ON THE ERGODIC THEORY OF TANAKA–ITO TYPE α-CONTINUED FRACTIONS

HITOSHI NAKADA AND WOLFGANG STEINER

ABSTRACT. We show the ergodicity of Tanaka–Ito type α-continued fraction maps and construct their natural extensions. We also discuss the relation between entropy and the size of the natural extension domain.

1. Introduction and main results

In 1981, two types of α-continued fraction maps were defined by [6, 12]: For $\alpha \in [0, 1]$,

- the first author considered in [6] the map

$$T_\alpha(x) = \left|\frac{1}{x}\right| - \left\lfloor \left|\frac{1}{x}\right| + 1 - \alpha \right\rfloor, \tag{1}$$

- S. Tanaka and S. Ito [12] studied

$$T_\alpha(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor, \tag{2}$$

where $0 \neq x \in [\alpha - 1, \alpha)$ and $T_\alpha(0) = 0$.

The main aim of these papers was the derivation of the density functions of the absolutely continuous invariant measure by constructing the natural extension of a 1-dimensional continued fraction map as a planar map. For the map (2), this was successful only for $\frac{1}{2} \le \alpha \le \frac{\sqrt{5}-1}{2}$, though for $\frac{1}{2} \le \alpha \le 1$ in case (1). In [5], this was extended to all $\alpha \in (0, 1]$ in case (1). Here, we show that this method also works for $\alpha > \frac{\sqrt{5}-1}{2}$ in case (2). In the sequel, the map T_α denotes the second type in the above, except where specified otherwise. Then T_α is symmetric w.r.t. $\frac{1}{2}$. Therefore, we can assume that $\frac{1}{2} \le \alpha \le 1$, and it is easy to extend our results to $0 \le \alpha \le \frac{1}{2}$. Since there were no proofs of the existence of the absolutely continuous invariant measure for $\alpha > \frac{\sqrt{5}-1}{2}$ and for the ergodicity w.r.t. this measure for $\alpha > \frac{1}{2}$, we give these proofs for all α in $[\frac{1}{2}, 1]$.

In §2, we give some basic properties of T_α, in particular that the set of full cylinders generates the Borel algebra (Proposition 1). In §3, we show the existence of the absolutely continuous invariant probability measure μ_α for T_α by the classical method (see [8]) based on Propositions 1 and 2. We note that Rychlik's result [10] implies the existence of the absolutely continuous invariant measure; however, Propositions 1 and 2 show both the existence of the absolutely continuous invariant measure and its ergodicity altogether.

Theorem 1. *There is an ergodic invariant probability measure μ_α for the dynamical system $([\alpha - 1, \alpha), T_\alpha)$ which is equivalent to the Lebesgue measure.*

Recall that an ergodic measure preserving map $\hat{S}$ is the natural extension of an ergodic measure preserving map S if $\hat{S}$ is invertible and any invertible extension of S is an extension of $\hat{S}$. We give the natural extension of T_α as a planar map

$$\mathcal{T}_\alpha(x, y) = \left(\frac{1}{x} - \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor, \frac{1}{y + \lfloor \frac{1}{x} + 1 - \alpha \rfloor}\right),$$

Date: September 27, 2020.

2010 *Mathematics Subject Classification.* 11K50, 11J70.

This work was supported by the Agence Nationale de la Recherche, project CODYS (ANR-18-CE40-0007).

with $\mathcal{T}_\alpha(0,y) = (0,0)$, and the natural extension domain

$$\Omega_\alpha = \bigcup_{n\geq 0} \overline{\mathcal{T}_\alpha^n\big([\alpha{-}1,\alpha) \times \{0\}\big)}.$$

Then $\frac{\mathrm{d}x\,\mathrm{d}y}{(1+xy)^2}$ gives an absolutely continuous invariant measure $\hat\mu$ of $(\Omega_\alpha, \mathcal{T}_\alpha)$, and we denote by $\hat\mu_\alpha$ the corresponding probability measure. The main problem here is to show that Ω_α has positive Lebesgue measure. We show the following theorem, where the density function of μ_α is given by

$$\frac{1}{\hat\mu(\Omega_\alpha)} \int_{y:\,(x,y)\in\Omega_\alpha} \frac{1}{(1+xy)^2}\,\mathrm{d}y.$$

Theorem 2. *For $\alpha \in (g,1]$, Ω_α has positive Lebesgue measure and thus $(\Omega_\alpha, \mathcal{T}_\alpha, \hat\mu_\alpha)$ is a natural extension of $([\alpha-1,\alpha), T_\alpha, \mu_\alpha)$.*

We note that the existence of μ_α follows directly from the result in §4 but we need the ergodicity proved in Theorem 1 for the concept of the natural extension.

In §5, we give a selfcontained proof that Rokhlin's formula

$$h(T_\alpha) = \int_{[\alpha-1,\alpha)} -2\log|x|\,\mathrm{d}\mu_\alpha$$

holds for T_α (Proposition 6); we refer to [13] for the general case of one dimensional maps. In this paper, we use Propositions 1 and 5 with the Shannon–McMillan–Breiman–Chung theorem; see [2, 4]. Moreover, we show that

$$-2\lim_{n\to\infty}\frac{1}{n}\log|q_{\alpha,n}(x)| = h(T_\alpha)$$

for almost all $x \in [\alpha-1,\alpha)$, where $q_{\alpha,n}(x)$ is the denominator of the n-th convergent of x given by T_α; note that Tanaka and Ito [12] mentioned this fact for $\alpha = 1/2$.

The behavior of the entropy as a function of α will be discussed in the forthcoming paper [3]. In the case of T_α defined by (1), it was shown in [5, Theorem 2] that $h(T_\alpha)\hat\mu(\Omega_\alpha) = \pi^2/6$ for all $\alpha \in (0,1]$, where $\hat\mu$ is the invariant measure of the natural extension given by $\frac{\mathrm{d}x\,\mathrm{d}y}{(1+xy)^2}$ (without normalization). For T_α defined by (2), this does not hold: for $\alpha = 1$, the maps defined by (1) and (2) are equal and we have thus $h(T_1)\hat\mu(\Omega_1) = \pi^2/6$ in both cases; for $\alpha = 1/2$, the maps defined by (1) and (2) produce the same continued fraction expansions and have thus the same entropy, but $\Omega_{1/2}$ for (2) is equal to $\Omega_{1/2} \cup (-\Omega_{1/2})$ for (1), hence we have $h(T_{1/2})\hat\mu(\Omega_{1/2}) = \pi^2/3$ in case (2). For case (2), we have the following.

Theorem 3. *The function*

$$\alpha \mapsto h(T_\alpha)\,\hat\mu(\Omega_\alpha)$$

is a monotonically decreasing function of $\alpha \in [\frac{1}{2}, 1]$.

2. Some definitions and notation

We start with basic definitions. Since we discuss a fixed α, we omit α from the index. We define

$$a_k(x) = \left\lfloor \frac{1}{T_\alpha^{k-1}(x)} + 1 - \alpha \right\rfloor, \quad k \geq 1,$$

when $T_\alpha^{k-1}(x) \neq 0$. We put $a_k(x) = 0$ if $T_\alpha^{k-1}(x) = 0$. Then we have

$$x = \frac{1|}{|a_1(x)} + \frac{1|}{|a_2(x)} + \cdots + \frac{1|}{|a_n(x)} + \cdots,$$

and the right hand side terminates at some positive integer n if and only if x is a rational number. As usual we put

$$\begin{pmatrix} p_{n-1}(x) & p_n(x) \\ q_{n-1}(x) & q_n(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1(x) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2(x) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n(x) \end{pmatrix}, \tag{3}$$

when $a_n(x) \neq 0$. It is well-known that

$$\frac{p_n(x)}{q_n(x)} = \frac{1}{|\,a_1(x)} + \frac{1}{|\,a_2(x)} + \cdots + \frac{1}{|\,a_n(x)}$$

and we call $\frac{p_n(x)}{q_n(x)}$ the n-th convergent of the α-continued fraction expansion of x. It is easy to see that $T_\alpha^n(x)$ is a linear fractional transformation defined by the inverse of (3), where (3) is the same matrix for all x in the same cylinder set of length n. Then we see that

$$p_n(x) = a_n(x)p_{n-1}(x) + p_{n-2}(x), \qquad q_n(x) = a_n(x)q_{n-1}(x) + q_{n-2}(x), \tag{4}$$

$$x = \frac{p_{n-1}(x)T_\alpha^n(x) + p_n(x)}{q_{n-1}(x)T_\alpha^n(x) + q_n(x)}, \tag{5}$$

and

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| = \left| \frac{T_\alpha^n(x)}{q_n(x) \cdot (q_{n-1}(x)T_\alpha^n(x) + q_n(x))} \right|;$$

here we note that the determinants of all matrices in (3) are ± 1.

In general we use the notation $\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}$ without x when $a_1, \ldots, a_n$ is given without x. For a given sequence of non-zero integers, $a_1, a_2, \ldots, a_n$, we denote by $\langle a_1, a_2, \ldots, a_n \rangle$ the associated cylinder set, i.e.,

$$\langle a_1, a_2, \ldots, a_n \rangle = \{x \in [\alpha - 1, \alpha) : a_1(x) = a_1, \ldots, a_n(x) = a_n\}.$$

A sequence $a_1, a_2, \ldots, a_n$ is said to be admissible if the associated cylinder set has an inner point; here we note that any cylinder set is an interval. A cylinder set is said to be full if

$$T_\alpha^n(\langle a_1, a_2, \ldots, a_n \rangle) = [\alpha - 1, \alpha).$$

Because of the definition (3) we see that

$$\frac{q_{n-1}(x)}{q_n(x)} = \frac{1}{|\,a_n(x)} + \frac{1}{|\,a_{n-1}(x)} + \cdots + \frac{1}{|\,a_1(x)}.$$

We set

$$g = \frac{\sqrt{5} - 1}{2}.$$

Lemma 1. *For any cylinder set $\langle a_1, a_2, \ldots, a_n \rangle$, we have*

$$\lambda(\langle a_1, a_2, \ldots, a_n \rangle) \leq g^{-2(n-1)}/2,$$

where λ denotes the Lebesgue measure.

Proof. For $|x| \leq g$, we have $|T_\alpha'(x)| = \frac{1}{x^2} \geq \frac{1}{g^2}$. For $x \geq g$, we have $\left|(T_\alpha^2)'(x)\right| = \frac{1}{(x\,T_\alpha(x))^2} \geq \frac{1}{g^4}$. Since the cylinder of length 0 has measure 1 and each cylinder of length 1 has measure at most 1/2, this shows the assertion of this lemma. □

Proposition 1. *The set of full cylinders generates the Borel algebra of $[\alpha - 1, \alpha)$.*

Proof. Fix $n \geq 1$. If

$$T_\alpha^k(\langle a_1, a_2, \ldots, a_k \rangle) \neq [\alpha - 1, \alpha) \quad \text{for all } 1 \leq k \leq n, \tag{6}$$

then $(a_1, a_2, \ldots, a_n)$ is a concatenation of sequences of the form $(a_1(\alpha), a_2(\alpha), \ldots, a_j(\alpha))$ or $(a_1(\alpha-1), a_2(\alpha-1), \ldots, a_j(\alpha-1))$, $1 \leq j \leq n$. This implies that the number of admissible sequences satisfying (6) is at most 2^n. We put

$$B_n = \bigcup_{(a_1,\ldots,a_n) \text{ with } (6)} \langle a_1, a_2, \ldots, a_n \rangle \quad \text{and} \quad B = \bigcap_{n=1}^{\infty} B_n.$$

From Lemma 1, we have

$$\lambda(B_n) \leq (2g^2)^{-n+1}/4, \tag{7}$$

and then $\lambda(B)=0$ since $2g^2<1$. Then we see that

$$\lambda\bigg(\bigcup_{n=1}^{\infty} T_\alpha^{-n}(B)\bigg)=0.$$

This implies that for a.e. $x\in[\alpha-1,\alpha)$ we have $T_\alpha^n(x)\notin B$ for all $n\geq 1$, hence there exists a sequence $n_1<n_2<\cdots$ (depending on x) such that $T_\alpha^{n_k}(\langle a_1(x),a_2(x),\ldots,a_{n_k}(x)\rangle)$ is a full cylinder for any $k\geq 1$. This shows the assertion of this proposition. □

The following lemma is essential in this paper.

Lemma 2. *Let $(a_1,\ldots,a_n)$ be an admissible sequence. If $\alpha\in[\frac{1}{2},g]$, then we have $|q_n|>|q_{n-1}|$. If $\alpha\in(g,1]$, then we have $-\frac{1}{2}<\frac{q_{n-1}}{q_n}<2$, with $\frac{q_{n-1}}{q_n}\geq 1$ only if $a_n=1$.*

Proof. We proceed by induction on n. Since $q_0=1$, $q_1=a_1$, $|a_n|\geq 2$ when $\alpha\in[\frac{1}{2},g]$, $a_n\geq 1$ or $a_n\leq -3$ when $\alpha\in(g,1]$, the statements hold for $n=1$. Suppose that they hold for $n-1$ and recall that $\frac{q_n}{q_{n-1}}=a_n+\frac{q_{n-2}}{q_{n-1}}$ by (4).

If $\alpha\in[\frac{1}{2},g]$, then $|a_n|\geq 2$ gives that $|\frac{q_n}{q_{n-1}}|>1$; see also [12, Remark 2.1].

Let now $\alpha\in(g,1]$. If $a_n<0$, then we have $a_n\leq -3$ and $a_{n-1}\neq 1$, thus $\frac{q_n}{q_{n-1}}<-2$. If $a_n>0$, then we have $\frac{q_n}{q_{n-1}}>\frac{3}{2}$ when $a_n\geq 2$, and $\frac{q_n}{q_{n-1}}>\frac{1}{2}$ when $a_n=1$. □

We now define the jump transformation of T_α, which we will use to show the existence of the absolutely continuous invariant measure. From Proposition 1, for a.e. $x\in[\alpha-1,\alpha)$ there exists $n\geq 1$ such that $T_\alpha^n\langle a_1(x),\ldots,a_n(x)\rangle=[\alpha-1,\alpha)$. We denote the minimum of those n by $N(x)$. If there is no such n, then we put $N(x)=0$. The jump transformation of T_α is

$$\mathring{T}_\alpha:\ [\alpha-1,\alpha)\to[\alpha-1,\alpha),\quad x\mapsto T_\alpha^{N(x)}(x).$$

Note that $y\in\langle a_1(x),\ldots,a_n(x)\rangle$ means that $a_j(y)=a_j(x)$ for all $1\leq j\leq n$. Hence we see that $N(y)=N(x)$. Thus there exists a countable partition $\mathcal{J}=\{J_k:k\geq 1\}$ of $[\alpha-1,\alpha)$ such that each J_k is a cylinder set of length N_k with $\mathring{T}_\alpha(x)=T_\alpha^{N_k}(x)$ for $x\in J_k$ and $T_\alpha^j J_k\neq[\alpha-1,\alpha)$, $1\leq j<N_k$, $T_\alpha^{N_k}J_k=[\alpha-1,\alpha)$. Obviously, $\mathring{T}_\alpha$ is a piecewise linear fractional map of the form

$$\frac{q_{N_k}x-p_{N_k}}{-q_{N_k-1}x+p_{N_k-1}}$$

for $x\in J_k$, and it is bijective from J_k to $[\alpha-1,\alpha)$.

3. Existence of the absolutely continuous invariant measure and ergodicity

We first prove the following.

Proposition 2. *For any admissible sequence $a_1,\ldots,a_n$,*

$$\frac{1}{9q_n^2}<\left|\psi'_{a_1,\ldots,a_n}(y)\right|<\frac{1}{g^4q_n^2}$$

holds for all $y\in T_\alpha^n\langle a_1,\ldots,a_n\rangle$, where $\psi_{a_1,\ldots,a_n}$ is the local inverse of T_α^n restricted to $\langle a_1,\ldots,a_n\rangle$.

Proof. From (5), we see that

$$\psi_{a_1,\ldots,a_n}(y)=\frac{p_{n-1}\,y+p_n}{q_{n-1}\,y+q_n}$$

and then

$$\left|\psi'_{a_1,\ldots,a_n}(y)\right|=\frac{1}{(q_{n-1}y+q_n)^2}$$

for $y\in T_\alpha^n\langle a_1,\ldots,a_n\rangle$. If $\alpha\in[1/2,g]$, then we have $y\in[-1/2,g)$ and thus

$$g^2<1+y\frac{q_{n-1}}{q_n}<1+g$$

by Lemma 2. If $\alpha\in(g,1]$, then we have $y\in(-g^2,1]$, with $y>0$ if $a_n=1$, thus

$$\frac{1}{2}<1+y\frac{q_{n-1}}{q_n}<3.$$ □

Proposition 3. *There exists an invariant probability measure ν for $\overset{\circ}{T}_\alpha$ that is equivalent to the Lebesgue measure.*

Proof. For any cylinder set J of length n such that $T_\alpha^n J = [\alpha-1,\alpha)$, the size of J is

$$\left| \frac{p_{n-1}(\alpha)T_\alpha^n(\alpha)+p_n(\alpha)}{q_{n-1}(\alpha)T_\alpha^n(\alpha)+q_n(\alpha)} - \frac{p_{n-1}(\alpha-1)T_\alpha^n(\alpha-1)+p_n(\alpha-1)}{q_{n-1}(\alpha-1)T_\alpha^n(\alpha-1)+q_n(\alpha-1)} \right|$$

From Lemma 2 and Proposition 2, this is $\sim q_n^2$ since the condition on J implies that $|a_n| \geq 2$. Then there exists a constant $C_1 > 1$ such that for any measurable set $A \subset [\alpha-1,\alpha)$

$$C_1^{-1}\lambda(A) < \lambda\big(\overset{\circ}{T}_\alpha{}^{-m}(A)\big) < C_1\lambda(A).$$

By the Dunford–Miller theorem we have that

$$\nu(A) = \lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \lambda\big(\overset{\circ}{T}_\alpha{}^{-m}(A)\big)$$

exists for any measurable subset A. It follows from the above estimate that

$$C_1^{-1}\lambda(A) \leq \nu(A) \leq C_1\lambda(A),$$

hence ν is a finite measure which is equivalent to Lebesgue measure. □

Proposition 4. *The map $\overset{\circ}{T}_\alpha$ is ergodic w.r.t. the Lebesgue measure.*

Proof. Suppose that A is an invariant set of $\overset{\circ}{T}_\alpha$ with $\lambda(A) > 0$. For any $\varepsilon > 0$, there exists a full-cylinder set J of length n such that

$$\frac{\lambda(A\cap J)}{\lambda(J)} > 1-\varepsilon.$$

Then we see that there exists a constant $C_2 > 0$ such that

$$\overset{\circ}{T}{}_\alpha^n(A\cap J) > 1 - C_2\varepsilon \qquad \text{and} \qquad A \supset \overset{\circ}{T}{}_\alpha^n(A\cap J).$$

This shows $\lambda(A) = 1$. □

We can now prove the ergodicity of T_α.

Proof of Theorem 1. We refer to [11] for determining the absolutely continuous invariant measure for T_α from that of $\overset{\circ}{T}_\alpha$ and the fact that the ergodicity of $\overset{\circ}{T}_\alpha$ implies that of T_α. Indeed we put

$$\mu_0(A) = \sum_{n=0}^{\infty} \nu(T_\alpha^{-n}A \cap B_n) \tag{8}$$

which is an invariant measere for T_α. Then the porperty

$$\sum_{n=1}^{\infty} \lambda(B_n) < \infty,$$

see (7), ensures the finiteness of the absolutely continuous invariant measure. Hence we have the invariant probability measure μ by normalization of μ_0. Since μ is equivalent to ν, it is equivalent to the Lebesgue measure λ. Thus from Proposition 4, it is easy to see that T_α is ergodic w.r.t. μ. □

Corollary 1. *The map T_α is exact w.r.t. μ, i.e. the σ-algebra $\cap_{n=0}^{\infty} T_\alpha^{-n}\mathfrak{B}$ consists of sets of μ-measures 0 and 1.*

Proof. For any interval $I \subset [\alpha-1,\alpha)$, we have

$$\lim_{n\to\infty} T_\alpha^n(I) = [\alpha-1,\alpha).$$

Indeed, from the proof of Proposition 1, we can choose an innner point x of I so that $\langle a_1(x), a_2(x), \ldots, a_n(x)\rangle$ is a full cylinder. This shows the assertion of this corollary; see [9]. □

Remark. It is possible to show that T_α is weak Bernoulli following the idea of the proof by R. Bowen [1], and the proof is similar to the case of other α-continued fraction maps; see [7].

4. Planar natural extension

We consider the planar natural extension map

$$\mathcal{T}_\alpha : (x,y) \mapsto \left(\frac{1}{x} - \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor, \frac{1}{y + \lfloor \frac{1}{x} + 1 - \alpha \rfloor}\right),$$

with $\mathcal{T}_\alpha(0,y) = (0,0)$, and the natural extension domain

$$\Omega_\alpha = \bigcup_{n\ge 0} \overline{\mathcal{T}_\alpha^n\big([\alpha{-}1,\alpha) \times \{0\}\big)}.$$

It is well known that $\Omega_1 = [0,1]^2$. It is easy to see that $\left(\Omega_\alpha, \mathcal{T}_\alpha, \frac{dx\,dy}{(1+xy)^2}\right)$ is a natural extension of T_α if Ω_α has positive (two-dimensional) Lebesgue measure; see [5, Theorem 1]. The invariance of the measure $\hat\mu$ given by $\mathrm{d}\hat\mu = \frac{\mathrm{d}x\,\mathrm{d}y}{(1+xy)^2}$ is proved in the same way as those in [6, 12].

The shape of Ω_α was determined by Tanaka and Ito [12] for $\alpha \in [1/2, g]$. In particular, we have

$$\Omega_g = [-g^2, g^2] \times [1-\sqrt{2}, \tfrac{1}{\sqrt{2}} - 1] \cup [-g^2, g] \times [\tfrac{1}{\sqrt{2}} - 1, 2 - \sqrt{2}]; \tag{9}$$

see Figure 1. The main purpose of this section is to prove that Ω_α has positive measure for $\alpha > g$. To this end, we show that Ω_α is contained in a certain polygon X_α, and then we relate Ω_α to Ω_g.

Lemma 3. *Let $\alpha \in (g,1)$ and $d = -a_1(\alpha-1)$. We have $\Omega_\alpha \subset X_\alpha$ with*

$$X_\alpha = \big[\alpha{-}1, T_\alpha(\alpha{-}1)\big] \times \big[\tfrac{1}{2-\sqrt{2}-d}, \tfrac{1}{1-\sqrt{2}-d}\big] \cup \big[\alpha{-}1, \alpha\big] \times \big[\tfrac{1}{1-\sqrt{2}-d}, 2{-}\sqrt{2}\big] \cup \big[\tfrac{1}{\alpha}{-}1, \alpha\big] \times \big[2{-}\sqrt{2}, \sqrt{2}\big].$$

Proof. We see that $\mathcal{T}_\alpha(X_\alpha) \subset X_\alpha$ by determining the images of rectangles

$$\begin{aligned}
\mathcal{T}_\alpha\Big(\big[\alpha-1, \tfrac{1}{\alpha-d-1}\big] \times \big[1-\sqrt{2}, 2-\sqrt{2}\big]\Big) &= \big[\alpha-1, T_\alpha(\alpha-1)\big] \times \big[\tfrac{1}{2-\sqrt{2}-d}, \tfrac{1}{1-\sqrt{2}-d}\big], \\
\mathcal{T}_\alpha\Big(\big(\tfrac{1}{\alpha-d-1}, 0\big) \times \big[1-\sqrt{2}, 2-\sqrt{2}\big]\Big) &= \big[\alpha-1, \alpha\big) \times \big[\tfrac{1}{1-\sqrt{2}-d}, 0\big), \\
\mathcal{T}_\alpha\Big(\big(0, \tfrac{1}{\alpha+2}\big] \times \big[1-\sqrt{2}, \sqrt{2}\big]\Big) &= \big[\alpha-1, \alpha\big) \times \big(0, \tfrac{1}{4-\sqrt{2}}\big], \\
\mathcal{T}_\alpha\Big(\big(\tfrac{1}{\alpha+2}, \tfrac{1}{\alpha+1}\big] \times \big[\tfrac{1}{\sqrt{2}} - 1, \sqrt{2}\big]\Big) &= \big[\alpha-1, \alpha\big) \times \big[1 - \tfrac{1}{\sqrt{2}}, 2-\sqrt{2}\big], \\
\mathcal{T}_\alpha\Big(\big(\tfrac{1}{\alpha+1}, \alpha\big] \times \big[\tfrac{1}{\sqrt{2}} - 1, \sqrt{2}\big]\Big) &= \big[\tfrac{1}{\alpha} - 1, \alpha\big) \times \big[\sqrt{2}-1, \sqrt{2}\big],
\end{aligned}$$

and by using that $\frac{1}{2-\sqrt{2}-d} = \frac{-1}{1+\sqrt{2}} = 1 - \sqrt{2}$ if $d = 3$, $\frac{1}{2-\sqrt{2}-d} \ge \frac{-1}{2+\sqrt{2}} = \frac{1}{\sqrt{2}} - 1$ if $d \ge 4$, $T_\alpha(\alpha-1) = \frac{1}{\alpha-1} + 3 < \frac{1}{\alpha+2}$ if $d = 3$, and $\frac{1}{4-\sqrt{2}} < \sqrt{2} - 1$. This implies that $\Omega_\alpha \subset X_\alpha$. □

We establish a relation between α-expansions for different α; see also [3].

Lemma 4. *Let $g \le \alpha \le \beta \le 1$, $x \in [\alpha-1, \alpha)$, $z \in [\beta-1, \beta)$.*

1. *If $x = z$ or $(x+1)(1-z) = 1$ or $(1-x)(z+1) = 1$, then $T_\beta(z) - T_\alpha(x) \in \{0,1\}$.*
2. *If $x + z = 0$ or $(x+1)(z+1) = 1$, then $T_\alpha(x) + T_\beta(z) \in \{0,1\}$.*
3. *If $z - x = 1$, then $(x+1)(T_\beta(z)+1) = 1$.*
4. *If $x + z = 1$, then*

$$\begin{cases} (T_\alpha(x)+1)(1-z) = 1 & \text{if } x > \frac{1}{\alpha+1}, \\ (1-x)(T_\beta(z)+1) = 1 & \text{if } z > \frac{1}{\beta+1}, \\ \big(T_\alpha(x)+1\big)\big(T_\beta(z)+1\big) = 1 & \text{otherwise.} \end{cases}$$

Proof. In case (1), we have $\frac{1}{x} - \frac{1}{z} \in \{-1,0,1\}$ or $x = z = 0$, thus $T_\beta(z) - T_\alpha(x) \in \mathbb{Z}$. We clearly have $T_\beta(z) - T_\alpha(x) \in (\beta-\alpha-1, \beta-\alpha+1) \subset (-1, 2-g)$, thus $T_\beta(z) - T_\alpha(x) \in \{0,1\}$.

In case (2), $\frac{1}{x} + \frac{1}{z} \in \{-1,0\}$ or $x = z = 0$ gives that $T_\alpha(x) + T_\beta(z) \in \mathbb{Z} \cap [\alpha+\beta-2, \alpha+\beta) = \{0,1\}$.

In case (3), we have $z = x + 1 \ge \alpha \ge g$, thus $T_\beta(z) = \frac{1}{z} - 1$ and $(x+1)(T_\beta(z)+1) = 1$.

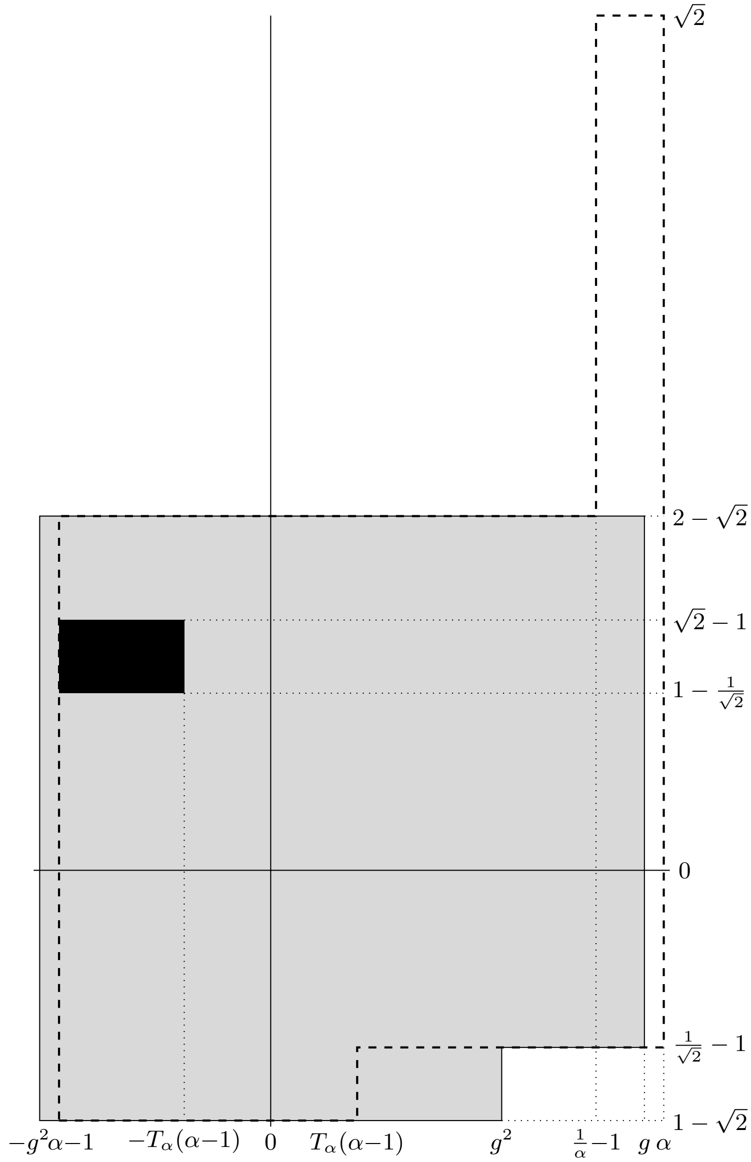

FIGURE 1. The natural extension domain Ω_g is in grey; for $\alpha = 13/20$, Ω_α is contained in the dashed polygon X_α and contains the black rectangle.

Finally, in case (4), if $x > \frac{1}{\alpha+1}$, then $T_\alpha(x) = \frac{1}{x}-1$ and $(T_\alpha(x)+1)(1-z) = 1$. Similarly, $z > \frac{1}{\beta+1}$ implies that $(1-x)(T_\beta(z)+1) = 1$. If $x \le \frac{1}{\alpha+1}$ and $z \le \frac{1}{\beta+1}$, then $x = 1-z \ge \frac{\beta}{\beta+1} \ge \frac{1}{g+2} \ge \frac{1}{\alpha+2}$ and $z = 1-x \ge \frac{\alpha}{\alpha+1} \ge \frac{1}{g+2} \ge \frac{1}{\beta+2}$. We cannot have $x = \frac{1}{\alpha+2}$ because this would imply that $\alpha = g = \beta = z$, contradicting that $z < \beta$. Similarly, we cannot have $z = \frac{1}{\beta+2}$. From $x \in (\frac{1}{\alpha+2}, \frac{1}{\alpha+1}]$ and $z \in (\frac{1}{\beta+2}, \frac{1}{\beta+1}]$, we infer that $(T_\alpha(x)+1)(T_\beta(z)+1) = (\frac{1}{x}-1)(\frac{1}{z}-1) = 1$. □

Lemma 5. *Let $g \le \alpha < \beta \le 1$, $x \in [\alpha-1, \alpha)$, $z \in [\beta-1, \beta)$, with $z-x \in \{0,1\}$ or $x+z \in \{0,1\}$. Let $n \ge 1$ be such that $T_\alpha^{n-1}(x) < \frac{\beta}{\beta+1}$. Then there is some $k \ge 1$ such that $T_\beta^k(z) - T_\alpha^n(x) \in \{0,1\}$ or $T_\alpha^n(z) + T_\beta^k(x) \in \{0,1\}$.*

Proof. Denote $x_j = T_\alpha^j(x)$ and $z_j = T_\beta^j(z)$. By Lemma 4 and since $x_{n-1} < \frac{\beta}{\beta+1} < \frac{1}{\alpha+1}$, we have $z_k - x_n \in \{0,1\}$ or $x_n + z_k \in \{0,1\}$ or $(x_n+1)(z_k+1) = 1$ for some $k \ge 1$. If $(x_n+1)(z_k+1) = 1$, then $z_{k-1} - x_n = 1$ (and $k \ge 2$) because $1 - x_{n-1} = z_{k-1} \le \frac{1}{\beta+1}$ would contradict $x_{n-1} < \frac{\beta}{\beta+1}$. □

Define

$$S(x,y) = \{(x,y), (-x,-y), (x+1, \tfrac{y}{1-y}), (1-x, \tfrac{-y}{y+1})\}.$$

Lemma 6. *Let $g \le \alpha < \beta \le 1$, $(x,y) \in \Omega_\alpha$, $(\tilde{x},\tilde{y}) \in S(x,y)$, $(x_n,y_n) = \mathcal{T}_\alpha^n(x,y)$ for some $n \ge 1$. If $\tilde{x} \in [\beta-1,\beta)$ and $y_n < 1 - \frac{1}{\sqrt{2}}$, then there is some $k \ge 1$ such that $\mathcal{T}_\beta^k(\tilde{x},\tilde{y}) \in S(x_n,y_n)$.*

Proof. Since $\Omega_\alpha \subset X_\alpha$ by Lemma 3, $y_n < 1 - \frac{1}{\sqrt{2}}$ implies that $a_n(x) \ge 3$ or $a_n(x) < 0$, i.e., $T_\alpha^{n-1}(x) \le \frac{1}{\alpha+2} < \frac{\beta}{\beta+1}$. Therefore, by Lemma 5, we have some $k \ge 1$ such that $T_\beta^k(\tilde{x}) - T_\alpha^n(x) \in \{0,1\}$ or $T_\alpha^n(\tilde{z}) + T_\beta^k(x) \in \{0,1\}$. Considering the associated linear fractional transformations, we obtain that $\mathcal{T}_\beta^k(\tilde{x},\tilde{y}) \in S\mathcal{T}_\alpha^n(x,y)$. □

Lemma 7. *Let $g \le \alpha < \beta \le 1$, $(x,y) \in \Omega_\alpha$ with $y < 1 - \frac{1}{\sqrt{2}}$. Then we have $S(x,y) \cap \Omega_\beta \ne \emptyset$.*

Proof. Assume first that $(x,y) = \mathcal{T}_\alpha^n(z,0)$ for some $n \ge 0$, $z \in [\alpha-1,\alpha)$, and choose $\tilde{z} \in [\beta-1,\beta)$ such that $(\tilde{z},0) \in S(z,0)$. Since $y < 1 - \frac{1}{\sqrt{2}}$, Lemma 6 gives some $k \ge 0$ such that $\mathcal{T}_\alpha^k(\tilde{z},0) \in S(x,y)$, thus $S(x,y) \cap \Omega_\beta \ne \emptyset$. As each $(x,y) \in \Omega_\alpha$ is the limit of points $\mathcal{T}_\alpha^n(z,0)$, this proves the lemma. □

From Lemma 7 with $\alpha = g$, we can easily conclude that Ω_β has positive Lebesgue measure, and the following lemma provides rectangles in the natural extension domain.

Lemma 8. *Let $\alpha \in (g,1)$, $d = -a_1(\alpha-1)$, $b = \lfloor T_\alpha(\alpha-1) + \alpha \rfloor$. We have $Y_\alpha \subset \Omega_\alpha$, with*

$$Y_\alpha = \big[\alpha-1, b - T_\alpha(\alpha-1)\big] \times \big[\tfrac{1}{d+\sqrt{2}-1-b}, \tfrac{1}{d+\sqrt{2}-2-b}\big] \cup \big[\alpha-1,\alpha\big] \times \big(\tfrac{1}{d+\sqrt{2}-2-b}, \sqrt{2}-1\big].$$

Proof. Let $(x,y) \in \Omega_g \setminus \Omega_\alpha$ with $y < 0$. Then Lemma 7 gives that $(-x,-y) \in \Omega_\alpha$ or $(x+1, \frac{y}{1-y}) \in \Omega_\alpha$ or $(1-x, \frac{-y}{y+1}) \in \Omega_\alpha$. We have thus $(-x,-y) \in \Omega_\alpha$ when $|x| < 1-\alpha$, $(1-x, \frac{-y}{y+1}) \in \Omega_\alpha$ when $x > 1-\alpha$. If $x \le \alpha - 1$ and $y < \frac{1}{2-\sqrt{2}-d}$, then we also have $(-x,-y) \in \Omega_\alpha$ because $x+1 \ge g$ and $\frac{y}{1-y} < \frac{1}{1-\sqrt{2}-d}$ imply that $(x+1, \frac{y}{1-y}) \notin \Omega_\alpha$ by Lemma 3.

From Lemma 3 and equation (9), we get that

$$\big([-g^2,g] \times \big[1-\sqrt{2}, \tfrac{1}{2-\sqrt{2}-d}\big) \cup (T_\alpha(\alpha-1), g] \times \big[\tfrac{1}{2-\sqrt{2}-d}, \tfrac{1}{1-\sqrt{2}-d}\big)\big) \setminus (g^2,g] \times \big[1-\sqrt{2}, \tfrac{1}{\sqrt{2}}-1\big) \subset \Omega_g \setminus \Omega_\alpha.$$

Considering points (x,y) with $x < 1-\alpha$ in this union of rectangles, we obtain that

$$\big(\alpha-1, g^2\big] \times \big(\tfrac{1}{d+\sqrt{2}-2}, \sqrt{2}-1\big] \cup \big(\alpha-1, \max\{\alpha-1, -T_\alpha(\alpha-1)\}\big) \times \big(\tfrac{1}{d+\sqrt{2}-1}, \tfrac{1}{d+\sqrt{2}-2}\big] \subset \Omega_\alpha.$$

If $d \ge 4$, then points (x,y) with $x > 1-\alpha$ and $y \ge \frac{1}{\sqrt{2}} - 1$ provide that

$$\big[g^2, \alpha\big) \times \big(\tfrac{1}{d+\sqrt{2}-3}, \sqrt{2}-1\big] \cup \big[g^2, \min\{\alpha, 1 - T_\alpha(\alpha-1)\}\big) \times \big(\tfrac{1}{d+\sqrt{2}-2}, \tfrac{1}{d+\sqrt{2}-3}\big] \subset \Omega_\alpha.$$

By distinguishing the cases $T_\alpha(\alpha-1) < 1-\alpha$, i.e., $b = 0$, and $T_\alpha(\alpha-1) \ge 1-\alpha$, i.e., $b = 1$, we get that $Y_\alpha \subset \Omega_\alpha$. (Note that Ω_α is a closed set.) □

Since for $\alpha \in (g,1)$ we have $d \ge 3$, with $b = 0$ if $d = 3$, Lemma 8 shows in particular that

$$\big[\alpha-1, \min\{\alpha, \tfrac{1}{1-\alpha} - 3\}\big] \times \big[1 - \tfrac{1}{\sqrt{2}}, \sqrt{2}-1\big] \subset \Omega_\alpha \tag{10}$$

(with $\frac{1}{1-\alpha} - 3 > \alpha - 1$). Theorem 2 is a direct consequence of this inclusion.

5. Entropy

From (10), we obtain the following proposition.

Proposition 5. *There exists a positive constant C_3 such that*

$$C_3^{-1}\lambda(A) < \mu_\alpha(A) < C_3\lambda(A)$$

for any measurable set $A \subset [\alpha-1,\alpha)$.

Proof. By Proposition 1, we have a full cylinder $\langle a_1(x),\dots,a_n(x)\rangle \subset \big[\alpha-1, \min\{1-\alpha, \frac{1}{1-\alpha}-3\}\big]$. Then there exists a real number y_0 and a positive number η such that

$$\mathcal{T}_\alpha^n\big(\langle a_1(x),\dots,a_n(x)\rangle \times \big[1-\tfrac{1}{\sqrt{2}}, \sqrt{2}-1\big]\big) = [\alpha-1,\alpha)\times[y_0, y_0+\eta].$$

This shows that there is a positive constant C_3' such that $\xi(x) > C_3'$, where

$$\xi(x) = \frac{1}{\hat\mu(\Omega_\alpha)}\int_{y\colon (x,y)\in\Omega_\alpha} \frac{1}{(1+xy)^2}\,\mathrm{d}y$$

is the density of μ_α. On the other hand, since $\Omega_\alpha \subset [\alpha-1,\alpha]\times[1-\sqrt{2},\sqrt{2}]$, we can find C_3'' such that $\xi(x) < C_3''$. Altogether, we have the assertion of this proposition. ☐

Let $h(T_\alpha)$ denote the entropy of T_α with respect to the invariant measure μ_α. The following shows that Rokhlin's formula holds, as mentioned at the end of §3.

Proposition 6. *For any $0<\alpha\le 1$, we have*

$$h(T_\alpha) = -\int_{[\alpha-1,\alpha)} \log x^2\,\mathrm{d}\mu_\alpha(x)$$

and

$$h(T_\alpha) = 2\lim_{n\to\infty}\frac{1}{n}\log|q_n(x)| \quad \textit{for a.e. } x\in[\alpha-1,\alpha).$$

Proof. Choose a generic point $x_0 \in [\alpha-1,\alpha)$ so that

- there exists a subsequence of natural numbers $(n_k)_{k\ge1}$ such that $\big\langle a_1(x_0),\dots,a_{n_k}(x_0)\big\rangle$ is a full cylinder for any $k\ge 1$,
- $-\lim_{n\to\infty}\frac{1}{n}\log\mu_\alpha\big(\big\langle a_1(x_0),\dots,a_n(x_0)\big\rangle\big) = h(T_\alpha)$,
- $\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\log\big|T_\alpha'(T_\alpha^n(x_0))\big| = -\int_{[\alpha-1,\alpha)}\log x^2\,\mathrm{d}\mu_\alpha(x)$.

For each n_k, we see that

$$\lambda\big(\big\langle a_1(x_0),\dots,a_{n_k}(x_0)\big\rangle\big) = \left|\frac{p_{n_k-1}\cdot\alpha + p_{n_k}}{q_{n_k-1}\cdot\alpha+q_{n_k}} - \frac{p_{n_k-1}\cdot(\alpha-1)+p_{n_k}}{q_{n_k-1}\cdot(\alpha-1)+q_{n_k}}\right|. \tag{11}$$

From Proposition 5, we have

$$\lim_{k\to\infty}\frac{1}{n_k}\log\mu_\alpha\big(\big\langle a_1(x_0),\dots,a_{n_k}(x_0)\big\rangle\big) = \lim_{k\to\infty}\frac{1}{n_k}\log\lambda\big(\big\langle a_1(x_0),\dots,a_{n_k}(x_0)\big\rangle\big).$$

Then by the mean-value theorem and (11) there exist $y_k\in[\alpha-1,\alpha)$ such that

$$h(T_\alpha) = -\lim_{k\to\infty}\frac{1}{n_k}\log\Big|\psi'_{a_1(x_0)\cdots a_{n_k}(x_0)}(y_k)\Big|.$$

From Proposition 2, we see

$$h(T_\alpha) = -\lim_{k\to\infty}\frac{1}{n_k}\log\Big|\psi'_{a_1(x_0)\cdots a_{n_k}(x_0)}(\tilde y_k)\Big|$$

for any $\tilde y_k \in [\alpha-1,\alpha)$. So we can choose $\tilde y_k = T_\alpha^{n_k}(x_0)$. Then

$$\psi'_{a_1(x_0)\cdots a_{n_k}(x_0)}(\tilde y_k) = \frac{1}{\left(T_\alpha^{n_k}\right)'(x_0)}$$

holds. Consequently by the choice of x_0 and the chain rule we have the first assertion of this proposition. The second assertion also follows from Proposition 2. ☐

Finally, we establish the monotonicity of the product $h(T_\alpha)\hat\mu(\Omega_\alpha)$.

Proof of Theorem 3. For each $\alpha \in [1/2, g]$, we have $h(T_\alpha) = \frac{\pi^2}{6}$ and $\hat\mu(\Omega_\alpha) = -2\log g$. Let now $g \le \alpha < \beta \le 1$, $d = -a_1(\alpha-1)$, $b = \lfloor T_\alpha(\alpha-1)+\alpha \rfloor$. Set

$$X_{\alpha,\beta} = \begin{cases} \big(\max\{1-\beta, \frac{1}{\beta-1}+d+1\}, \alpha\big) \times \big[\frac{1}{1-\sqrt{2}-d}, \frac{1}{-\sqrt{2}-d}\big] \cap \Omega_\alpha & \text{if } T_\alpha(\alpha-1) = \alpha-1, \\ \big(\max\{\alpha-1, \frac{1}{\beta-1}+d\}, T_\alpha(\alpha-1)\big] \times \big[\frac{1}{2-\sqrt{2}-d}, \frac{1}{1-\sqrt{2}-d}\big] \cap \Omega_\alpha & \text{if } \alpha-1 < T_\alpha(\alpha-1) \le 1-\beta, \\ \big(\max\{1-\beta, \frac{1}{\beta-1}+d\}, T_\alpha(\alpha-1)\big] \times \big[\frac{1}{2-\sqrt{2}-d}, \frac{1}{1-\sqrt{2}-d}\big] \cap \Omega_\alpha & \text{if } T_\alpha(\alpha-1) > 1-\beta. \end{cases}$$

Note that $X_{\alpha,\beta} \subset X_\alpha \setminus X_\beta$, and we have $\hat\mu(X_{\alpha,\beta}) > 0$ because of (10) together with $\mathcal{T}_\alpha(\Omega_\alpha) \subset \Omega_\alpha$,

$\mathcal{T}_\alpha([\alpha-1, x] \times [1-\sqrt{2}, 2-\sqrt{2}]) = [T_\alpha(x), T_\alpha(\alpha-1)] \times [\frac{1}{2-\sqrt{2}-d}, \frac{1}{1-\sqrt{2}-d}]$ for all $x \in (\alpha-1, \frac{1}{\alpha-d-1}]$,

and, in case $T_\alpha(\alpha-1) = \alpha-1$,

$\mathcal{T}_\alpha((\alpha-1, x] \times [1-\sqrt{2}, 2-\sqrt{2}]) = [T_\alpha(x), T_\alpha(\alpha-1)) \times [\frac{1}{1-\sqrt{2}-d}, \frac{1}{-\sqrt{2}-d}]$ for all $x \in (\alpha-1, \frac{1}{\alpha-d-2}]$.

Let

$$\varphi(x,y) = \begin{cases} (-x, -y) & \text{if } T_\alpha(\alpha-1) \in (1-\alpha, 1-\beta], \\ (1-x, \frac{-y}{y+1}) & \text{otherwise.} \end{cases}$$

Then we have $\hat\mu(\varphi(X_{\alpha,\beta})) = \hat\mu(X_{\alpha,\beta})$ and, by Lemma 8, $\varphi(X_{\alpha,\beta}) \subset \Omega_\beta$. Let $\widetilde{\mathcal{T}}_\alpha$ be the first return map of $\mathcal{T}_\alpha$ on $X_{\alpha,\beta}$, and let $\widetilde{\mathcal{T}}_\beta$ be the first return map of $\mathcal{T}_\beta$ on $\varphi(X_{\alpha,\beta})$. For $(x,y) \in X_{\alpha,\beta}$, we have, by Lemma 6, $\mathcal{T}_\beta^k \varphi(x,y) \in S\widetilde{\mathcal{T}}_\alpha(x,y)$ for some $k \ge 1$, thus $\mathcal{T}_\beta^k \varphi(x,y) = \varphi\widetilde{\mathcal{T}}_\alpha(x,y)$, hence $\varphi\widetilde{\mathcal{T}}_\alpha(x,y) = \widetilde{\mathcal{T}}_\beta^m \varphi(x,y)$ for some $m \ge 1$. This implies that $h(\widetilde{\mathcal{T}}_\beta) \le h(\widetilde{\mathcal{T}}_\alpha)$. Abramov's formula gives that

$$h(\widetilde{\mathcal{T}}_\alpha) = \frac{\hat\mu(\Omega_\alpha)}{\hat\mu(X_{\alpha,\beta})} h(\mathcal{T}_\alpha) \quad \text{and} \quad h(\widetilde{\mathcal{T}}_\beta) = \frac{\hat\mu(\Omega_\beta)}{\hat\mu(\varphi(X_{\alpha,\beta}))} h(\mathcal{T}_\beta),$$

thus $\hat\mu(\Omega_\beta)\, h(\mathcal{T}_\beta) \le \hat\mu(\Omega_\alpha)\, h(\mathcal{T}_\alpha)$. □

References

[1] R. Bowen, Bernoulli maps of the interval, Israel J. Math. 28 (1977), no. 1–2, 161-–168.
[2] L. Breiman, The individual ergodic theorem of information theory, Ann. Math. Statist. 28 (1957), 809–811.
[3] C. Carminati, N.D.S. Langeveld and W. Steiner, Tanaka–Ito α-continued fractions and matching, in preparation (2019).
[4] K. L. Chung, A note on the ergodic theorem of information theory, Ann. Math. Statist. 32 (1961), 612–614.
[5] C. Kraaikamp, T. A. Schmidt and W. Steiner, Natural extensions and entropy of α-continued fractions, Nonlinearity 25 (2012), no. 8, 2207–2243.
[6] H. Nakada, Metrical theory for a class of continued fraction transformations and their natural extensions, Tokyo J. Math. 4 (1981), 399–426.
[7] H. Nakada and R. Natsui, Some strong mixing properties of a sequence of random variables arising from α-continued fractions, Stoch. Dyn. 3 (2003), 463–476.
[8] A. Rényi, Representations for real numbers and their ergodic properties, Acta Math.Acad. Sci. Hungar. 8 (1957), 477–493.
[9] V. A. Rokhlin, Exact endomorphisms of Lebesgue spaces (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 25 (1961), 499–530, Translation in Amer. Math. Soc. Transl. Series 2, 39 (1964), 1–36.
[10] M. Rychlik, Bounded variation and invariant measures, Studia Math. 76 (1983), 69–80.
[11] F. Schweiger, Some remarks on ergodicity and invariant measures, Michigan Math. J. 22 (1975), 181–187.
[12] S. Tanaka and S. Ito, On a family of continued-fraction transformations and their ergodic properties, Tokyo J. Math. 4 (1981), no. 1, 153–175.
[13] R. Zweimüller, Ergodic properties of infinite measure preserving interval maps with indiferent fixed points, Ergodic Theory Dynam. Sys. 20 (2000), 1519–1549.

Department of Mathematics, Keio University, Yokohama, Japan
Email address: nakada@math.keio.ac.jp

Université de Paris, IRIF, CNRS, F–75006 Paris, France
Email address: steiner@irif.fr