta) = \bigcup_{n \geq 1} D_n(\theta)$, i.e., the set of all finite 0-1 sums in nonnegative powers of θ.

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Put

$$\ell(\theta) = \liminf_n (y_{n+1} - y_n),$$
$$L(\theta) = \limsup_n (y_{n+1} - y_n).$$

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Lemma (S+Solomyak)

$z_n(\lambda) \geq |\lambda|^{-n-1}$ *for all* $\lambda \in \mathbb{C}$ *with* $\frac{1}{2} < |\lambda| < 1$.

With this lemma, the proof of the theorem is fairly easy: assume first that $1 < \theta < |\alpha|$, where α is a conjugate of θ.

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From this, one can deduce that

$$z_n(\theta) \geq C \cdot \min\{\theta^{mn}, 2^n\} \gg \theta^n. \quad \square$$

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Hence $|\theta_i \theta_j| = \theta^2$, and $\max\{|\theta_i|, |\theta_j|\} \geq \theta$, i.e., θ is not Perron. Therefore, $\ell(\theta) = 0$.

Bonus: In all these results, if $\theta < \sqrt{2}$, then $L(\theta) = 0$. This is because $\ell(\theta^2) = 0$ implies $L(\theta) = 0$ (Erdős-Komornik).

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