A DYNAMICAL VIEW OF TIJDEMAN'S SOLUTION OF THE CHAIRMAN ASSIGNMENT PROBLEM

VALÉRIE BERTHÉ, OLIVIER CARTON, NICOLAS CHEVALLIER, WOLFGANG STEINER, AND REEM YASSAWI

Abstract. In 1980, R. Tijdeman provided an on-line algorithm that generates sequences over a finite alphabet with minimal discrepancy, that is, such that the occurrence of each letter optimally tracks its frequency. In this article, we define discrete dynamical systems generating these sequences. The dynamical systems are defined as exchanges of polytopal pieces, yielding cut and project schemes, and they code tilings of the line whose sets of vertices form model sets. We prove that these sequences of low discrepancy are natural codings of toral translations with respect to polytopal atoms, and that they generate a minimal and uniquely ergodic subshift with purely discrete spectrum. Finally, we show that the factor complexity of these sequences is of polynomial growth order n^{d-1}, where d is the cardinality of the alphabet.

1. Introduction

Quoting Tijdeman in [Tij80], the chairman assignment problem is stated as follows: "Suppose k states form a union and every year a union chairman has to be selected in such a way that at any time the accumulated number of chairmen from each state is proportional to its weight." The question is then to give a simple algorithm for a chairman assignment which guarantees a small discrepancy. The richness of this problem is that it can be reformulated in several ways, as a sequencing problem in operations research for optimal routing and scheduling, in terms of word combinatorics, symbolic dynamics and aperiodic order, and also as a discrepancy problem.

In this latter setting, the problem asks for the existence of very well distributed sequences such as introduced by Niederreiter in [Nie72b]. Given a finite alphabet $\mathcal{A}$ of cardinality d and a vector $\boldsymbol{\alpha}$ of frequencies for the letters of $\mathcal{A}$, the aim is to construct a sequence over $\mathcal{A}$ in which each letter occurs with its prescribed frequency as evenly as possible, i.e., where the occurrence of each letter optimally tracks its frequency; see Section 2 for the definition of frequency. More precisely, given a sequence $u = (u_k)_{k\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$, the *(letter) discrepancy*

Date: May 16, 2024.

Key words and phrases. Symbolic dynamics; discrepancy; billiard word; toral translation; natural coding.

The first, second, and fourth authors were supported by the Agence Nationale de la Recherche through the project "SymDynAr" (ANR-23-CE40-0024-01). The first and fourth authors were supported by the Agence Nationale de la Recherche through the project "IZES" (ANR-22-CE40-0011). The fifth author was supported by the EPSRC, grant number EP/V007459/2.

of u with respect to $\boldsymbol{\alpha}$ is defined as

$$\Delta_{\boldsymbol{\alpha}}(u) = \max_{i\in\mathcal{A}} \sup_{n\in\mathbb{N}} \left|\mathrm{Card}\{0 \le k < n \,:\, u_k = i\} - n\alpha_i\right|,$$

for a given vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (0,1)^d$ with $\sum_{i=1}^d \alpha_i = 1$. It is then natural to consider, for $d \ge 2$, the quantity

$$D_d = \sup_{\boldsymbol{\alpha}} \inf_u \Delta_{\boldsymbol{\alpha}}(u),$$

where the supremum is taken over the set of *frequency vectors* $\boldsymbol{\alpha}$ in the d-simplex, and the infimum is taken over the set of sequences u with values in an alphabet of cardinality d.

The question of finding sequences with minimal discrepancy was raised in [Nie72a, Nie72b]. Niederreiter proved in [Nie72b, Lemma 1] the existence of a sequence with $\Delta_{\boldsymbol{\alpha}}(u) \le d-1$; see also [MN72] for refinements. He also conjectured that $D_d \le 1$. Tijdeman answered this conjecture positively in a nonconstructive fashion in [Tij73] with a proof based on Hall's marriage theorem, and he also showed that $D_d \ge 1-\frac{1}{2d-2}$. Refining Tijdeman's proof, Meijer proved in [Mei73] that

$$D_d = 1 - \tfrac{1}{2d-2},$$

but the proof of this result was also nonconstructive. Lastly, Tijdeman provided in [Tij80] a linear time on-line algorithm to determine, for each d-dimensional vector $\boldsymbol{\alpha}$, a sequence u over a finite alphabet of cardinality d for which $\Delta_{\boldsymbol{\alpha}}(u) \le 1-\frac{1}{2d-2}$. We describe Tijdeman's construction in Section 3.3.

These sequences are the object of the present paper. We call a sequence constructed by Tijdeman's algorithm a *Tijdeman sequence* with frequency $\boldsymbol{\alpha}$, and we call sequences satisfying $\Delta_{\boldsymbol{\alpha}}(u) \le 1-\frac{1}{2d-2}$ *fairly distributed*. The full definition of Tijdeman sequences (see Definition 3.7) depends on the frequency $\boldsymbol{\alpha}$ and on three other parameters: two constants C and C' and a starting point $\mathbf{x}_0$. These three parameters can be used to optimize the discrepancy.

As recalled in [Tij82], the algorithm proposed by Tijdeman in [Tij80] for the chairman assignment problem is closely related to the quota-method of Balinski and Young [BY75, BY77, BY85] for the (discrete) apportionment problem; see [CCV22] for more references on the subject. This problem, which has its origins in the problem of seat assignments to the house of representatives in the United States, consists in allocating seats in a proportional way. See also [Li22] for the connection with apportionment problems and Just-In-Time sequencing problems (with maximal deviation JIT scheduling), such as considered in [AGH00, BC04, BJ08], and see the survey [Vui03] for more references. See in addition [CNP+11], where it is proved that the greedy algorithm is optimal among online algorithms for the chairman assignment problem.

As mentioned above, the richness of this problem is that it goes well beyond the scheduling framework: indeed, it can be reformulated combinatorially in terms of balance in word combinatorics, dynamically in terms of symbolic codings of toral translations, arithmetically in terms of bounded remained sets or else in terms of cut and project schemes, within the setting of aperiodic order; the latter corresponds to the mathematical formalization of quasicrystals. Let us discuss now these complementary and intricately connected notions.

In combinatorial terms, discrepancy is closely related to the notion of balance. Given a finite alphabet $\mathcal{A}$, a sequence $u \in \mathcal{A}^{\mathbb{N}}$ is said to be *B-balanced* if there exists a constant B such that for every letter $i \in \mathcal{A}$ and for every pair (w, w') of factors of u of the same length, the difference between the number of occurrences of i in w and w' differs by at most B. Balance was first studied in the form of 1-balance for binary sequences by Morse and Hedlund in the seminal papers [MH38, MH40], in which they laid the basis for symbolic dynamics (see Section 2.1): the binary 1-balanced aperiodic sequences are exactly the *Sturmian sequences*. The notion of balance was then considered for larger alphabets, for B-balance, with $B > 1$, and for factors instead of letters. For more on the subject, see the survey [Vui03]. Words over a larger alphabet that are 1-balanced have been characterized in [Hub00] (see also [Gra73]) and shown to be closely related to Sturmian words. Let us observe that their letter frequencies are rationally dependent: there exists γ such that $\alpha_i \in \mathbb{Z} + \gamma\mathbb{Z}$, for $1 \leq i \leq d$, by [Hub00, Lemma 4.1]. Note that, in ergodic terms, balance can be interpreted as an optimal speed of convergence of Birkhoff sums toward frequencies of words. A sequence is B-balanced if and only if it has has finite discrepancy (see [Ada03, Ada04]), and a fairly distributed sequence over two letters must be 1-balanced; see Proposition 2.1.

There is a classical way of constructing balanced sequences, and in particular Sturmian sequences, in terms of *cutting sequences*. Specifically, Sturmian sequences, which are defined on binary alphabets, are codings of trajectories of billiards on a square table with an irrational direction and, by unfolding trajectories, they code whether a horizontal or a vertical side of a lattice square is hit. They have been generalized to larger alphabets as hypercubic billiard sequences, as presented in [AMST94a, AMST94b]; see also Section 3.2. An equivalent description is in terms of natural codings of toral translations; see Definition 2.2. Indeed, Sturmian sequences are known to be symbolic codings of a special kind, namely a Sturmian sequence codes an exchange of two intervals, which happens to be a translation of the one-dimensional torus $\mathbb{R}/\mathbb{Z}$.

We focus here on the case where the vector of frequencies $\boldsymbol{\alpha}$ has linearly independent entries over the rationals. Under this hypothesis, a lower bound for the discrepancy is given in [Sch96]: one has $\Delta_{\boldsymbol{\alpha}}(u) \geq 1-\frac{1}{d}$ for all sequences u [Sch96, Theorem 2]. Thus, when $d = 2$ ($D_2 = \frac{1}{2}$), this gives $\Delta_{\boldsymbol{\alpha}}(u) \geq \frac{1}{2}$. This bound is achieved by sequences defined in terms either of Beatty sequences, or, in some equivalent way, of Sturmian sequences (see in particular Remark 3.4 below): for $\boldsymbol{\alpha} = (\alpha, 1-\alpha)$, the sequence $(u_k)_{k\in\mathbb{N}}$, defined by $u_k = 1$ if and only if $k = \lceil (n-\frac{1}{2})/\alpha \rceil$ for some nonnegative n, satisfies $\Delta_{\boldsymbol{\alpha}}(u) = 1/2$, with the sequence $(\lceil (n-\frac{1}{2})/\alpha \rceil)_n$ being known as a *Beatty sequence*. We recall that the first difference sequence of a Beatty sequence is a Sturmian sequence; and that Sturmian sequences are 1-balanced, as discussed above.

The aim of this paper is to provide a similar dynamical description for Tidjeman sequences, which, as mentioned above, are among the sequences having the lowest letter discrepancy $\Delta_{\boldsymbol{\alpha}}(u)$ with $\Delta_{\boldsymbol{\alpha}}(u) \le D_d = 1-\frac{1}{2d-2}$, for vectors of frequencies $\boldsymbol{\alpha}$ with rationally independent coordinates. We present them as *natural codings* of a dynamical system defined as a higher dimensional domain exchange with polytopal pieces, which turns out to be a translation of the torus; see Definition 2.2.

Let $\boldsymbol{\alpha}$ be a frequency vector, i.e., a vector in $(0,1)^d$ with $\sum_{i=1}^d \alpha_i = 1$. We say that $\boldsymbol{\alpha}$ is *totally irrational* if it has rationally independent coordinates. This is equivalent to demanding that the coordinates of $(\alpha_1, \dots, \alpha_{d-1}, 1)$ are rationally independent. We consider the (minimal[1]) toral translation by $(\alpha_1, \dots, \alpha_{d-1})$, denoted as $T_{\boldsymbol{\alpha}}$, defined on $\mathbb{T}^{d-1} = \mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$ by[2]

$$T_{\boldsymbol{\alpha}}:\ \mathbb{T}^{d-1} \to \mathbb{T}^{d-1}, \quad \mathbf{x} \mapsto \mathbf{x} + (\alpha_1, \dots, \alpha_{d-1}) \pmod{\mathbb{Z}^{d-1}}.$$

The codings we consider are called natural, by which we mean that these sequences code translations $T_{\boldsymbol{\alpha}}$ of the torus with respect to (polytopal) partitions of $\mathbb{T}^{d-1}$ that come from domain exchanges in $\mathbb{R}^{d-1}$ with translation vectors equal to $(\alpha_1, \dots, \alpha_{d-1}) \bmod \mathbb{Z}^{d-1}$; see Figure 1 and also Definition 2.2. These codings are even *bounded* natural codings, in the sense that the pieces are bounded as subsets of $\mathbb{R}^{d-1}$. This dynamical description allows us to deduce estimates on the factor complexity of Tijdeman sequences, in particular proving that the factor complexity is of order n^{d-1}, when defined over an alphabet of cardinality d.

Our main result is

Theorem 1.1. *Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ be a totally irrational frequency vector. Then there exist Tijdeman parameters generating a sequence u with $\Delta_{\boldsymbol{\alpha}}(u) \le 1-\frac{1}{2d-2}$, and such that u is the bounded natural coding of $T_{\boldsymbol{\alpha}}$, via a partition of a fundamental domain of $\mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$ into d finite unions of convex polytopes.[3] Furthermore, the shift defined by u is minimal, uniquely ergodic, has purely discrete spectrum and factor complexity of order n^{d-1}.*

Theorem 1.1 induces the existence of a "good" symbolic coding for any minimal toral translation $T_{\boldsymbol{\alpha}}$ such that the $d-1$ first coordinates of $\boldsymbol{\alpha}$ are in the polyhedron $\mathcal{T} = \{(\alpha_1, \dots, \alpha_{d-1}) : \alpha_i \ge 0 \text{ and } \sum_{i=1}^{d-1} \alpha_i \le 1\}$. By changing the signs of the coordinates of $(\alpha_1, \dots, \alpha_{d-1})$, we also obtain good codings for all $\boldsymbol{\alpha}$ with $\sum_{i=1}^{d-1} |\alpha_i| \le 1$. Hence, when $d-1 = 2$, we obtain a good coding for any minimal toral translation. However, when $d > 3$, we do not obtain all the toral translations. Neverthless, thanks to the following standard argument (see for example [BST23, Remark 3.4] or [FN20, Section 10.2]), it is possible to find a good coding for every minimal translation even if $d > 3$. Indeed, the cube $[0,1]^{d-1}$ is a union of $(d-1)!$ images of the polyhedron $\mathcal{T}$ by some transformations in $\mathrm{GL}_{d-1}(\mathbb{Z})$, so that any translation $T_{\boldsymbol{\beta}} : \mathbb{T}^d \to \mathbb{T}^d$ can be written as $T_{\boldsymbol{\beta}} = g T_{\boldsymbol{\alpha}} g^{-1}$ for some $\boldsymbol{\alpha} \in \mathcal{T}$ and $g \in \mathrm{GL}_{d-1}(\mathbb{Z})$. This implies that the codings associated with $T_{\boldsymbol{\alpha}}$ with respect to some

[1]See Section 2.1 for the definition of minimality.

[2]Observe that the translation $T_{\boldsymbol{\alpha}}$ is defined on $\mathbb{T}^{d-1}$, and not on $\mathbb{T}^d$; this is due to the fact that $\boldsymbol{\alpha}$ being assumed to be a frequency vector, $\sum_{i=1}^d \alpha_i = 1$.

[3]We follow the usual convention that a convex polytope is the convex hull of a finite number of points. Hence here the fundamental domains under consideration are bounded.

partition of the torus share the same properties as the codings associated with $T_{\boldsymbol{\beta}}$ and the image by g of this partition.

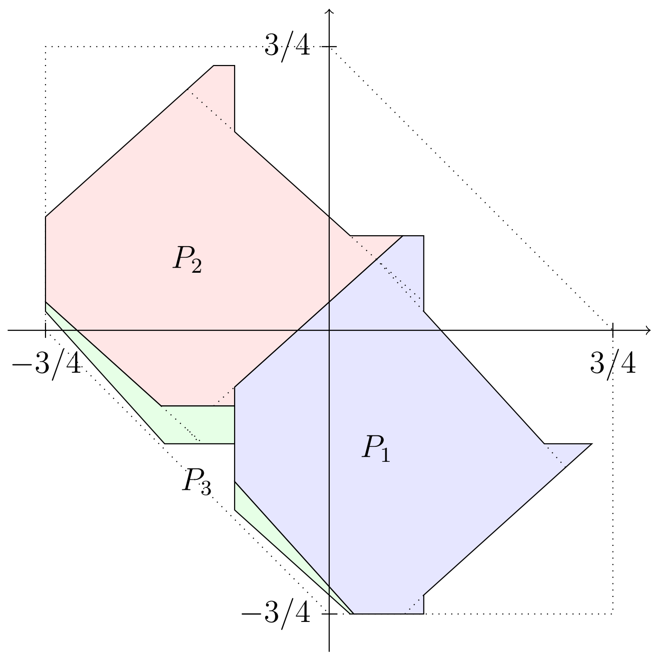

FIGURE 1. A fundamental domain of $\mathbb{R}^2/\mathbb{Z}^2$ and its partition by finite unions of polygons such that the natural codings of the action of $T_{\boldsymbol{\alpha}}$ are Tijdeman sequences with $\boldsymbol{\alpha} \approx (0.5, 0.45, 0.05)$, $C = C' = D_3 = 3/4$.

The fact that the factor complexity is bounded below by Cn^{d-1} implies that a Tijdeman sequence cannot generate a primitive substitution shift for $d \geq 3$, since points in primitive substitution shifts have linear factor complexity; see e.g. [Que10]. Note also that similar results have been stated in terms of estimation of balance in order to get sequences having low balance. See for example [DMP23] for a construction of 2-balanced sequences over a three-letter and a four-letter alphabet based on Sturmian sequences; see also [BCS13].

Let us explain why the toral translation $T_{\boldsymbol{\alpha}}$ intervenes naturally in the present setting. In order to produce a sequence u with a small discrepancy, one constructs, step-by-step, a half broken line $\mathbf{L}_u$ (a half discrete line) whose vertices belong to $\mathbb{N}^d$ and that "optimally" approximates (in a sense to be defined) the half line $\mathbb{R}^+\boldsymbol{\alpha}$. Given a d-letter alphabet $\mathcal{A}$, we associate with each distinct letter of $\mathcal{A}$ a vector of the canonical basis for $\mathbb{R}^d$. Using this association, each sequence $u \in \mathcal{A}^{\mathbb{N}}$ defines a half broken line $\mathbf{L}_u$ that lives in $\mathbb{N}^d$; see Figure 2 for an illustration. To define a sequence that does not stray from $\mathbb{R}^+\boldsymbol{\alpha}$, we must establish a strategy that allows us to optimally choose the value of u_n when $u_0 \cdots u_{n-1}$ are determined. Geometrically, the discrepancy $\Delta_{\boldsymbol{\alpha}}(u)$ measures how far the vertices of the broken line are from the half line $\mathbb{R}^+\boldsymbol{\alpha}$. Consider indeed the projection along the direction $\mathbb{R}^+\boldsymbol{\alpha}$ onto some given transverse hyperplane that does not contain $\boldsymbol{\alpha}$. The discrepancy is calculated by measuring the distance between the projection of the set of vertices of the broken line and the origin. For suitable choices of sequences u, the closure of the projection of the set of vertices of the broken line form (after some natural change of variables) a fundamental domain for the action of the lattice $\mathbb{Z}^{d-1}$ on $\mathbb{R}^{d-1}$. Moreover, moving on the broken line by one step, i.e., by some canonical vector, consists in moving in the fundamental domain by $T_{\boldsymbol{\alpha}}$. Thus the sequence u codes the action of $T_{\boldsymbol{\alpha}}$ with respect to a finite partition. The atom P_a of the partition of the fundamental domain is obtained by taking the closure of the projection of the vertices of the broken line when the last letter equals a; see Figure 1 as an illustration of the atoms of the partition. Our approach has to be compared to the one developed in [Che09] which holds in dimension $d = 3$ and where the choice of the vertices of the broken line is done with respect to the Euclidean norm (whereas here we rely on the supremum norm).

Bounded remainder sets also play a central role here. A *bounded remainder set* for the translation $T_{\boldsymbol{\alpha}}$ acting on the $(d-1)$-dimensional torus $\mathbb{T}^{d-1}$ is a measurable subset A of $\mathbb{T}^{d-1}$ for which there exists $C > 0$ such that, for any point $\mathbf{x} \in \mathbb{T}^{d-1}$ and any $n \in \mathbb{N}$,

$$|\mathrm{Card}\{0 \leq k < n \,:\, T_{\boldsymbol{\alpha}}^k(\mathbf{x}) \in A\} - n\mu(A)| \leq C.$$

These are sets having bounded local discrepancy, i.e., the difference between the number of visits to this particular set and its expected value is bounded. Their study started with the work of Schmidt in his series of papers on irregularities of distributions initiated in [Sch68]. Grepstad and Lev have given in [GL15] a particularly nice family of bounded remainder sets for the minimal translation $T_{\boldsymbol{\alpha}}$ as the parallelotopes in $\mathbb{R}^{d-1}$ spanned by vectors belonging to $\mathbb{Z}(\alpha_1, \cdots, \alpha_{d-1}) + \mathbb{Z}^{d-1}$. This family generalizes Kesten's characterization of the intervals that are bounded remainder sets of the unit circle for an irrational translation by α as the intervals of length in $\alpha\mathbb{Z} + \mathbb{Z}$. The atoms of the partition from Theorem 1.1 are bounded remainder sets.

Lastly, let us briefly reinterpret the previous notions in terms of cut and project schemes and model sets. Model sets play a prominent role as mathematical models for quasicrystals. They have been introduced by Meyer in [Mey72]. For more details, see for instance [KW21] and the references therein. We recall here the corresponding definition in the simple Euclidean setting of the present paper.

We start by defining a *cut and project scheme*. We consider the full rank lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$, together with the decomposition of $\mathbb{R}^d$ as the direct sum of two subspaces, namely the hyperplane $\mathbf{1}^\perp$ made of vectors whose sum of coordinates equals 0 (called the *internal space*) and a line $\mathbf{L}$ directed by $\boldsymbol{\alpha}$ (called the *physical space*). The lattice $\mathbb{Z}^d$ will be projected onto the line $\mathbf{L}$, and the inner space $\mathbf{1}^\perp$ determines the direction of the projection map. Let $\tilde{\pi}_{\boldsymbol{\alpha}}$ denote the corresponding projection. We also consider the projection $\pi_{\boldsymbol{\alpha}}$ onto $\mathbf{1}^\perp$ along $\mathbf{L}$. We further assume that the restriction of $\tilde{\pi}_{\boldsymbol{\alpha}}$ to $\mathbb{Z}^d$ is injective and that $\tilde{\pi}_{\boldsymbol{\alpha}}(\mathbb{Z}^d)$ is dense in $\mathbf{1}^\perp$. This holds true if $\boldsymbol{\alpha}$ is a totally irrational frequency vector, by Kronecker's theorem. Model sets are then formed by projections together with a way of selecting points (the cutting part), and this selection is done thanks to an *acceptance window* W that lives in the internal space $\mathbf{1}^\perp$. Then, a subset Λ of $\mathbb{R}^d$ is a *model set* (associated with the cut and project scheme developed above) if there exists a precompact set W of the internal

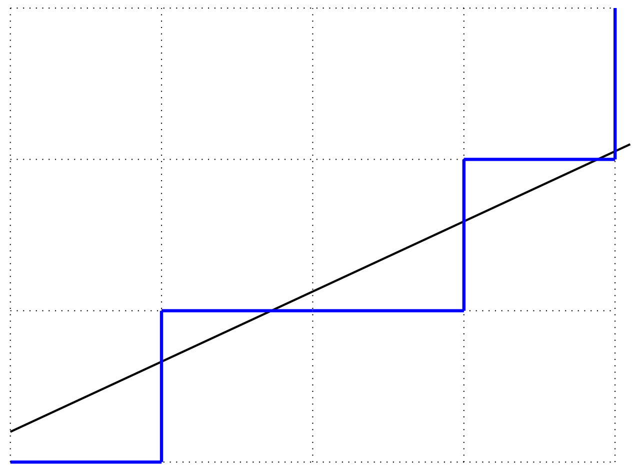

Figure 2. A piece of a broken line associated with the sequence $1211212\cdots$.

space $\mathbf{1}^\perp$ with nonempty interior such that

$$\Lambda = \{\tilde{\pi}_{\boldsymbol{\alpha}}(\mathbf{x}) \,:\, \mathbf{x} \in \mathbb{Z}^d,\ \pi_{\boldsymbol{\alpha}}(\mathbf{x}) \in W\}.$$

We choose here suitable acceptance windows (as finite unions of convex polytopes) such that the set of points $\{\mathbf{x} \,:\, \mathbf{x} \in \mathbb{N}^d,\ \pi_{\boldsymbol{\alpha}}(\mathbf{x}) \in W\}$ forms a half broken line $\mathbf{L}_u$ associated with some sequence u, with also the sequence u having a small discrepancy. The half broken line $\mathbf{L}_u$ discussed above is the set of points $\mathbf{x}$ in $\mathbb{Z}^d$ such that $\pi_{\boldsymbol{\alpha}}(\mathbf{x}) \in W$. The subset Λ will then be the projection by $\tilde{\pi}_{\boldsymbol{\alpha}}$ of the set of vertices of the broken line.

We now sketch the contents of this paper. In Section 2, we recall basic definitions, in particular basics from symbolic dynamics, and the notions of exchange of domains and that of a natural coding. In Section 3, we recall and compare two constructions of sequences with small discrepancy, hypercubic billiard sequences in Section 3.2, and then Tijdeman's construction from [Tij80] in Section 3.3. We prove that both constructions are obtained by coding the same toral translation $T_{\boldsymbol{\alpha}}$ with respect to partitions by finite unions of convex polytopes. The proof of Theorem 1.1 is given in Section 4. We end this paper with questions in Section 5.

Acknowledgements. The authors thank J. Abou Samra for his careful reading and J. Walton for his insight on factor complexity for cut and project sets.

2. Notation and basic definitions

2.1. **Word combinatorics and symbolic dynamics.** Let $\mathcal{A} = \{1, 2, \ldots, d\}$ be a finite alphabet. We denote by ε the empty word of the free monoid $\mathcal{A}^*$, and by $\mathcal{A}^{\mathbb{N}}$ the set of sequences over $\mathcal{A}$. For $i \in \mathcal{A}$ and for $w \in \mathcal{A}^*$, let $|w|_i$ denote the number of occurrences of the letter i in the word w, and let $|w|$ denote the length of w. The k-th letter of w is denoted as w_k, where we always label indices starting at 0, i.e., $w = w_0 w_1 \cdots w_{|w|-1}$. If w is a word or a sequence, and $j \leq k$ are nonnegative integers, let $w_{[j,k)} := w_j \cdots w_{k-1}$

(where $w_{[k,k)}$ is the empty word). In a finite word or sequence u, any word of the form $u_{[j,k)}$ is called a *factor*. The set of factors $\mathcal{L}(u)$ of a sequence u is called its *language*. The *factor complexity* of the sequence u is the function which, given $n \in \mathbb{N}$, counts the number of factors of u of length n. Given a word w and a letter i, let $|w|_i$ denote the number of occurrences of i in w, and for $w \in \mathcal{A}^*$, let

$$\mathbf{p}(w) = (|w|_i)_{i\in\mathcal{A}}$$

denote its *Parikh vector*.

Shifts. Let S denote the *shift map* acting on $\mathcal{A}^{\mathbb{N}}$, i.e., $S((u_n)_{n\in\mathbb{N}}) = (u_{n+1})_{n\in\mathbb{N}}$. A *shift* is a pair (X, S) where X is a closed shift-invariant subset of some $\mathcal{A}^{\mathbb{N}}$; X is called a *shift space*. Here, $\mathcal{A}$ is equipped with the discrete topology, and $\mathcal{A}^{\mathbb{N}}$ is equipped with the product topology. One associates with any sequence $u \in \mathcal{A}^{\mathbb{N}}$ the symbolic dynamical system (X_u, S), where the shift space $X_u \subset \mathcal{A}^{\mathbb{N}}$ is defined as $X_u = \{v \in \mathcal{A}^{\mathbb{N}} : \mathcal{L}(v) \subset \mathcal{L}(u)\}$. A shift (X, S) is said to be *minimal* if X admits no nontrivial closed and shift-invariant subset. If X is a shift space, then its *language* $\mathcal{L}(X)$ is defined as the set of factors of elements of X. For any $n \geq 1$, we let $\mathcal{L}_n(X)$ denote the set of factors of length n of elements in X. The *factor complexity* $p_X(n)$ of a minimal shift (X, S) is defined as $p_X(n) = \mathrm{Card}\mathcal{L}_n(X)$.

Frequencies and invariant measures. Let u be a sequence in $\mathcal{A}^{\mathbb{N}}$ and let $v \in \mathcal{A}^*$. If the limit $\lim_{n\to+\infty} \big(|u_{[0,n)}|_v\big)/n$ exists, then we call it the *frequency* of v in u, and denote it by α_v (suppressing the dependence on u). Assume that u is such that the frequencies of the factors of u all exist. Then u is said to have *uniform frequencies* if, for every word v, the convergence $|u_{[k,n)}|_v/(n-k) \to \alpha_v$ is uniform in k.

Let (X, S) be a shift with $X \subset \mathcal{A}^{\mathbb{N}}$. A probability measure μ on X is said to be *S-invariant* if $\mu(S^{-1}B) = \mu(B)$ for every Borel set $B \subset X$. The shift (X, S) is *uniquely ergodic* if there exists a unique shift-invariant probability measure on X; this is the case if and only if every word $u \in X$ has uniform factor frequencies [BR10, Proposition 7.2.10]. In that case, one recovers the frequency α_v of a factor $v = v_0 \cdots v_{n-1}$ as $\alpha_v = \mu([v])$, with the *cylinder* $[v] := \{u \in X : u_{[0,n)} = v\}$. For more on invariant measures and ergodicity, we refer to [Que10] and [BR10, Chap. 7].

Balance and discrepancy. Let $u \in \mathcal{A}^{\mathbb{N}}$. A sequence u is said to be *B-balanced* if there exists a constant B such that, for every letter $i \in \mathcal{A}$ and for every pair (w, w') of words in $\mathcal{L}(u)$ with $|w| = |w'|$, we have $|w|_i - |w'|_i \leq B$. We then define

$$B_u = \max_{i\in\mathcal{A}} \sup_{w,w'\in\mathcal{L}(u)\,:\,|w|=|w'|} \big(|w|_i - |w'|_i\big).$$

Note that the first papers devoted to balance were concerned with 1-balance for letters; see e.g. [Lot02].

Let $u \in \mathcal{A}^{\mathbb{N}}$ and assume that letters i admit frequencies α_i in u. The *(letter) discrepancy* of u is defined as the (possibly infinite) quantity

$$\Delta_{\boldsymbol{\alpha}}(u) = \sup_{n\in\mathbb{N}} \|\mathbf{p}(u_{[0,n)}) - n\boldsymbol{\alpha}\|_\infty,$$

with the frequency vector $\boldsymbol{\alpha} = (\alpha_i)_{i\in\mathcal{A}}$, and $n\boldsymbol{\alpha} - \mathbf{p}(u_{[0,n)})$ is called a *discrepancy vector*.

These definitions extend to any minimal shift (X,S) in a straightforward way.

Let u be a sequence for which the letter frequencies exist. It has bounded letter balance if and only if the discrepancy is finite; see [Ada03, Ada04]. Moreover, one has

$$\Delta_{\boldsymbol{\alpha}}(u) \le B_u \le 4\Delta_{\boldsymbol{\alpha}}(u)$$

by the triangle inequality; see [Ada03, Proposition 7 and Remark 8]. One advantage of the notion of balance is that one does not need to know in advance the frequency vector $\boldsymbol{\alpha}$. Balance is also equivalently formulated as having bounded abelian complexity, where the abelian complexity counts the number of distinct Parikh vectors of factors of a given length; see e.g. [RSZ11]. For related results on the abelian complexity, see [DMP23].

The following holds for fairly distributed sequences over small alphabets for rationally independent coordinates. By Remark 3.4, not all Sturmian sequences are fairly distributed.

Proposition 2.1. *Let $\boldsymbol{\alpha}$ be a totally irrational frequency vector, and let u be a fairly distributed sequence with letter frequency $\boldsymbol{\alpha}$, i.e., $\Delta_{\boldsymbol{\alpha}}(u) \le 1-\frac{1}{2d-2}$. If $d=2$, then u is Sturmian and its balance B_u equals 1. If $d=3$, then $B_u \le 2$.*

Proof. Consider two factors w and w' of u of the same length and a letter a in the alphabet $\mathcal{A}$ on which u is defined. Let n be such that both factors w and w' occur in u at indices smaller than n. Since α is irrational, one has, for $0 \le k \le n$, $|u_0 \cdots u_{k-1}|_a - k\alpha_a$ irrational, hence the value $1-\frac{1}{2d-2}$ is not attained and $|u_0 \cdots u_{n-1}|_a - n\alpha_a < 1-\frac{1}{2d-2}$. By the triangle inequality,

$$\big||w|_a - |w'|_a\big| \le 4 \max_{0\le k \le n} \big||u_0 \cdots u_{k-1}|_a - k\alpha_a\big| < 4\left(1-\tfrac{1}{2d-2}\right).$$

Assume $d=2$. Since $|w|_a - |w'|_a$ takes integer values, $|w|_a - |w'|_a \le 1$, and so its balance B_u satisfies $B_u \le 1$. Since u is one-sided and $\boldsymbol{\alpha}$ is totally irrational, then $B_u = 1$ and u is a Sturmian sequence. If $d=3$, we similarly get that $B_u \le 2$. □

Discrete spectrum. Recall that two measure preserving systems (X,S,μ) and (Y,T,ν) are *measurably conjugate* if there are measurable sets $X_0 \subset X$ and $Y_0 \subset Y$, each of measure 1, and a measurable bijection $\Phi : X_0 \to Y_0$ which intertwines the action, $T \circ \Phi = \Phi \circ S$. A shift (X,S,μ) has *purely discrete spectrum* if the measurable eigenfunctions of the *Koopman operator* $U_S : L^2(X,S,\mu) \to L^2(X,S,\mu)$, $f \mapsto f \circ S$, span $L^2(X,S,\mu)$. This is equivalent to the shift being measurably conjugate to a translation on a compact abelian group. If the system (X,S) is uniquely ergodic, then we write (X,S) instead of (X,S,μ). Here, the shifts under consideration will be conjugate to $(\mathbb{T}^{d-1}, T_{\boldsymbol{\alpha}})$.

2.2. **Exchange of pieces, toral translations and natural codings.** Let $\boldsymbol{\alpha}$ be a frequency vector, i.e., a vector in $(0,1)^d$ with $\sum_{i=1}^d \alpha_i = 1$. We recall that $\boldsymbol{\alpha}$ is said to be totally irrational if it has rationally independent coordinates. We consider a frequency vector $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_d) \in (0,1)^d$ and the toral translation by $\boldsymbol{\alpha}$, denoted as $T_{\boldsymbol{\alpha}}$, defined on $\mathbb{T}^{d-1} = \mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$ by

$$T_{\boldsymbol{\alpha}} : \mathbb{T}^{d-1} \to \mathbb{T}^{d-1}, \quad \mathbf{x} \mapsto \mathbf{x} + (\alpha_1,\ldots,\alpha_{d-1}) \pmod{\mathbb{Z}^{d-1}}. \tag{2.1}$$

A *topological dynamical system*, i.e., a pair (X,T) where X is a compact metric space and $T : X \to X$ is a homeomorphism, is *minimal* when it does not contain any non-empty proper closed T-invariant subset. In other words, minimality for $(\mathbb{T}^{d-1}, T_{\boldsymbol{\alpha}})$ means that the orbit of any point is dense in $\mathbb{T}^{d-1}$; it is equivalent to the fact that $\boldsymbol{\alpha}$ is totally irrational (i.e., the coordinates of $(\alpha_1, \ldots, \alpha_{d-1}, 1)$ are rationally independent). We recall that a minimal translation is also uniquely ergodic; see e.g. [Wal82].

We want to provide symbolic codings of the translation $T_{\boldsymbol{\alpha}}$ with respect to finite partitions (by polytopes) of fundamental domains of $\mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$. We consider in particular partitions of fundamental domains of $\mathbb{T}^{d-1}$ that are well adapted to the action of $T_{\boldsymbol{\alpha}}$, in the sense that *on each atom the map $T_{\boldsymbol{\alpha}}$ is a translation by a vector*. They are called *natural partitions*. Let us state a few definitions in order to make this notion more precise. For the next definition, we follow [BST23, Section 2.4], originally stated in the case that $V = \mathbb{R}^{d-1}$ and $\Lambda = \mathbb{Z}^{d-1}$. We let denote by Leb_V, or Leb when there is no ambiguity, the Lebesgue measure on a finite dimensional real vector space V. The interior of a set P is denoted as $\mathring{P}$, its closure as $\overline{P}$, and its boundary as ∂P.

Definition 2.2 (Fundamental domains and natural partitions). Let V be a finite dimensional real vector space, Λ a full rank lattice in V, and $\boldsymbol{\alpha} \in V$. A *measurable fundamental domain* of the torus V/Λ is a measurable set $P \subset V$ that satisfies

$$P + \Lambda = V \text{ and } \mathrm{Leb}_V(P \cap (P + \mathbf{n})) = 0 \text{ for all } \mathbf{n} \in \Lambda \setminus \{0\}.$$

Let P be a measurable fundamental domain of the torus V/Λ, We consider the translation

$$T_{\boldsymbol{\alpha}} : V/\Lambda \to V/\Lambda,\ x \mapsto x + \boldsymbol{\alpha} \mod \Lambda,$$

which we assume to be minimal. A collection $\{P_1, \ldots, P_h\}$ is said to be a *natural partition* (it is a partition up to zero measure sets) of P with respect to $T_{\boldsymbol{\alpha}}$ if

(1) $\bigcup_{i=1}^h P_i = P$;
(2) $\mathrm{Leb}_V(P_i \cap P_j) = 0$ for all $i \neq j$, $1 \leq i, j \leq h$;
(3) each P_i, $1 \leq i \leq h$, is the closure of its interior and $\mathrm{Leb}_V(\partial P_i) = 0$;
(4) there exist $\mathbf{t}_1, \ldots, \mathbf{t}_h \in V$ with $\mathbf{t}_i \equiv \boldsymbol{\alpha} \bmod \Lambda$ such that $\mathbf{t}_i + P_i \subset P$, $1 \leq i \leq h$.

A natural partition is called *bounded* if the set P is bounded.

In the following, we shall consider fundamental domains and natural partitions in the vector space $V = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_{i=1}^d x_i = 0\}$, with $\Lambda = V \cap \mathbb{Z}^d$, V/Λ being isomorphic to $\mathbb{T}^{d-1}$, and we shall also project these fundamental domains and natural partitions onto $\mathbb{R}^{d-1}$. Note that we keep the same notation as before for $T_{\boldsymbol{\alpha}}$ in order to simplify the notation. Note also that we focus here on partitions by finite unions of convex polytopes (e.g. as in Figure 1), hence Condition (3) is satisfied.

Such a natural partition $\{P_1, \ldots, P_h\}$ allows us to define a.e. on P a map $\tau_{\mathbf{t}} : P \to P$ as an *exchange of domains* (which depends on the partition) by

$$\tau_{\mathbf{t}}(\mathbf{x}) = \mathbf{x} + \mathbf{t}_i \quad \text{whenever } \mathbf{x} \in \mathring{P}_i. \tag{2.2}$$

The map $\tau_{\mathbf{t}}$ is defined on $P \setminus \bigcup_{i=1}^h \partial P_i$, hence, it is defined almost everywhere. The dynamical system $(P, \tau_{\mathbf{t}}, \mathrm{Leb}|_P)$ is measurably conjugate to $(V/\Lambda, T_{\boldsymbol{\alpha}})$ (endowed with the Haar measure). One has for a.e. $\mathbf{x} \in P$, $\tau_{\mathbf{t}}(\mathbf{x}) \equiv T_{\boldsymbol{\alpha}}(\mathbf{x}) \bmod \Lambda$. The collection

$\{P_1{+}\mathbf{t}_1, \dots, P_h{+}\mathbf{t}_h\}$ also forms a measurable natural partition of P, hence the terminology exchange of domains.

Now that we have seen how exchanges of pieces act as toral translations, we discuss their symbolic codings. We continue with the notation of Definition 2.2.

Definition 2.3 (Natural coding). A sequence $(u_n)_{n\in\mathbb{N}} \in \{1, \dots, h\}^{\mathbb{N}}$ is said to be a *natural coding* of the minimal toral translation $(V/\Lambda, T_{\boldsymbol{\alpha}})$ w.r.t. the natural partition $\{P_1, \dots, P_h\}$ if there exists $\mathbf{x} \in P$ such that $(u_n)_{n\in\mathbb{N}}$ codes the orbit of $\mathbf{x}$ under the action of $\tau_{\mathbf{t}}$, i.e.,

$$\tau_{\mathbf{t}}^n(\mathbf{x}) = \mathbf{x} + \sum_{k=0}^{n-1} \mathbf{t}_{u_k} \in \mathring{P}_{i_n}$$

for all $n \in \mathbb{N}$; note that $T_{\boldsymbol{\alpha}}^n(\mathbf{x}) \equiv \tau_{\mathbf{t}}^n(\mathbf{x}) \bmod \Lambda$. If u is a natural coding of $(V/\Lambda, T_{\boldsymbol{\alpha}})$ w.r.t. a natural partition $\{P_1, \dots, P_h\}$ whose elements $P_1, \dots, P_h$ are bounded, we call u a *bounded natural coding*.

Remark 2.4. Bédaride and Bertazzon [BB13] have shown that a natural partition associated with a minimal translation $T_{\boldsymbol{\alpha}}$ of $\mathbb{T}^{d-1}$ has at least d pieces, hence the alphabet of a natural coding has at least d letters.

Remark 2.5. When P is bounded, the natural codings of two different points $\mathbf{x}, \mathbf{y} \in P$ cannot be equal. Indeed, if they were equal, we would have by definition $\tau_{\mathbf{t}}^n(\mathbf{y}) = \tau_{\mathbf{t}}^n(\mathbf{x}){+}\mathbf{y}{-}\mathbf{x}$ for all $n \in \mathbb{N}$, but using the minimality of the toral translation $T_{\boldsymbol{\alpha}}$, we can see that, for any $\mathbf{u} \neq \mathbf{0}$, there exists some $n \in \mathbb{N}$ such that $\tau_{\mathbf{t}}^n(\mathbf{x}){+}\mathbf{u} \notin P$, so that $\tau_{\mathbf{t}}^n(\mathbf{y}) = \tau_{\mathbf{t}}^n(\mathbf{x}){+}\mathbf{y}{-}\mathbf{x} \notin P$ for some n, a contradiction.

The shift (X_u, S) generated by a natural coding u of the minimal translation $(V/\Lambda, T_{\boldsymbol{\alpha}})$ is minimal, uniquely ergodic, and has purely discrete spectrum according to [BST23, Lemma 5.12] or [Che09, Theorems A and B]. Hence the two main steps in the proof of Theorem 1.1 are, firstly, to exhibit the partition providing a natural coding (see Section 3) and, secondly, to estimate the factor complexity (see Sections 4.1 to 4.4).

2.3. **Hypercubic fundamental domains.**

We now illustrate the formalism developed in the previous section with a very simple choice of a fundamental domain for $\mathbb{T}^{d-1}$, following e.g. [AMST94a]. This will provide a simple geometric model for $T_{\boldsymbol{\alpha}}$ defined in (2.1) as an exchange of pieces, that will play a crucial role in Section 3.

Define

$$F_i = \{(x_1, \dots, x_d) \in [0,1]^d : x_i = 1\} \qquad (1 \le i \le d) \tag{2.3}$$

to be an *upper face* of the d-dimensional unit cube, and similarly

$$\tilde{F}_i = \{(x_1, \dots, x_d) \in [0,1]^d : x_i = 0\}$$

as a *lower face*. As an illustration, the union of the three lower faces when $d = 3$ is depicted as , see also Figure 4, and Figure 3 for $d = 2$. Let $\mathbf{1}$ be the vector all of whose entries equal one and define its orthogonal complement

$$\mathbf{1}^\perp := \Big\{(x_1,\ldots,x_d) \in \mathbb{R}^d \,:\, \sum_{i=1}^d x_i = 0\Big\}.$$

Denote by

$$\pi_{\boldsymbol\alpha} : \mathbb{R}^d \to \mathbf{1}^\perp \text{ the projection along } \mathbb{R}\boldsymbol\alpha \text{ onto } \mathbf{1}^\perp,$$

i.e., for $1 \le i \le d$,

$$\pi_{\boldsymbol\alpha}(\mathbf{e}_i) = \mathbf{e}_i - \boldsymbol\alpha. \tag{2.4}$$

Let

$$E_{\boldsymbol\alpha} = \bigcup_{i=1}^d E_{\boldsymbol\alpha,i} \quad \text{with} \quad E_{\boldsymbol\alpha,i} = \pi_{\boldsymbol\alpha}(F_i). \tag{2.5}$$

First observe that

$$E_{\boldsymbol\alpha} = \pi_{\boldsymbol\alpha}([0,1]^d).$$

Indeed, we have $E_{\boldsymbol\alpha} \subseteq \pi_{\boldsymbol\alpha}([0,1]^d)$, and for $\mathbf{x} = \sum_{i=1}^d x_i \pi_{\boldsymbol\alpha}(\mathbf{e}_i)$, $x_i \in [0,1]$, we use that

$$\sum_{i=1}^d \alpha_i \pi_{\boldsymbol\alpha}(\mathbf{e}_i) = \sum_{i=1}^d \alpha_i\,(\mathbf{e}_i - \boldsymbol\alpha) = \boldsymbol\alpha - \boldsymbol\alpha = \mathbf{0}$$

to obtain that, for j satisfying $\frac{1-x_j}{\alpha_j} = \min\{\frac{1-x_i}{\alpha_i} : 1 \le i \le d\}$,

$$\mathbf{x} = \sum_{i=1}^d x_i \pi_{\boldsymbol\alpha}(\mathbf{e}_i) + \frac{1-x_j}{\alpha_j} \sum_{i=1}^d \alpha_i \pi_{\boldsymbol\alpha}(\mathbf{e}_i) = \sum_{i=1}^d \Big(x_i + \frac{1-x_j}{\alpha_j}\alpha_i\Big)\pi_{\boldsymbol\alpha}(\mathbf{e}_i) \in \pi_{\boldsymbol\alpha}(F_j) \subset E_{\boldsymbol\alpha}.$$

We also obtain that

$$E_{\boldsymbol\alpha,j} \cap E_{\boldsymbol\alpha,k} = \bigg\{ \sum_{i=1}^d x_i\, \pi_{\boldsymbol\alpha}(\mathbf{e}_i) \,:\, x_j = x_k = 1,\, x_i \in [0,1] \text{ for } i \notin \{j,k\} \bigg\}$$

is a $(d{-}2)$-dimensional subset of $\mathbf{1}^\perp$, hence $\bigcup_{i=1}^d E_{\boldsymbol\alpha,i}$ forms a *topological partition* of $E_{\boldsymbol\alpha}$, i.e., the atoms $E_{\boldsymbol\alpha,i}$ have disjoint interiors.

We then consider the polyhedral exchange map

$$\tilde{T}_{\boldsymbol\alpha} : E_{\boldsymbol\alpha} \to E_{\boldsymbol\alpha}, \quad \mathbf{x} \mapsto \mathbf{x} + \boldsymbol\alpha - \mathbf{e}_i \quad \text{if } \mathbf{x} \in E_{\boldsymbol\alpha,i}. \tag{2.6}$$

Here and in the following, we neglect the intersections $E_{\boldsymbol\alpha,i} \cap E_{\boldsymbol\alpha,j}$, $i \ne j$. Then we have

$$\tilde{T}_{\boldsymbol\alpha}(E_{\boldsymbol\alpha,i}) = E_{\boldsymbol\alpha,i} - \pi_{\boldsymbol\alpha}(\mathbf{e}_i) = \pi_{\boldsymbol\alpha}(\tilde{F}_i).$$

Since the $\tilde{F}_i$'s are lower faces of $[0,1]^d$, we have $E_{\boldsymbol\alpha} = \bigcup_{i=1}^d \tilde{T}_{\boldsymbol\alpha}(E_{\boldsymbol\alpha,i})$, hence the map $\tilde{T}_{\boldsymbol\alpha}$ is an *exchange of pieces* (in $\mathbf{1}^\perp$); see Figure 3 for $d = 2$ as an illustration.

Since $\{\mathbf{x}{+}\pi_{\boldsymbol\alpha}([0,1]^d) : \mathbf{x} \in \mathbb{Z}^d \cap \mathbf{1}^\perp\}$ forms a tiling of $\mathbf{1}^\perp$, the set $E_{\boldsymbol\alpha}$ is a measurable fundamental domain of $\mathbf{1}^\perp/(\mathbb{Z}^d \cap \mathbf{1}^\perp)$. Indeed, let us first show that the translates of $E_{\boldsymbol\alpha}$ cover $\mathbf{1}^\perp$. Let $\mathcal{Q}^+$ be the union of all the unit hypercubes in the lattice that are included in the closed half space $H^+ = \{\mathbf{x} \in \mathbb{R}^d : l(\mathbf{x}) = x_1{+}\cdots{+}x_d \ge 0\}$. Since $\pi_\alpha(\partial\mathcal{Q}^+) = \mathbf{1}^\perp$,

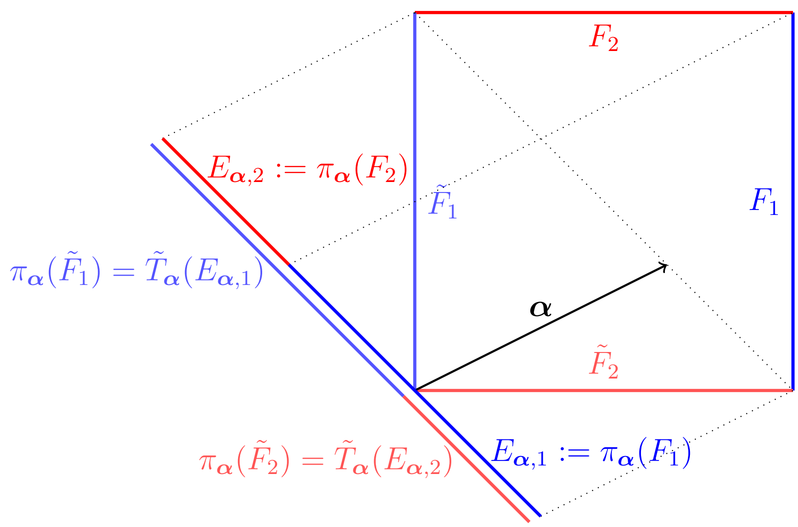

FIGURE 3. Exchange of pieces, $d = 2$.

it is enough to show that the boundary $\partial\mathcal{Q}^+$ is included in the union of all the lower faces of the lattice unit hypercubes contained in H^+ with one vertex in $\mathbf{1}^\perp$. On the one hand, $\partial\mathcal{Q}^+$ is contained in H^+ and is a union of $(d-1)$-dimensional faces of unit lattice hypercubes. On the other hand, $\partial\mathcal{Q}^+ = \partial\mathcal{Q}^-$ where $\partial\mathcal{Q}^-$ is the union of all the lattice unit hypercubes not included in H^+, and therefore $\partial\mathcal{Q}^+$ is a union of faces of lattice unit hypercubes with at least one vertex in $\{\mathbf{x} \in \mathbb{R}^d : l(\mathbf{x}) \leq -1\}$, which in turn implies that $\partial\mathcal{Q}^+ \subset \{\mathbf{x} \in \mathbb{R}^d : l(\mathbf{x}) \leq d-1\}$. It follows that each face of $\partial\mathcal{Q}^+$ has a vertex in $\mathbf{1}^\perp$. This shows that the translates of $E_{\boldsymbol{\alpha}}$ cover $\mathbf{1}^\perp$. Lastly, by inspection of neighbouring tiles, one checks that they intersect on a set of measure zero.

To obtain the toral translation $T_{\boldsymbol{\alpha}}$ (see (2.1)) from $\tilde{T}_{\boldsymbol{\alpha}}$ (defined in (2.6)), we now omit the last coordinate, i.e., we consider the conjugation by

$$\iota : \mathbf{1}^\perp \to \mathbb{R}^{d-1}, \quad (x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_{d-1}).$$

Then $\iota(E_{\boldsymbol{\alpha}})$ is a measurable fundamental domain of $\mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$, and $\iota \circ \tilde{T}_{\boldsymbol{\alpha}} = T_{\boldsymbol{\alpha}} \circ \iota$. Moreover, the collection $\{\iota(E_{\boldsymbol{\alpha},1}), \ldots, \iota(E_{\boldsymbol{\alpha},d})\}$ is a bounded natural partition with respect to $T_{\boldsymbol{\alpha}}$, according to Definition 2.2. Indeed, the map $\iota \circ \tilde{T}_{\boldsymbol{\alpha}}$ coincides with the map $\tau_{\mathbf{t}}$ from (2.2) with vectors $\mathbf{t}_1 = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_{d-1})$, …, $\mathbf{t}_{d-1} = (\alpha_1, \ldots, \alpha_{d-2}, \alpha_{d-1} - 1)$, and $\mathbf{t}_d = (\alpha_1, \alpha_2, \ldots, \alpha_{d-1})$. They all satisfy $\mathbf{t}_i \equiv (\alpha_1, \ldots, \alpha_{d-1}) \mod \mathbb{Z}^{d-1}$.

In fact, we have shown the following proposition.

Proposition 2.6. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (0,1)^d$ be a totally irrational frequency vector. Then $\iota(E_{\boldsymbol{\alpha}})$ is a measurable fundamental domain of $\mathbb{T}^{d-1}$ admitting the bounded natural partition $\iota(E_{\boldsymbol{\alpha}}) = \bigcup_{i=1}^d \iota(E_{\boldsymbol{\alpha},i})$. Moreover, the map $\tilde{T}_{\boldsymbol{\alpha}} : E_{\boldsymbol{\alpha}} \to E_{\boldsymbol{\alpha}}$ is conjugate to the minimal translation $T_{\boldsymbol{\alpha}}$ on $\mathbb{T}^{d-1}$.*

The above proof shows that $\iota(E_{\boldsymbol{\alpha}})$ is a measurable fundamental domain even if $\alpha_1, \ldots, \alpha_d$ are dependent over $\mathbb{Q}$. See Figure 4 for an illustration of $\iota(E_{\boldsymbol{\alpha}})$ and the image by $\tilde{T}_{\boldsymbol{\alpha}}$ for $d = 3$.

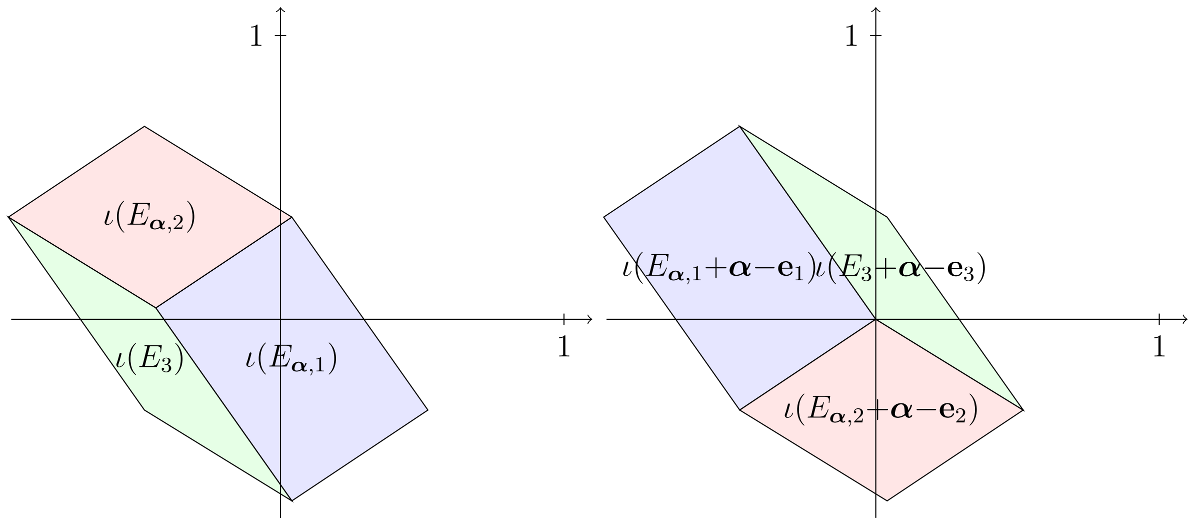

FIGURE 4. The (ι-representation of the) parallelepipeds $E_{\boldsymbol{\alpha},i}$ and their $\tilde{T}_{\boldsymbol{\alpha}}$-images for $d = 3$, $\boldsymbol{\alpha} = (0.48, 0.32, 0.2)$.

2.4. **Construction of natural partitions.** We have considered in the previous section a very simple choice of a fundamental domain. In the present section, we consider a more general situation, and we prove two technical propositions that allow the construction of a measurable fundamental domain (Proposition 2.8), and of a natural partition with respect to a toral translation (Proposition 2.9). This rather general construction will be used with Tijdeman sequences (see Section 3.3). Among others, a technical aspect comes from the fact that we are handling maps that are not continuous, but only piecewise continuous; see Remark 2.7 below.

Similarly to Definition 2.2, we consider a finite dimensional real vector space V, a full rank lattice Λ in V and $\boldsymbol{\alpha} \in V$. Let R be a nonempty subset of V with a finite partition $R = R_1 \cup R_2 \cup \cdots \cup R_h$. For each $i \in \{1, \ldots, h\}$, fix some $\mathbf{n}_i \in \Lambda$. Consider the maps

$$T_i : V \to V,\ \mathbf{x} \mapsto \mathbf{x} + \boldsymbol{\alpha} + \mathbf{n}_i \quad \text{and} \quad \widehat{T} : R \to V,\ \mathbf{x} \mapsto T_i(\mathbf{x}) \quad \text{when } \mathbf{x} \in R_i.$$

Finally, let $\mathbf{x}_0 \in R$. We make the following assumptions.

(1) The translation $T_{\boldsymbol{\alpha}} : V/\Lambda \to V/\Lambda$, $x \mapsto x+\boldsymbol{\alpha} \mod \Lambda$ is minimal;
(2) the set R is closed and $T_i(R_i) \subset R$ for all $i \in \{1, \ldots, h\}$;
(3) $\mathrm{Leb}_V(\overline{R}_i \cap \overline{R}_j) = 0$ for all $i \neq j \in \{1, \ldots, h\}$;
(4) there exist a compact set $K \subset R$ and a nonempty open set $U \subset K$ such that K contains the orbit $\{\widehat{T}^n(\mathbf{x}_0) : n \in \mathbb{N}\}$, and moreover any point $\mathbf{x} \in U$ has only one

representative mod Λ in K, i.e.,

$$\mathbf{x} + \mathbf{n} \notin K \text{ for all } \mathbf{n} \in \Lambda \setminus \{\mathbf{0}\};$$

(5) for all $i \in \{1, \dots, h\}$, R_i is a finite union of nearly convex sets, with nonempty interiors, that are finite intersections of open half spaces. (Recall that $A \subset V$ is *nearly convex* if there exists a convex set $B \subset V$ such that $B \subset A \subset \overline{B}$.)

The following proposition allows us to cope with the fact that $\widehat{T}$ is only piecewise continuous. A crucial argument in the proof is that, thanks to Assumptions (1) and (4), any $\widehat{T}$-backward orbit that lies in K must enter the open set U.

Remark 2.7. If (X, T) is a compact continuous dynamical system and if $x \in X$, then any point $y \in \overline{\{T^n(x) : n \in \mathbb{N}\}} \setminus \{x\}$ is the image by T of some element in $\overline{\{T^n(x) : n \in \mathbb{N}\}}$; in other words, the image by T of the closure of the orbit of x contains the closure of the orbit except perhaps x. This property no longer holds in the following case with T not being continuous. Let $T : [0, 1] \to [0, 1]$ be defined by $T(x) = x+\alpha$ for $x \in [0, 1-\alpha]$ and $T(x) = x+\alpha-1$ for $x \in (1-\alpha, 1]$, where $\alpha \in [0, 1] \setminus \mathbb{Q}$. One has $0 \notin T(X)$, whereas the orbit closure of any point is X. However, the following proof shows that we can overcome this up to a negligible set.

Proposition 2.8. *If Assumptions (1)–(4) hold, then* $D = \overline{\{\widehat{T}^n(\mathbf{x}_0) : n \in \mathbb{N}\}}$ *is a measurable fundamental domain of the torus* V/Λ*.*

Proof. The projection of D onto the torus V/Λ is a compact set that contains the sequence of projections of the points $\widehat{T}^n(\mathbf{x}_0)$, $n \in \mathbb{N}$, hence the projection of D is the whole torus by the minimality assumption from (1). It follows that $\bigcup_{\mathbf{n}\in\Lambda}(D+\mathbf{n}) = V$.

We now want to prove that $\mathbf{y}-\mathbf{y}' \in \Lambda$ implies $\mathbf{y} = \mathbf{y}'$ for all $\mathbf{y}, \mathbf{y}' \in D \setminus \mathcal{N}$, where $\mathcal{N}$ is Lebesgue-null. If we can find a Lebesgue-null set $\mathcal{N}$ such that for all $y \in D \setminus \mathcal{N}$ and all $n \in \mathbb{N}$, there exists $\mathbf{x} \in D$ such that $\widehat{T}^n(\mathbf{x}) = y$, we are done. Indeed, let $\mathbf{y}, \mathbf{y}' \in D \setminus \mathcal{N}$ be such that $\mathbf{y}-\mathbf{y}' \in \Lambda$. Since the translation $T_{\boldsymbol{\alpha}}$ is minimal and since U is a nonempty open set, there exists $m \in \mathbb{N}$ such that $\mathbf{y}-m\boldsymbol{\alpha} \in U+\Lambda$. By our assumption on $\mathcal{N}$, there exist $\mathbf{z}, \mathbf{z}' \in D$ such that $\widehat{T}^m(\mathbf{z}) = \mathbf{y}$ and $\widehat{T}^m(\mathbf{z}') = \mathbf{y}'$. By definition of $\widehat{T}$, we have $\mathbf{z} = \mathbf{y}-m\boldsymbol{\alpha} \mod \Lambda$ and $\mathbf{z}' = \mathbf{y}'-m\boldsymbol{\alpha} \mod \Lambda$, so that $\mathbf{z}-\mathbf{z}' \in \Lambda$. Now $\mathbf{z} \in U \mod \Lambda$ and $\mathbf{z}, \mathbf{z}' \in D \subset K$, hence by Assumption (4), $\mathbf{z} = \mathbf{z}'$ which in turn implies $\mathbf{y} = \mathbf{y}'$.

It remains to define $\mathcal{N}$. Let

$$\mathcal{N} = \bigcup_{k\geq 0} \widehat{T}^k \mathcal{N}_0, \quad \text{with} \quad \mathcal{N}_0 = \big\{\widehat{T}^n(\mathbf{x}_0) : n \in \mathbb{N}\big\} \cup \bigcup_{i,j\in\{1,\dots,h\},\, i\neq j} T_i\big(\overline{R}_i \cap \overline{R}_j\big).$$

Thanks to Assumption (3), $\mathcal{N}$ is a null set. Let us show first that, if $\mathbf{y} \in D \setminus \mathcal{N}_0$, then $\widehat{T}(\mathbf{x}) = \mathbf{y}$ for some $\mathbf{x} \in D$. Since $\mathbf{y} \notin \bigcup_{i\neq j} T_i(\overline{R}_i \cap \overline{R}_j)$, $\varepsilon = \mathrm{d}(\mathbf{y}, \bigcup_{i\neq j} T_i(\overline{R}_i \cap \overline{R}_j)) > 0$. Since $\mathbf{y} \neq \widehat{T}^n(\mathbf{x}_0)$ for all $n \in \mathbb{N}$, there exists an increasing sequence of integers $(n_k)_k$ such that $\mathbf{y} = \lim_{k\to\infty} \widehat{T}^{n_k}(\mathbf{x}_0)$ and $\widehat{T}^{n_k}(\mathbf{x}_0) \in B(\mathbf{y}, \varepsilon/2)$ for all integers k. By passing to a subsequence, we can suppose that $\widehat{T}^{n_k}(\mathbf{x}_0) = T_{i_0}(\widehat{T}^{n_k-1}(\mathbf{x}_0))$ for all k and some fixed $i_0 \in \{1, \dots, h\}$. For all $k \in \mathbb{N}$ and all $j \in \{1, \dots, h\}$ with $j \neq i_0$, since $\widehat{T}^{n_k}(\mathbf{x}_0) \in B(\mathbf{y}, \varepsilon/2)$

and $\mathrm{d}(\mathbf{y}, \bigcup_{i\neq j} T_i(\overline{R}_i \cap \overline{R}_j)) = \varepsilon$, we have $\mathrm{d}(T_{i_0}(\widehat{T}^{n_k-1}(\mathbf{x}_0)), T_{i_0}(\overline{R}_{i_0} \cap \overline{R}_j)) \geq \varepsilon/2$, hence $\mathrm{d}(\widehat{T}^{n_k-1}(\mathbf{x}_0), \overline{R}_{i_0} \cap \overline{R}_j) \geq \varepsilon/2$. Again by passing to a subsequence, we can suppose that $\lim_{k\to\infty} \widehat{T}^{n_k-1}(\mathbf{x}_0) = \mathbf{x} \in \overline{R}_{i_0} \cap R$. Since $\mathrm{d}(\mathbf{x}, \overline{R}_{i_0} \cap \overline{R}_j) \geq \varepsilon/2$ for all $j \neq i_0$, it follows that $\mathbf{x} \in R_{i_0}$. Therefore, $\widehat{T}(\mathbf{x}) = T_{i_0}(\lim_{k\to\infty} \widehat{T}^{n_k-1}(\mathbf{x}_0)) = \lim_{k\to\infty} T_{i_0}(\widehat{T}^{n_k-1}(\mathbf{x}_0)) = \mathbf{y}$.

By induction, we see that for all $k \in \mathbb{N}$ and all $\mathbf{y} \in D \setminus \bigcup_{i=0}^{k} \widehat{T}^i \mathcal{N}_0$, there exists $\mathbf{x} \in D$ such that $\widehat{T}^{k+1}(\mathbf{x}) = \mathbf{y}$. Therefore, for all $y \in D \setminus \mathcal{N}$, a backward $\widehat{T}$-orbit of y is in D. $\square$

Proposition 2.9. *Suppose that Assumptions (1)–(5) hold. Let*

$$P = \overline{\mathring{D}} \quad \text{and} \quad P_i = \overline{\mathring{P} \cap \overline{\mathring{R}}_i},\ i \in \{1, \ldots, h\}.$$

Then

(a) P is a measurable fundamental domain of the torus V/Λ;

(b) $P = \bigcup_{i\in\{1,\ldots,h\}} P_i$.

Moreover, for each $i \in \{1, \ldots, h\}$ such that $P_i \neq \emptyset$, one has

(c) $P_i \subset \overline{R}_i$ and $T_i(P_i) \subset P$;

(d) P_i is a finite union of convex polytopes with nonempty interiors;

(e) P_i is the closure of its interior, $\mathrm{Leb}(\partial P_i) = 0$ and $\mathrm{Leb}(P_i \cap P_j) = 0$ for all $j \neq i$.

Thus, $P = \{P_1, \ldots, P_h\}$ is a natural partition with respect to $T_{\boldsymbol{\alpha}}$.

Remark 2.10. We will use the following observations several times: If Q is a nearly convex set with nonempty interior then $Q \subset \overline{\mathring{Q}}$ and $\mathrm{Leb}(\partial Q) = 0$, and if Q' is another nearly convex set with nonempty interior, then $\mathring{Q} \cap Q' \neq \emptyset$ implies $\mathring{Q} \cap \mathring{Q}' \neq \emptyset$, which in turn implies $\overline{Q \cap Q'} = \overline{Q} \cap \overline{Q'}$.

Proof. Preliminaries.
By Assumption (5), for all $i \in \{1, \ldots, h\}$, $R_i = \bigcup_{j\in J_i} R_{ij}$ is a finite union of nearly convex sets, with nonempty interiors, that are finite intersections of open half spaces. Let $\mathcal{K} = \{R_{ij} : i \in \{1, \ldots, h\},\ j \in J_i\}$. For each $K \in \mathcal{K}$, let $i(K)$ denote the unique integer in $\{1, \ldots, h\}$ such that $K \subset R_{i(K)}$. For a nearly convex set Q with nonempty interior, let

$$\mathcal{K}(Q) = \{K \in \mathcal{K} \,:\, \mathring{Q} \cap \mathring{K} \neq \emptyset\}.$$

By the above remark, for any nearly convex set $Q \subset R$ with nonempty interior,

$$\mathring{Q} = \bigcup_{\mathcal{K}\in K(Q)} \mathring{Q} \cap K \quad \text{and} \quad Q = \bigcup_{K\in\mathcal{K}(Q)} Q \cap \overline{K}.$$

Observe that $\mathrm{Leb}(T_{i(K)}(D\cap K)\setminus D) = 0$ for each $K \in \mathcal{K}$. Indeed, if $\mathrm{Leb}(T_{i(K)}(D\cap K)\setminus D) > 0$, then $\mathrm{Leb}(T_{i(K)}(D \cap \mathring{K}) \setminus D) > 0$ and, since $\mathring{K} \cap T_{i(K)}^{-1}(V \setminus D)$ is open, there would be n such that $\mathbf{x} = \widehat{T}^n(\mathbf{x}_0) \in D \cap \mathring{K} \cap T_{i(K)}^{-1}(V \setminus D)$ and thus $\widehat{T}^{n+1}(\mathbf{x}_0) = \widehat{T}(\mathbf{x}) = T_{i(K)}(\mathbf{x}) \notin D$, a contradiction.

Let $Q_0 \subset U$, with U as in Assumption (4), be a closed hypercube with nonempty interior. By minimality, there exists $N \in \mathbb{N}$ such that $\bigcup_{n=0}^{N} \widehat{T}^n(\mathring{Q}_0) + \Lambda = V$.

We define by induction a sequence $(\mathcal{R}_n)_n$ of sets. Let $\mathcal{R}_0 = \{Q_0\}$. Suppose $\mathcal{R}_n$ is defined and let

$$\mathcal{R}_{n+1} = \bigcup_{Q\in\mathcal{R}_n} \big\{T_{i(K)}(\overline{\mathring{Q} \cap \mathring{K}}) : K \in \mathcal{K}(Q)\big\}.$$

By induction, we see that each $Q \in \mathcal{R}_n$ is a convex polytope with nonempty interior. Let

$$\mathcal{R} = \bigcup_{i=0}^{N} \mathcal{R}_n \qquad \text{and} \qquad P' = \bigcup_{Q\in\mathcal{R}} Q.$$

We show by induction that, for all $Q \in \mathcal{R}_n$, we have $Q \subset D$. Let $Q \in \mathcal{R}_n$, $K \in \mathcal{K}(Q)$ and $Q' = T_{i(K)}(\overline{\mathring{Q} \cap \mathring{K}})$. If Q' were not included in D, we would have $\mathrm{Leb}(T_{i(K)}(Q\cap K)\setminus D) > 0$ because D is closed, but by the induction hypothesis $Q \subset D$, thus $\mathrm{Leb}(T_{i(K)}(Q\cap K)\setminus D) \leq \mathrm{Leb}(T_{i(K)}(D \cap K) \setminus D)$, contradicting $\mathrm{Leb}(T_{i(K)}(D \cap K) \setminus D) = 0$. It follows that $P' \subset D$.

• To see (a), let us show by induction that $\widehat{T}^n(\mathring{Q}_0) \subset \cup_{Q\in\mathcal{R}_n} Q + \Lambda$. Let $\mathbf{x} \in \widehat{T}^n(\mathring{Q}_0)$ and let $\mathbf{x}' \in Q$, $Q \in \mathcal{R}_n$ be such that $\mathbf{x}'-\mathbf{x} \in \Lambda$. Since $Q = \bigcup_{K\in\mathcal{K}(Q)} Q\cap\overline{K}$, we have $\mathbf{x}' \in Q\cap\overline{K}$ for some $K \in \mathcal{K}(Q)$. It follows that

$$T_{i(K)}(\mathbf{x}') \in T_{i(K)}(Q \cap \overline{K}) \in \mathcal{R}_{n+1},$$

which in turn implies $\widehat{T}(\mathbf{x}) \in T_{i(K)}(Q \cap \overline{K}) + \Lambda \subset \bigcup_{Q\in\mathcal{R}_{n+1}} Q + \Lambda$.

It follows that $P' + \Lambda = V$. Furthermore, for all $n \leq N$ and $Q \in \mathcal{R}_n$, we have $Q \subset D$ and $\overline{\mathring{Q}} = Q$, therefore $P' \subset \overline{\mathring{D}} = P$. On the other hand, if $\mathring{D} \setminus P' \neq \emptyset$, then it has nonzero Lebesgue measure, which is not possible for $\mathrm{Leb}(P') \geq 1$ and $\mathrm{Leb}(D) = 1$. It follows that $P' = \overline{\mathring{D}} = P$. Furthermore, by the previous proposition, since $P \subset D$, $\mathrm{Leb}(P\cap(P+\mathbf{n})) = 0$ for all $\mathbf{n} \in \Lambda \setminus \{\mathbf{0}\}$.

• To see (b), let us show that $P = \bigcup_{i\in\{1,\dots,h\}} P_i$. Let $Q \in \mathcal{R}_n$ for some $n \leq N$. We have $Q \cap \overline{K} \subset \overline{\mathring{Q} \cap \mathring{K}} \subset \overline{\mathring{P} \cap \mathring{\overline{R}}_{i(K)}} = P_{i(K)}$ for each $K \in \mathcal{K}(Q)$, therefore $Q \subset \cup_{i\in I} P_i$.

• We verify (c). Clearly $P_i \subset P \cap \overline{R}_i$. Next, to show that $T_i(P_i) \subset P$, it is enough to prove that $T_i(\mathring{P} \cap \mathring{\overline{R}}_i) \subset P$. Now for all $K \in \mathcal{K}$, $\mathrm{Leb}(T_{i(K)}(D \cap K) \setminus D) = 0$ and $\overline{R}_i = \bigcup_{K\in\mathcal{K}:i(K)=i} \overline{K}$, therefore $\mathrm{Leb}(T_i(D \cap \overline{R}_i) \setminus D) = 0$. Since $\mathrm{Leb}(D \setminus P) = 0$, it follows that $\mathrm{Leb}(T_i(P \cap \overline{R}_i) \setminus P) = 0$, which implies that $T_i(\mathring{P} \cap \mathring{\overline{R}}_i) \subset P$.

• To see (d) and (e), we have to show that each P_i is a finite union of convex polytopes with non empty interiors, for, this implies that P_i is the closure of its interior and that $\mathrm{Leb}(\partial P_i) = 0$. It is enough to show that

$$\mathring{P} \cap \mathring{\overline{R}}_i \subset \bigcup_{(K,Q)\in\mathcal{K}\times\mathcal{R}:\, i(K)=i, K\in\mathcal{K}(Q)} \overline{\mathring{Q} \cap \mathring{K}}.$$

Let $\mathbf{x} \in \mathring{P} \cap \mathring{\overline{R}}_i$ and let $Q \in \mathcal{R}$ containing $\mathbf{x}$. Consider the set $\mathcal{K}_{\mathbf{x}}$ of $K \in \mathcal{K}$ such that $\mathbf{x} \in \overline{K}$. There is an open ball $B(\mathbf{x}, r)$ with $r > 0$ such that $B(\mathbf{x}, r) \subset \mathring{P} \cap \mathring{\overline{R}}_i$ and such that

$B(\mathbf{x},r) \cap \overline{K} = \emptyset$ for all $K \in \mathcal{K} \setminus \mathcal{K}_{\mathbf{x}}$. The ball $B(\mathbf{x},r)$ is included in the union of the $\overline{K}$, $K \in \mathcal{K}_{\mathbf{x}}$, therefore there exists $K \in \mathcal{K}_{\mathbf{x}}$ such that $\mathring{Q} \cap \mathring{K} \neq \emptyset$. Since $\mathring{Q} \cap \mathring{K} \neq \emptyset$, we have $\overline{\mathring{Q} \cap \mathring{K}} = \overline{\mathring{Q}} \cap \overline{\mathring{K}} = Q \cap \overline{K}$ and hence $\mathbf{x} \in \overline{\mathring{Q} \cap \mathring{K}}$.

Finally, $\mathrm{Leb}(P_i \cap P_j) = 0$ for all $j \neq i$ follows from Assumption (3) and $P_i \subset \overline{R}_i$. □

3. Two constructions of sequences with small discrepancy

3.1. **Strategy.** We now have gathered all that is needed in terms of notation and concepts for describing in detail the strategy for the constructions of sequences developed in the present section.

We represent an infinite word u as the set $\{\mathbf{p}(u_{[0,n)}) : n \in \mathbb{N}\}$ of vertices of a broken line $\mathbf{L}_u$; see Figure 2. The discrepancy $\Delta_{\boldsymbol{\alpha}}(u)$ can be seen as a measure of the distance (with respect to the supremum norm) of the vertices of this broken (half) line to the (half) line $\mathbb{R}^+\boldsymbol{\alpha}$. Our aim is thus to control the supremum of the discrepancy vectors

$$n\boldsymbol{\alpha} - \mathbf{p}(u_{[0,n)}),\ n \in \mathbb{N}.$$

One notices that $n\boldsymbol{\alpha}-\mathbf{p}(u_{[0,n)}) = -\pi_{\boldsymbol{\alpha}}(\mathbf{p}(u_{[0,n)})$ and thus $\Delta_{\boldsymbol{\alpha}}(u) = \sup_n \|\pi_{\boldsymbol{\alpha}}(\mathbf{p}(u_{[0,n)})\|$. Hence projecting along $\mathbb{R}\boldsymbol{\alpha}$ onto a transverse hyperplane such as the hyperplane $\mathbf{1}^\perp$ allows one to understand how far the vertices of the broken line are from the line $\mathbb{R}^+\boldsymbol{\alpha}$. The strength of Tijdeman's construction relies on the fact that the set $\overline{\{-\pi_{\boldsymbol{\alpha}}(\mathbf{p}(u_{[0,n)}) : n \in \mathbb{N}\}}$ forms a fundamental domain of $\mathbf{1}^\perp/(\mathbb{Z}^d \cap \mathbf{1}^\perp)$, and that we can partition this fundamental domain into atoms $\overline{\{-\pi_{\boldsymbol{\alpha}}(\mathbf{p}(u_{[0,n)}) : u_n = i,\ n \in \mathbb{N}\}}$, for each letter i. This then allows us to relate the dynamics of the shift with the dynamics of $T_{\boldsymbol{\alpha}}$. The sequence u is then a bounded natural coding according to Definition 2.3 (whose associated natural partition is obtained by application of the map ι).

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (0,1)^d$ be a totally irrational frequency vector. The first construction in Section 3.2 produces classical hypercubic billiard sequences as presented in [AMST94a]. The second one, given in Section 3.3, corresponds to Tijdeman's construction in [Tij80] and is obtained by introducing more specification stated in terms of lower and upper bounds for the supremum norm of the discrepancy vectors. Both constructions are obtained by coding the same toral translation $T_{\boldsymbol{\alpha}}$ with respect to finite partitions by polytopes. In particular, Section 3.2, which is devoted to hypercubic billiard sequences, aims at explaining that they do not have the lowest possible discrepancy (see Proposition 3.2) and to prepare the main construction from Section 3.3, which can be seen as an improvement of the hypercubic billiard codings.

3.2. **Hypercubic billiard sequences.** In this section, we consider cutting words associated with the hypercubic billiard. In Proposition 3.2, we recall that their discrepancy is generally not minimal for $d > 2$; see also [AV22]. The description of these cutting words will help with the understanding of Section 3.3, where we recall Tijdeman's construction. Tijdeman's construction can be seen as an improvement of the present construction for hypercubic billiard sequences.

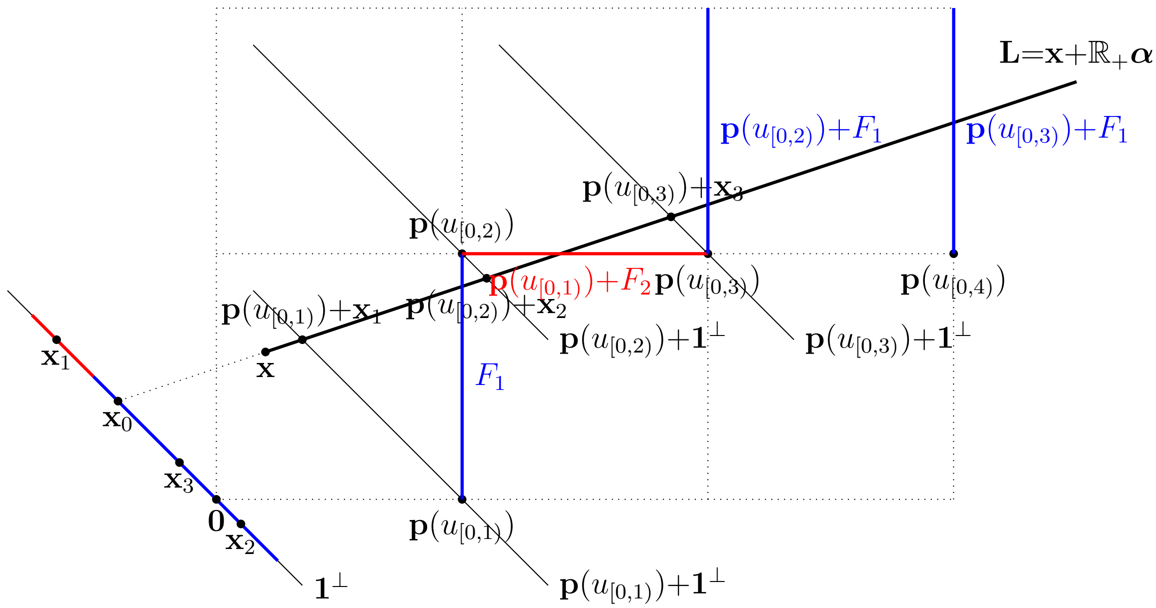

FIGURE 5. Notation for hypercubic billiard sequences. Here, $\boldsymbol{\alpha} \approx (0.75, 0.25)$, $\mathbf{x} \approx (0.2, 0.6)$, thus $\mathbf{x}_0 \approx (-0.4, 0.4)$. If $\mathbf{x}_n \in \{(-a,a) : -\alpha_1 < a < 1-2\alpha_1\}$, then $u_n = 2$, and if $\mathbf{x}_n \in \{(-a,a) : 1-2\alpha_1 < a < 1-\alpha_1\}$, then $u_n = 1$. Here $u_0u_1 \cdots = 1211\ldots$.

We follow the approach of [AMST94a] adapted to the present context. Travelling along a half line $\mathbf{L} = \mathbf{x}+\mathbb{R}_+\boldsymbol{\alpha}$, $\mathbf{x} \in [0,1)^d$, one meets the faces of the unit hypercubes that are located at the grid defined by the set $\mathbb{Z}^d$ of integer points. The *cutting word* $u = u_0u_1\cdots \in \{1,\ldots,d\}^{\mathbb{N}}$ codes the sequence of upper faces (of unit hypercubes) that are met by $\mathbf{L}$, where if the line hits a face parallel to $\mathbf{e}_j^\perp$, we code this intersection with the letter j; see Figure 5 for an illustration. More precisely, starting at $\mathbf{x}$ and given $u_0\cdots u_{n-1}$, the letter u_n is the coding of the upper face of $\mathbf{p}(u_{[0,n)})+[0,1]^d$ (as defined in (2.3)) that is intersected by $\mathbf{x}+\mathbb{R}\boldsymbol{\alpha}$. In the latter description, we can replace $\mathbf{x}$ by any point on $\mathbf{x}+\mathbb{R}\boldsymbol{\alpha}$, in particular by $\mathbf{x}_0 = \pi_{\boldsymbol{\alpha}}(\mathbf{x}) \in E_{\boldsymbol{\alpha}}$. We call such a coding sequence a *hypercubic billiard sequence with frequency* $\boldsymbol{\alpha}$ *and initial condition* $\mathbf{x}$ *(or* $\mathbf{x}_0$*)*.

Note that the half line intersects the lattice $\mathbb{Z}^d$ at most once since $\boldsymbol{\alpha}$ is totally irrational; we neglect the starting points $\mathbf{x}$ for which $\mathbf{L}\setminus\{\mathbf{x}\}$ may intersect the lattice $\mathbb{Z}^d$. We denote by $\mathcal{B}_{\boldsymbol{\alpha}}$ the set of hypercubic billiard sequences with frequency $\boldsymbol{\alpha}$ and initial condition $\mathbf{x}$ such that $\mathbf{L}\setminus\{\mathbf{x}\}$ does not intersect the lattice $\mathbb{Z}^d$.

For $\mathbf{x}_0 \in E_{\boldsymbol{\alpha}}$, we have $u_0 = i$ if $(\mathbf{x}_0+\mathbb{R}_+\boldsymbol{\alpha}) \cap F_i \neq \emptyset$, i.e., if $\mathbf{x}_0 \in E_{\boldsymbol{\alpha},i}$. For $n \geq 1$, set

$$u_n = i \quad \text{if } (\mathbf{x}_0+\mathbb{R}_+\boldsymbol{\alpha}) \cap (\mathbf{p}(u_{[0,n)})+F_i) \neq \emptyset,$$

i.e., $u_n = i$ if $\mathbf{x}_0 - \pi_{\boldsymbol{\alpha}}(\mathbf{p}(u_{[0,n)})) \in E_{\boldsymbol{\alpha},i}$. By (2.4), we have

$$\mathbf{x}_n := \mathbf{x}_0 - \pi_{\boldsymbol{\alpha}}(\mathbf{p}(u_{[0,n)})) = \mathbf{x}_0 + n\boldsymbol{\alpha} - \mathbf{p}(u_{[0,n)}).$$

Since $u_n = i$ if $\mathbf{x}_n \in E_{\boldsymbol{\alpha},i}$, we have by (2.6) that $\mathbf{x}_{n+1} = \tilde{T}_{\boldsymbol{\alpha}}(\mathbf{x}_n)$ for all $n \in \mathbb{N}$. Thus u is the coding of $\mathbf{x}_0$ w.r.t. $\tilde{T}_{\boldsymbol{\alpha}}$, which is the exchange of pieces w.r.t. the partition $\{E_{\boldsymbol{\alpha},i} : 1 \le i \le d\}$: one has $u_n = i$ if and only if $\tilde{T}^n_{\boldsymbol{\alpha}}(\mathbf{x}_0) \in E_{\boldsymbol{\alpha},i}$. Since ι is bijective, u is also the bounded natural coding of $\iota(\mathbf{x}_0)$ w.r.t. the toral translation $T_{\boldsymbol{\alpha}}$ and the partition $\{\iota(E_{\boldsymbol{\alpha},i}) : 1 \le i \le d\}$ (in the sense of Definition 2.3).

Note that $\mathbf{x}_n - \mathbf{x}_0 = n\boldsymbol{\alpha} - \mathbf{p}(u_{[0,n)})$ is a discrepancy vector of u, thus the discrepancy is $\Delta_{\boldsymbol{\alpha}}(u) = \sup_n \|\mathbf{x}_n - \mathbf{x}_0\|_\infty$. Writing $\mathbf{x}_n = (x_{n,1}, \ldots, x_{n,d})$, and letting $t_{n,i} = \frac{1-x_{n,i}}{\alpha_i}$, we have

$$\mathbf{x}_0 + (n+t_{n,i})\boldsymbol{\alpha} \in \mathbf{p}(u_{[0,n)}) + \mathbf{e}_i + \mathbf{e}_i^\perp.$$

In other words, $t_{n,i}$ is the time needed on the line $\mathbf{x}_0 + \mathbb{R}\boldsymbol{\alpha}$ to go from $\mathbf{x}_0 + n\boldsymbol{\alpha}$ to the hyperplane $\mathbf{p}(u_{[0,n)}) + \mathbf{e}_i + \mathbf{e}_i^\perp$. We have $u_n = i$ if we hit $\mathbf{p}(u_{[0,n)}) + F_i$, which is equivalent to $t_{n,i} = \min\{t_{n,j} : 1 \le j \le d\}$. This is to be compared to (3.2) below, where the hyperplane to be hit will be of the form $\mathbf{p}(u_{[0,n)}) + C\mathbf{e}_j + \mathbf{e}_j^\perp$. To construct Tijdeman sequences in Section 3.3, we will optimize the choice of u_n with respect to two criteria expressed in terms of the values taken by the $t_{n,i}$'s.

Remark 3.1. This construction can be interpreted in terms of model sets with the acceptance window being given by $E_{\boldsymbol{\alpha}}$; one has

$$-\mathbf{x}_0 + \{\mathbf{x} \in \mathbb{Z}^d \,:\, \pi_{\boldsymbol{\alpha}}(\mathbf{x}) \in E_{\boldsymbol{\alpha}},\, \langle \mathbf{x}, \mathbf{1}^\perp \rangle \ge 0\} = \{-\mathbf{p}(u_{[0,n)}), n \in \mathbb{N}\}.$$

Indeed, we have proved the inclusion $\{-\mathbf{p}(u_{[0,n)}), n \in \mathbb{N}\} \subset -\mathbf{x}_0 + \{\mathbf{x} \in \mathbb{Z}^d : \pi_{\boldsymbol{\alpha}}(\mathbf{x}) \in E_{\boldsymbol{\alpha}},\, \langle \mathbf{x}, \mathbf{1} \rangle \ge 0\}$ above. The reverse inclusion comes from the denseness of the set of points $\pi_{\boldsymbol{\alpha}}(\mathbf{x})$, $\mathbf{x} \in \mathbb{N}^d$, together with the fact that there is only one point with integer coordinates on each half line $\mathbf{p}(u_{[0,n)}) + \mathbb{R}\boldsymbol{\alpha}$. The (half) broken line $\mathbf{L}_u$ associated with u is thus exactly the set of points $\mathbf{x} \in \mathbb{N}^d$ such that $\pi_{\boldsymbol{\alpha}}(\mathbf{x}) \in E_{\boldsymbol{\alpha}}$.

The next proposition provides estimates on the discrepancy of hypercubic billiard sequences; see also [Vui03] expressed in terms of balance and [AV22] for the case of particular $\boldsymbol{\alpha}$, where it is proved that for $d \ge 5$ and for every $k \in \{3, \ldots, d-1\}$, there exists a hypercubic k-balanced billiard word (with $\boldsymbol{\alpha}$ a totally irrational frequency vector).

We recall that $\mathcal{B}_{\boldsymbol{\alpha}}$ stands for the set of hypercubic billiard sequences with frequency $\boldsymbol{\alpha}$ and initial condition $\mathbf{x}_0 \in E_{\boldsymbol{\alpha}}$ such that $\mathbf{L} \setminus \{\mathbf{x}_0\}$ does not intersect the lattice $\mathbb{Z}^d$.

Proposition 3.2. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (0,1)^d$ be a totally irrational frequency vector. Then*

$$\{\Delta_{\boldsymbol{\alpha}}(u) \,:\, u \in \mathcal{B}_{\boldsymbol{\alpha}}\} = \left[\tfrac{1}{2}(1 + (d-2)\|\boldsymbol{\alpha}\|_\infty), 1 + (d-2)\|\boldsymbol{\alpha}\|_\infty\right].$$

Moreover,

$$\inf_{\boldsymbol{\alpha}} \inf_{u \in \mathcal{B}_{\boldsymbol{\alpha}}} \Delta_{\boldsymbol{\alpha}}(u) = 1 - \frac{1}{d}.$$

where the supremum is taken over totally irrational frequency vectors $\boldsymbol{\alpha} \in (0,1)^d$.

Remark 3.3. We remark that, when $d \ge 3$, there exist hypercubic billiard sequences u in $\mathcal{B}_{\boldsymbol{\alpha}}$ that are fairly distributed, i.e., $\Delta_{\boldsymbol{\alpha}}(u) \le D_d = 1 - \frac{1}{2d-2}$. Indeed, it suffices to take $\boldsymbol{\alpha}$ such that $\|\boldsymbol{\alpha}\|_\infty$ is close to $\frac{1}{d}$ and $\mathbf{x}_0$ such that $\Delta_{\boldsymbol{\alpha}}(u)$ is close to $\frac{1}{2}(1 + (d-2)\|\boldsymbol{\alpha}\|_\infty)$.

Remark 3.4. When $d=2$, a fairly distributed sequence with totally irrational frequency $\boldsymbol{\alpha}$ is Sturmian; see Proposition 2.1. Moreover, Proposition 3.2 indicates that not all Sturmian sequences have the same discrepancy, although they are all 1-balanced. Indeed, when $d=2$, Proposition 3.2 gives that the range of values taken by $\Delta_{\boldsymbol{\alpha}}(u)$ is the whole segment $[1/2, 1]$.

Proof of Proposition 3.2. Let $u \in \mathcal{B}_{\boldsymbol{\alpha}}$ be a hypercubic billiard sequence with a totally irrational frequency vector $\boldsymbol{\alpha}$ and initial condition $\mathbf{x}_0 \in E_{\boldsymbol{\alpha}}$. Since $\tilde{T}_{\boldsymbol{\alpha}}$ is minimal, the sequence $(\tilde{T}^n_{\boldsymbol{\alpha}}(\mathbf{x}_0))_n$ is dense in $E_{\boldsymbol{\alpha}}$ and

(3.1)
$$\Delta_{\boldsymbol{\alpha}}(u) = \sup_{n\in\mathbb{N}} \|n\boldsymbol{\alpha} - \mathbf{p}(u_{[0,n)})\|_\infty = \sup_{n\in\mathbb{N}} \|\mathbf{x}_n - \mathbf{x}_0\|_\infty = \sup_{n\in\mathbb{N}} \|\tilde{T}^n_{\boldsymbol{\alpha}}(\mathbf{x}_0) - \mathbf{x}_0\| = \sup_{\mathbf{y}\in E_{\boldsymbol{\alpha}}} \|\mathbf{y} - \mathbf{x}_0\|_\infty.$$

Then elements of $E_{\boldsymbol{\alpha}} = \pi_\alpha([0,1]^d)$ are of the form $\sum_{i=1}^d y_i(\mathbf{e}_i - \boldsymbol{\alpha})$, $y_i \in [0,1]$. Therefore, the i-th coordinate is in $[-(d-1)\alpha_i, 1-\alpha_i]$, where the endpoints of the interval are attained by $\sum_{j\neq i} \pi_{\boldsymbol{\alpha}}(\mathbf{e}_j)$ and $\pi_{\boldsymbol{\alpha}}(\mathbf{e}_i)$ respectively. Therefore, the diameter of $E_{\boldsymbol{\alpha}}$ is $1+(d-2)\|\boldsymbol{\alpha}\|_\infty$. This implies that
$$\Delta_{\boldsymbol{\alpha}}(u) \leq 1 + (d-2)\|\boldsymbol{\alpha}\|_\infty \leq d-1$$
and, by the triangle inequality,
$$\Delta_{\boldsymbol{\alpha}}(u) \geq \frac{1+(d-2)\|\boldsymbol{\alpha}\|_\infty}{2} \geq \frac{d-1}{d}.$$

For $\mathbf{x}_0 = \pi_{\boldsymbol{\alpha}}(\mathbf{e}_i)$ such that $\alpha_i = \|\boldsymbol{\alpha}\|_\infty$, we have $\Delta_\alpha(u) = 1+(d-2)\|\boldsymbol{\alpha}\|_\infty$. For $\mathbf{x}_0 = \frac{1}{2}\pi_{\boldsymbol{\alpha}}(\mathbf{1}) = \frac{1}{2}(\mathbf{1} - d\boldsymbol{\alpha})$ and $1 \leq i \leq d$, the i-th coordinates of $\pi_{\boldsymbol{\alpha}}(\mathbf{e}_i) - \mathbf{x}_0$ and of $\mathbf{x}_0 - \sum_{j\neq i} \pi_{\boldsymbol{\alpha}}(\mathbf{e}_j)$ are $\frac{1}{2}(1+(d-2)\alpha_i)$. Moreover, the absolute value of the i-th coordinate of $\mathbf{x}_0 - \sum_{j\in I} \mathbf{e}_j$, for $\emptyset \neq I \subset \{1,\ldots,d\}$, is smaller than or equal to $\frac{1}{2}(1+(d-2)\alpha_i)$. Thus $\Delta_\alpha(u) = \frac{1}{2}(1+(d-2)\|\boldsymbol{\alpha}\|_\infty)$. Now taking convex combinations of $\pi_{\boldsymbol{\alpha}}(\mathbf{e}_i)$ and $\frac{1}{2}\pi_{\boldsymbol{\alpha}}(\mathbf{1})$ as initial condition $\mathbf{x}_0$, the discrepancy takes all values in the interval $\left[\frac{1}{2}(1+(d-2)\|\boldsymbol{\alpha}\|_\infty), 1+(d-2)\|\boldsymbol{\alpha}\|_\infty\right]$. The smallest value in this interval is obtained by $\boldsymbol{\alpha} \approx (\frac{1}{d}, \ldots, \frac{1}{d})$, so that $\Delta_{\boldsymbol{\alpha}}(u) \geq \frac{d-1}{d}$. □

3.3. **Tijdeman's construction.** Our aim now is to produce sequences u on a d-letter alphabet such that $\mathbf{x}_0 - \pi_{\boldsymbol{\alpha}}(\mathbf{p}(u_{[0,n)})) \in [-C', C]^d$ for all n, for given constants C, C'. In the previous section (with $C=1$, $C' \geq (d-1)\|\boldsymbol{\alpha}\|_\infty$, see Remark 3.6 below), the word u was defined by the sequence of faces met by a half line $\mathbf{L}$, which gave us the fundamental domain of Figure 4. In order to reduce the discrepancy, we work with different fundamental domains; see Figure 6 for an illustration. More precisely, we modify the construction in order to obtain a fundamental domain in $[-C', C]^d$ for arbitrary C, C' with
$$C, C' \geq 1 - \frac{1}{d}, \quad C \leq 1, \quad \text{and} \quad C + C' \geq 2 - \frac{1+\min_i \alpha_i}{d-1},$$
and we call these sequences Tidjeman sequences; see Definition 3.7 and Proposition 3.8. In particular, the choice $C = C' = 1 - \frac{1+\min_i \alpha_i}{2d-2}$ minimizes $\max(C, C')$ under the constraint $C+C' \geq 2 - \frac{1+\min \alpha_i}{d-1}$, given that the other constraints hold when $d \geq 3$ because $1 - \frac{1+\min_i \alpha_i}{2d-2} \geq 1 - \frac{1+1/d}{2d-2} \geq 1 - \frac{1}{d}$. We will show that these sequences have discrepancy at most $C+C'$; see Proposition 3.10.

Similarly to hypercubic billiard sequences, we consider the first time that $\mathbf{x}_0+\mathbb{R}_+\boldsymbol{\alpha}$ hits a hyperplane with the given properties, for $\mathbf{x}_0 \in \mathbf{1}^\perp$ in a certain neighbourhood of $\mathbf{0}$ that we specify later. We define the word $u = (u_n)_n \in \{1, \ldots, d\}^{\mathbb{N}}$ as follows. Given $u_{[0,n)}$, $n \geq 0$, we first consider $t_{n,i} \in \mathbb{R}$, $1 \leq i \leq d$, such that

$$\mathbf{x}_0 + (n+t_{n,i})\boldsymbol{\alpha} \in \mathbf{p}(u_{[0,n)}) + C\mathbf{e}_i + \mathbf{e}_i^\perp. \tag{3.2}$$

With $\mathbf{x}_n := \mathbf{x}_0+n\boldsymbol{\alpha}-\mathbf{p}(u_{[0,n)}) \in \mathbf{1}^\perp$, we have

$$\mathbf{x}_n + t_{n,i}\boldsymbol{\alpha} \in C\mathbf{e}_i + \mathbf{e}_i^\perp.$$

Writing $\mathbf{x}_n = (x_{n,1}, \ldots, x_{n,d})$, we obtain that

$$t_{n,i} = \frac{C - x_{n,i}}{\alpha_i} \quad (1 \leq j \leq d). \tag{3.3}$$

We now have to choose the index i that will be the value of u_n. We do this according to two criteria.

We first want that

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \boldsymbol{\alpha} - \mathbf{e}_{u_n} \in [-C', \infty)^d.$$

The i-th coordinate of $\mathbf{x}_{n+1}$ is $x_{n,i}+\alpha_i-1$ if $u_n = i$. Thus we consider only indices i such that

$$x_{n,i} + \alpha_i - 1 \geq -C', \quad \text{i.e.,} \quad t_{n,i} \leq 1 + \frac{C + C' - 1}{\alpha_i}. \tag{3.4}$$

Remark 3.5. If $C' \geq 1 - \frac{1}{d}$, then some i satisfying (3.4) always exists because $x_{n,i}+\alpha_j-1 < -C'$ for all i would imply that $\sum_{i=1}^d x_{n,i} < d\,(1-C') - 1 < 0$ by the assumption $C' \geq 1-\frac{1}{d}$, contradicting that $\mathbf{x}_n \in \mathbf{1}^\perp$.

The second condition will be to take the index i providing the smallest value of $t_{n,i}$ fulfilling (3.4). We thus set

$$u_n = i \quad \text{if} \quad t_{n,i} = \min\{t_{n,j} \,:\, 1 \leq j \leq d,\, x_{n,i} + \alpha_i - 1 \geq -C'\}. \tag{3.5}$$

When the minimum is attained for several i, we choose e.g. the smallest i. We will see in Proposition 3.8 that this yields a sequence satisfying $\mathbf{x}_n \in [-C', C]^d$ for all n.

Remark 3.6. Note that for $C = 1$, $C' \geq (d-1)\|\boldsymbol{\alpha}\|_\infty$, and thus $C' \geq 1-\frac{1}{d}$, we obtain the billiard sequence u (with $\mathbf{x}_0 = \mathbf{0}$), by Remark 3.5, together with the fact that we took $t_{n,i} = \min\{t_{n,j} : 1 \leq j \leq d\}$ for the classical billiards. For general $C \geq 0$, $C' \geq 1-\frac{1}{d}$, the construction ensures that $\mathbf{x}_n \in [-C', \infty)^d$ for all $n \in \mathbb{N}$. Indeed, we have $x_{n+1,u_n} \geq -C'$ and $x_{n+1,j} = x_{n,j}+\alpha_j \geq x_{n,j}$ for $j \neq u_n$.

Before showing that $\mathbf{x}_n \in [-C', C]^d$ for all n (see Proposition 3.8), we define Tijdeman's sequences in terms of dynamical systems that generate them. For $C \geq 0$, $C' \geq 1-\frac{1}{d}$, let

$$\begin{aligned} S_{\boldsymbol{\alpha},C,C',i} = \{(x_1, \ldots, x_d) \in \mathbf{1}^\perp \cap [-C', \infty)^d \,:\, x_i + \alpha_i - 1 \geq -C' \text{ and} \\ \tfrac{C-x_i}{\alpha_i} \leq \tfrac{C-x_j}{\alpha_j} \text{ for all } 1 \leq j \leq d \text{ such that } x_j + \alpha_j - 1 \geq -C'\} \end{aligned}$$

and set

$$\hat{T}_{\boldsymbol{\alpha},C,C'} : \mathbf{1}^\perp \cap [-C',\infty)^d \to \mathbf{1}^\perp \cap [-C',\infty)^d, \quad \mathbf{x} \mapsto \mathbf{x} + \boldsymbol{\alpha} - \mathbf{e}_i \quad \text{if } \mathbf{x} \in S_{\boldsymbol{\alpha},C,C',i},$$

where we choose the smallest such i in case that $\mathbf{x} \in S_{\boldsymbol{\alpha},C,C',i} \cap S_{\boldsymbol{\alpha},C,C',j}$ for some $i \neq j$.

Definition 3.7 (Tijdeman parameters and sequences). Let $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_d) \in (0,1)^d$ be a totally irrational frequency vector. Let $C \geq 0$, $C' \geq 1-\frac{1}{d}$, $\mathbf{x}_0 \in \mathbf{1}^\perp$. If $\hat{T}^n_{\boldsymbol{\alpha},C,C'}(\mathbf{x}_0) \in [-C',C]^d$ for all $n \geq 0$, then we call $(\boldsymbol{\alpha},C,C',\mathbf{x}_0)$ *Tijdeman parameters* and the sequence $u_0u_1\cdots$ such that $\hat{T}^{n+1}_{\boldsymbol{\alpha},C,C'}(\mathbf{x}_0) - \hat{T}^n_{\boldsymbol{\alpha},C,C'}(\mathbf{x}_0) = \boldsymbol{\alpha} - \mathbf{e}_{u_n}$ a *Tijdeman sequence*.

The sets $S_{\boldsymbol{\alpha},C,C',i}$ are infinite, and we are looking for bounded natural partitions as in Definition 2.2, obtained by application of ι. For Tijdeman parameters $(\boldsymbol{\alpha},C,C',\mathbf{0})$, let

$$D_{\boldsymbol{\alpha},C,C'} = \overline{\{\hat{T}^n_{\boldsymbol{\alpha},C,C'}(\mathbf{0}) : n \geq 0\}}, \qquad P_{\boldsymbol{\alpha},C,C'} = \overline{\mathring{D}}_{\boldsymbol{\alpha},C,C'}, \tag{3.6}$$

$$P_{\boldsymbol{\alpha},C,C',i} = \overline{\mathring{P}_{\boldsymbol{\alpha},C,C'} \cap \mathring{\overline{S}}_{\boldsymbol{\alpha},C,C',i}} \qquad (1 \leq i \leq d). \tag{3.7}$$

Using Proposition 2.9 and that $\{S_{\boldsymbol{\alpha},C,C',i} : 1 \leq i \leq d\}$ forms a partition of $\mathbf{1}^\perp \cap [-C',\infty)^d$ (up to the negligible intersections), we will see that $\{P_{\boldsymbol{\alpha},C,C',i} : 1 \leq i \leq d\}$ forms a natural partition of $P_{\boldsymbol{\alpha},C,C'}$.

Let us come back to the construction of the sequence u from (3.5). We have $t_{n,i} = \frac{C-x_{n,i}}{\alpha_i}$ for all $n \in \mathbb{N}$, $1 \leq i \leq d$. Therefore, when $\mathbf{x}_n \in S_{\boldsymbol{\alpha},C,C',i}$ we have $u_n = i$, i.e., $\mathbf{x}_n \in S_{\boldsymbol{\alpha},C,C',u_n}$, and $\hat{T}_{\boldsymbol{\alpha},C,C'}(\mathbf{x}_n) = \mathbf{x}_{n+1}$, which implies that $\mathbf{x}_n = \hat{T}^n_{\boldsymbol{\alpha},C,C'}(\mathbf{x}_0)$, $n \in \mathbb{N}$, and u is the coding of $\mathbf{x}_0$ w.r.t. the partition $\{S_{\boldsymbol{\alpha},C,C',i} : 1 \leq i \leq d\}$. Note that, for $\mathbf{x}_0 = \mathbf{0}$, $P_{\boldsymbol{\alpha},C,C',i}$ is the closure of the set of points $n\boldsymbol{\alpha} - \mathbf{p}(u_{[0,n)})$ with $u_n = i$.

We will see with Proposition 3.9 below that $P_{\boldsymbol{\alpha},C,C'}$ is a measurable fundamental domain of $\mathbf{1}^\perp/(\mathbb{Z}^d \cap \mathbf{1}^\perp)$ admitting the partition $P_{\boldsymbol{\alpha},C,C'} = \bigcup_{i=1}^d P_{\boldsymbol{\alpha},C,C',i}$. But first let us state the following key proposition, which provides Tijdeman parameters (see Definition 3.7) and is inspired by [Tij80, Theorem 1].

Proposition 3.8. *Let $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_d) \in (0,1)^d$ be a totally irrational frequency vector. Let $C, C' \geq 1-\frac{1}{d}$, $C \leq 1$, and $C+C' \geq 2-\frac{1+\min_i \alpha_i}{d-1}$. Let $\hat{T}_{\boldsymbol{\alpha},C,C'}$ be the map from Definition 3.7 and $\mathbf{x}_0 \in [C-1,C]^d \cap \mathbf{1}^\perp$. Then $\hat{T}^n_{\boldsymbol{\alpha},C,C'}(\mathbf{x}_0) \in [-C',C]^d$ for all $n \in \mathbb{N}$.*

Proof. Write $\hat{T}^n_{\boldsymbol{\alpha},C,C'}(\mathbf{x}_0) = \mathbf{x}_n = (x_{n,1},\ldots,x_{n,d})$ and let u_n be such that $\mathbf{x}_{n+1} = \mathbf{x}_n + \boldsymbol{\alpha} - \mathbf{e}_{u_n}$. Suppose that $\mathbf{x}_n \notin [-C',C]^d$ for some $n \geq 1$, and let n be minimal with this property.

We have seen above that $C' \geq 1-\frac{1}{d}$ implies that $\mathbf{x}_n \in [-C',\infty)^d$, thus there exists i such that $x_{n,i} > C$. Then $t_{n,i} < 0$ and $x_{n,i} + \alpha_i - 1 > C-1 \geq -C'$, hence also $t_{n,u_n} < 0$, i.e.,

$$x_{n,u_n} > C. \tag{3.8}$$

Then the set

$$W = \big\{ i \in \{1,\ldots,d\} : x_{n,i} < C-1 \big\}$$

is not empty since otherwise we would have

$$\sum_{i=1}^{d} x_{n,i} > C + (d-1)(C-1) \geq 0$$

(using that $C \geq 1-\frac{1}{d}$), contradicting that $\mathbf{x}_n \in \mathbf{1}^\perp$.

Let $m < n$ be maximal such that $u_m \in W$. Such an $m \geq 0$ exists because otherwise we have $x_{n,i} = x_{0,i} + n\alpha_i$ for all $i \in W$, giving the contradiction $C-1 \leq x_{0,i} < x_{n,i} < C-1$. Let

$$V = \{u_k \,:\, m < k < n\}.$$

Using $\mathbf{x}_m + (n-m)\boldsymbol{\alpha} = \mathbf{x}_n + \mathbf{p}(u_{[m,n)})$, we obtain that

$$x_{m,u_m} + (n-m)\alpha_{u_m} = x_{n,u_m} + 1 < C \qquad \text{(since } |u_{[m,n)}|_{u_m} = 1 \text{ and } u_m \in W\text{)}, \tag{3.9}$$

$$x_{m,u_n} + (n-m)\alpha_{u_n} \geq x_{n,u_n} > C \qquad \text{(since } |u_{[m,n)}|_{u_n} \geq 0 \text{ and (3.8) holds)}, \tag{3.10}$$

$$x_{m,i} + (n-m)\alpha_i \geq x_{n,i} + 1 \geq C \text{ for all } i \in V \qquad \text{(since } |u_{[m,n)}|_i \geq 1 \text{ and } i \notin W\text{)}. \tag{3.11}$$

For all $i \in V \cup \{u_n\}$, we have $t_{m,i} \leq n - m < t_{m,u_m}$ by (3.9)–(3.11), thus the definition of u_m implies that $x_{m,i} + \alpha_i - 1 < -C'$, hence $x_{m+1,i} < 1 - C'$. We obtain

$$x_{n,u_n} - x_{m+1,u_n} > C + C' - 1, \qquad x_{n,i} - x_{m+1,i} > C + C' - 2 \quad \text{for all } i \in V.$$

In particular, we have $n \neq m + 1$ since $C + C' \geq 1$, thus

$$x_{n,i} - x_{m+1,i} = (n-m-1)\alpha_i \geq \alpha_i \quad \text{for all } i \notin V.$$

Using that $u_m \notin V \cup \{u_n\}$, in particular that $V \setminus \{u_n\}$ has at most $d-2$ elements, and distinguishing between the cases $C + C' \geq 2$ and $C + C' < 2$, we obtain that

$$\begin{aligned} 0 = \sum_{i=1}^{d} x_{n,i} - \sum_{i=1}^{d} x_{m+1,i} &\geq x_{n,u_n} - x_{m+1,u_n} + x_{n,u_m} - x_{m+1,u_m} + \sum_{i\in V\setminus\{u_n\}} (x_{n,i} - x_{m+1,i}) \\ &> C + C' - 1 + \alpha_{u_m} + \min\big\{0, (d-2)(C+C'-2)\big\} \\ &\geq \min\{C + C' - 1, (d-1)(C+C'-2) + 1 + \min_i \alpha_i\}, \end{aligned}$$

which contradicts the assumptions $C+C' \geq 2\,(1-\frac{1}{d}) \geq 1$ and $C+C' \geq 2-\frac{1+\min_i \alpha_i}{d-1}$. This implies that $\mathbf{x}_n \in [-C', C]^d$ for all $n \in \mathbb{N}$. □

We can now conclude with the following two propositions.

Proposition 3.9. *Let $(\boldsymbol{\alpha}, C, C', \mathbf{0})$ be Tijdeman parameters with $C, C' \in \big[1-\frac{1+\min \alpha_i}{2d-2}, 1\big)$. Let $\hat{T}_{\boldsymbol{\alpha},C,C'}$ be the map from Definition 3.7 and let $P_{\boldsymbol{\alpha},C,C'}$ be defined as in (3.6). Then $P_{\boldsymbol{\alpha},C,C'}$ is a measurable fundamental domain of $\mathbf{1}^\perp/(\mathbb{Z}^d \cap \mathbf{1}^\perp)$ admitting the natural partition $P_{\boldsymbol{\alpha},C,C'} = \bigcup_{i=1}^{d} P_{\boldsymbol{\alpha},C,C',i}$. The restriction of $\hat{T}_{\boldsymbol{\alpha},C,C'}$ to $P_{\boldsymbol{\alpha},C,C'}$ is measurably conjugate to the translation $T_{\boldsymbol{\alpha}}$ on $\mathbb{T}^{d-1}$. Moreover, the $P_{\boldsymbol{\alpha},C,C',i}$ are finite unions of convex polytopes with nonempty interiors, and $(\boldsymbol{\alpha}, C, C', \mathbf{x}_0)$ are Tijdeman parameters for almost all $\mathbf{x}_0 \in P_{\boldsymbol{\alpha},C,C'}$.*

Proof. We use Propositions 2.8 and 2.9 with $V = \mathbf{1}^\perp$, $R = V \cap [-C', \infty)^d$, $R_i = S_i \setminus \cup_{1 \le j < i} S_j$ where $S_i = S_{\boldsymbol{\alpha}, C, C', i}$, $i \in \{1, \ldots, d\}$, $\widehat{T} = \hat{T}_{\boldsymbol{\alpha}, C, C'}$, $K = [-C', C]^d \cap V$ and $U = (C-1, 1-C')^d \cap V$. It is clear that Assumptions (1)–(3) from Section 2.4 hold.

Let us prove that Assumption (4) holds. On the one hand, by Proposition 3.8, $D_{\boldsymbol{\alpha}, C, C'} \subset K$ when $C, C' \ge 1 - \frac{1}{d}$, $C \le 1$, and $C + C' \ge 2 - \frac{1 + \min_i \alpha_i}{d-1}$. On the other hand, $U \neq \emptyset$ when $C, C' < 1$. Hence, Assumption (4) holds when $C, C' \in \left[1 - \frac{1 + \min \alpha_i}{2d-2}, 1\right)$.

We now prove that Assumption (5) holds. We want to prove that each R_i is is a finite union of nearly convex sets, with nonempty interiors that are finite intersections of open half spaces. Consider the sets $Q_i \subset S_i$, $i = 1, \ldots, d$, defined as

$$
\begin{aligned}
Q_i = \{(x_1, \ldots, x_d) \in R \,:\, & x_i + \alpha_i - 1 \ge -C', \\
& \tfrac{C - x_i}{\alpha_i} < \tfrac{C - x_j}{\alpha_j} \text{ for all } 1 \le j < i \text{ such that } x_j + \alpha_j - 1 \ge -C', \text{ and} \\
& \tfrac{C - x_i}{\alpha_i} \le \tfrac{C - x_j}{\alpha_j} \text{ for all } i < j \le d \text{ such that } x_j + \alpha_j - 1 \ge -C'\}.
\end{aligned}
$$

Clearly the sets $Q_1, \ldots, Q_d$ are disjoint. The sets Q_i can be decomposed into convex subsets:

$$
Q_i = \bigcup_{J \subset \{1, \ldots, d\},\ i \in J} Q_{i,J},
$$

where

$$
\begin{aligned}
Q_{i,J} = \{(x_1, \ldots, x_d) \in R \,:\, & x_j + \alpha_j - 1 \ge -C' \text{ iff } j \in J, \\
& \tfrac{C - x_i}{\alpha_i} < \tfrac{C - x_j}{\alpha_j} \text{ for all } 1 \le j < i \text{ with } j \in J, \text{ and} \\
& \tfrac{C - x_i}{\alpha_i} \le \tfrac{C - x_j}{\alpha_j} \text{ for all } i < j \le d \text{ with } j \in J \ \}.
\end{aligned}
$$

Clearly, each $Q_{i,J}$ is a finite intersection of half spaces, open or closed. Hence, each $\mathring{Q}_{i,J}$ is a finite intersection of half open spaces. Thus the sets

$$
Q'_i = \bigcup_{i \in J \subset \{1, \ldots, d\} : \mathring{Q}_{i,J} \neq \emptyset} \overline{Q_{i,J}}
$$

are finite unions of convex polytopes with nonempty interiors that are finite intersections of open half spaces. If we can show that, for each i, $S_i \subset Q'_i$, we are done. Indeed, for each pair (i, I) such that $\mathring{Q}_{i,I} \neq \emptyset$, and each pair (j, J) with $j \neq i$ and such that $\mathring{Q}_{i,I} \neq \emptyset$, the set $\overline{Q}_{i,I} \setminus \overline{Q}_{j,J}$ is a nearly convex set with nonempty interior, because $\mathring{Q}_{i,I} \subset Q_i$, $\mathring{Q}_{j,J} \subset Q_j$ and Q_i and Q_j are disjoint. It follows that $\overline{Q}_{i,I} \setminus \cup_{1 \le j < i} Q'_i$ is nearly convex with interior $\mathring{Q}_{i,I}$, which in turn implies that Assumption (5) holds for R_i.

So it remains to show that $S_i \subset Q'_i$. Let us show first that $\mathring{S}_i \subset Q_i$. By contradiction, suppose there exists $\mathbf{x} \in \mathring{S}_i \setminus Q_i$. By definition of S_i and Q_i, there exists $j < i$ such that $x_j + \alpha_j - 1 \ge -C'$ and $\frac{C - x_i}{\alpha_i} = \frac{C - x_j}{\alpha_j}$. Moreover, since $\mathbf{x} \in \mathring{S}_i$, for $t > 0$ small enough, $\mathbf{y}_t = x - te_i + te_j$ is in S_i. But the coordinates $y_{t,k}$ of $\mathbf{y}$ satisfy $y_{t,j} + \alpha_j - 1 \ge x_j + \alpha_j - 1 \ge -C'$ and $\frac{C - y_{t,i}}{\alpha_i} = \frac{C - x_i + t}{\alpha_i} > \frac{C - x_j - t}{\alpha_j} = \frac{C - y_{t,j}}{\alpha_i}$ a contradiction.

Next, let us show that $\mathring{S}_i \subset Q'_i$. Observe first, that if for some pair (i, J), $\mathring{Q}_{i,J} = \emptyset$, then $Q_{i,J}$ is contained in a hyperplane of V and therefore $\overline{\mathring{Q}}_{i,J} = \emptyset$. By contradiction, suppose that there exists $\mathbf{x} \in \mathring{S}_i \setminus Q'_i$. Since Q'_i is closed, there exists $r > 0$ such that $B(\mathbf{x}, r) \subset \mathring{S}_i \setminus Q'_i$. Now $\mathring{S}_i \subset Q_i$, so that $B(\mathbf{x}, r) \subset Q_i \setminus Q'_i$ which is impossible for the $Q_{i,J}$ that are missing in Q'_i have empty interiors.

Finally, let us show that $S_i \subset \overline{\mathring{S}}_i$, which will finish the proof because it implies $S_i \subset \overline{S}_i = \overline{\mathring{S}}_i \subset \overline{Q'_i} = Q'_i$. We can supppose $i = d$. Let $\mathbf{x} \in S_d$ and let $I(\mathbf{x}) = \{j < d : x_j + \alpha_j - 1 \geq -C'\}$. For all $j \in I(\mathbf{x})$, $\frac{C-x_j}{\alpha_j} \geq \frac{C-\sum_{i<d} x_i}{\alpha_d}$. Let $\mathbf{y}_t = (y_{t,1}, \ldots, y_{t,d}) = \mathbf{x} - t\sum_{j\in I(\mathbf{x})} e_j + t \operatorname{card} I(\mathbf{x}) e_d$. For $t > 0$, $I(\mathbf{y}_t) \subset I(\mathbf{x})$ and for all $j \in I(\mathbf{x})$, one has

$$\frac{C - y_{t,j}}{\alpha_j} > \frac{C - x_j}{\alpha_j} \geq \frac{C - x_d}{\alpha_d} > \frac{C - x_d - t \operatorname{card} I(\mathbf{x})}{\alpha_d}.$$

Therefore, for all $t > 0$ such that $y_{t,j} > -C'$ for all $j \in I(\mathbf{x})$, the point $\mathbf{y}_t$ is in S_d which in turn implies that $\mathbf{y}_t \in \mathring{S}_d$ for all $t > 0$ small enough because all the inequalities are strict. □

The following result on the discrepancy of Tijdeman sequences can be considered as a counterpart of Proposition 3.2. It provides suitable parameters that yield fairly distributed sequences.

Proposition 3.10. *Let u be a Tijdeman sequence with parameters $(\boldsymbol{\alpha}, C, C', \mathbf{x}_0)$, $C, C' \in \big[1 - \frac{1+\min \alpha_i}{2d-2}, 1\big)$. Then the sequence u is eventually a natural coding of the dynamical system $(P_{\boldsymbol{\alpha},C,C'}, \hat{T}_{\boldsymbol{\alpha},C,C'})$. Moreover, for the dynamical system $(P_{\boldsymbol{\alpha},C,C'}, \hat{T}_{\boldsymbol{\alpha},C,C'})$ and for any $\mathbf{x}_0 \in P_{\boldsymbol{\alpha},C,C'}$, the discrepancy of the coding u of $\mathbf{x}_0$ satisfies*

$$\Delta_{\boldsymbol{\alpha}}(u) \leq C + C',$$

and, for $\mathbf{x}_0 = \mathbf{0}$, the discrepancy satisfies

$$\Delta_{\boldsymbol{\alpha}}(u) \leq \max\{C, C'\}.$$

Hence, if $C = C' = 1 - \frac{1+\min_i \alpha_i}{2d-2}$ and $\mathbf{x}_0 = \mathbf{0}$, then

$$\Delta_{\boldsymbol{\alpha}}(u) \leq 1 - \frac{1 + \min_i \alpha_i}{2d - 2} < D_d.$$

Proof. If we start from some $\mathbf{x}_0 \notin P_{\boldsymbol{\alpha},C,C'}$, then by minimality $\hat{T}^n_{\boldsymbol{\alpha},C,C'}(\mathbf{x}_0) \in (C-1, 1-C')^d \cap \mathbf{1}^\perp \subset P_{\boldsymbol{\alpha},C,C'}$ for some $n \geq 1$. Next, by Lemma 4.4, for all n large enough, $\hat{T}_{\boldsymbol{\alpha},C,C'}(\mathbf{x}_0)$ is in none of the boundaries of the $P_{\boldsymbol{\alpha},C,C',i}$ because these boundaries are included in a finite union of subspaces of dimension $d-2$. Therefore, the sequence u is eventually a natural coding of the dynamical system $(P_{\boldsymbol{\alpha},C,C'}, \hat{T}_{\boldsymbol{\alpha},C,C'})$. We conclude about the discrepancy as in (3.1) in the proof of Proposition 3.2. □

Figure 6 shows a case where the sets $P_{\boldsymbol{\alpha},C,C'}$ are polygons that are more complicated than the sets $E_{\boldsymbol{\alpha},i}$ (which correspond to the case of the hypercubic billiard sequences such as developed in Section 3.2). For large C', we would be in the (shifted) billiard case, with $P_{\boldsymbol{\alpha},C,C'} = E_{\boldsymbol{\alpha},i} - (1-C)\pi_{\boldsymbol{\alpha}}(\mathbf{1})$. However, for the parameters of Figure 6, we do not have $E_{\boldsymbol{\alpha}} - (1-C)\pi_{\boldsymbol{\alpha}}(\mathbf{1}) \subset [-C', C]^d$. Here, the leftmost part of $E_{\boldsymbol{\alpha},1} - (1-C)\pi_{\boldsymbol{\alpha}}(\mathbf{1})$ is not in $S_{\boldsymbol{\alpha},C,C',1}$ but in $S_{\boldsymbol{\alpha},C,C',2} \cup S_{\boldsymbol{\alpha},C,C',3}$. (The figure depicts the ι-images of the sets.)

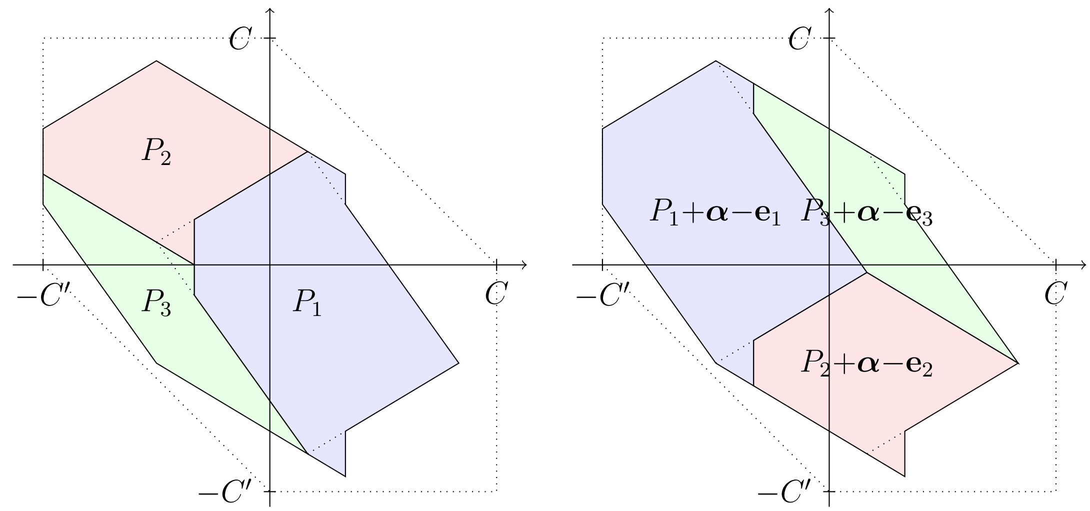

FIGURE 6. The polygons $P_i = P_{\boldsymbol{\alpha},C,C',i}$ for $\boldsymbol{\alpha} \approx (0.5, 0.3, 0.2)$, $C = C' = 3/4$.

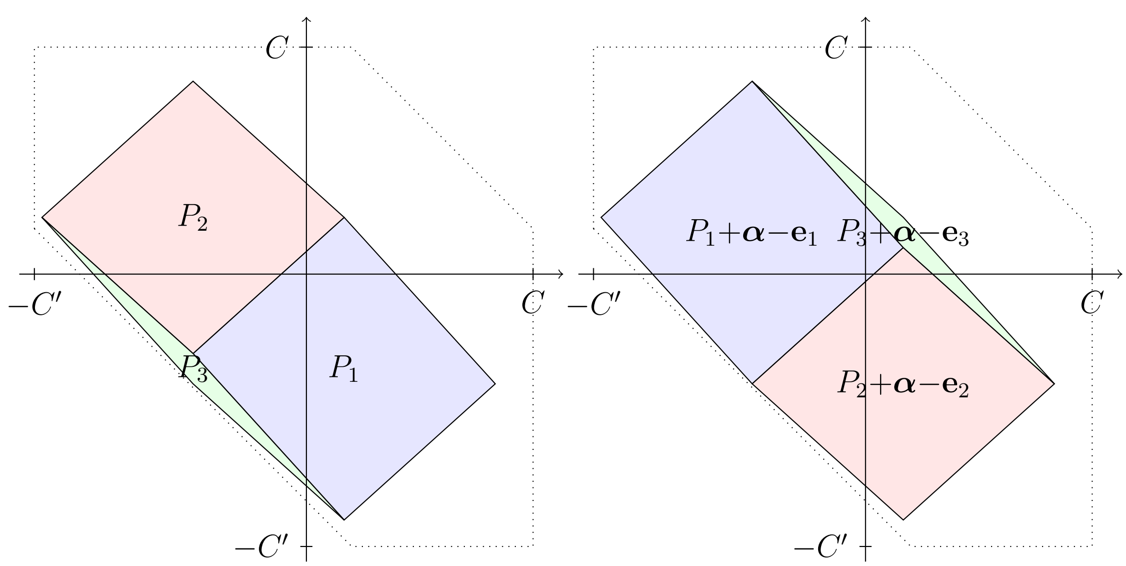

FIGURE 7. The parallelograms $P_i = P_{\boldsymbol{\alpha},C,C',i}$ for $d = 3$, $\boldsymbol{\alpha} \approx (0.5, 0.45, 0.05)$, $C = 3/4$, $C' = 9/10$.

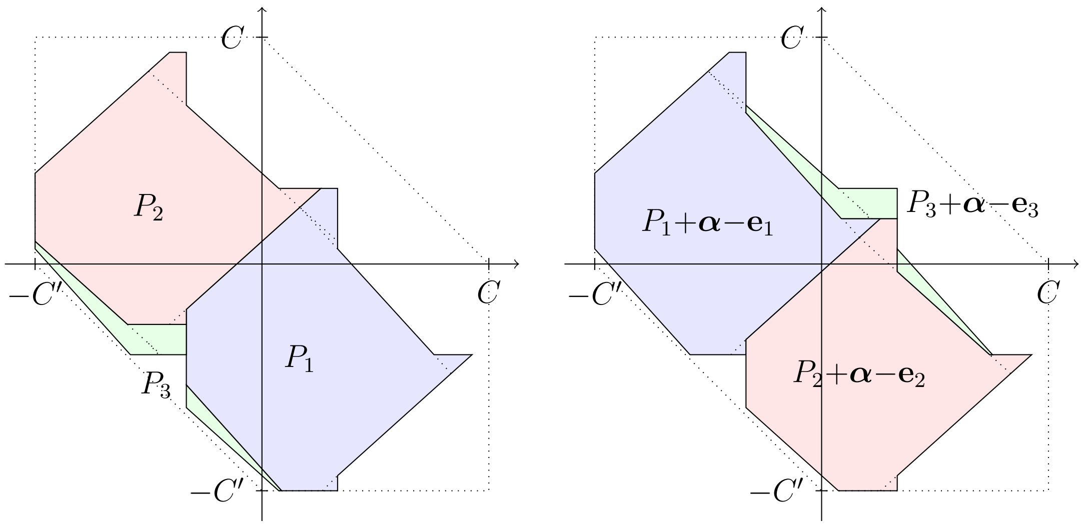

FIGURE 8. For $d = 3$, $\boldsymbol{\alpha} \approx (0.5, 0.45, 0.05)$, $C = C' = 3/4$, the $P_i = P_{\boldsymbol{\alpha},C,C',i}$ are unions of polygons.

4. Proof of Theorem 1.1

Let u be a generalized Tijdeman sequence with parameters $(\boldsymbol{\alpha}, C, C', \mathbf{x}_0)$ where $\boldsymbol{\alpha} = (\alpha_i)_{1\le i\le d}$ is a totally irrational frequency vector. Then u is eventually the bounded natural coding of $T_{\boldsymbol{\alpha}}$ via a partition of a measurable fundamental domain of $\mathbb{T}^{d-1}$ into d finite unions of convex polytopes, according to Definition 2.3 and Propositions 3.9 and 3.10. The fact that u is a bounded natural coding implies the dynamical properties stated in Theorem 1.1. Indeed, the shift (X, S) is minimal, uniquely ergodic, and has purely discrete spectrum according to [Che09, Theorems A and B] (for $d = 3$) and [BST23, Lemma 5.12] for $d > 3$.

It remains to prove the statement on the factor complexity. The proof runs from Section 4.1 to 4.4. The general strategy is described in Section 4.1. The upper bound is provided in Section 4.2 (see Proposition 4.2), and the lower bound is based on Theorem 4.3 (see Section 4.3). The proof of the lower bound lastly requires some assumption stated in Theorem 4.3, which is handled in Section 4.4.

Note that the result on the factor complexity is classical in the setting of cut and project schemes with a polytopal window; see e.g. [Che09, Jul10, KW21]. More precisely, convex polytopal windows are considered in [KW21] and [KW22, Theorem 6.1]; see also [Wal24] and the papers [Che09, Jul10] dealing with dimension $d = 3$. We provide here a proof that works in any dimension $d \ge 2$, and with finite unions of convex polytopes.

4.1. **General strategy for estimating the factor complexity.** In this and the two subsequent sections, we work with maps acting on the d-dimensional torus for the sake of simplicity in the notation, although we have considered so far the $(d-1)$-dimensional torus when handling an alphabet of size d.

Let $T_{\boldsymbol{\alpha}}$ be the translation defined by $T_{\boldsymbol{\alpha}} : \mathbf{x} \in \mathbb{T}^d \mapsto \mathbf{x}+\boldsymbol{\alpha} \in \mathbb{T}^d$ assumed to be minimal, i.e., the coordinates of $(1, \boldsymbol{\alpha})$ are rationally independent. Suppose that

$$\mathbb{T}^d = W_1 \cup W_2 \cup \cdots \cup W_k,$$

where the sets W_i, $i = 1, \ldots, k$, are closed sets with disjoint interiors. We want to estimate the complexity of the coding of a trajectory $(T_{\boldsymbol{\alpha}}^n(\mathbf{x}))_n$ with respect to the partition $\{W_1, W_2, \ldots, W_k\}$.

Let $F = \bigcup_{i=1}^k \partial W_i$ denote the union of the boundaries of the W_i. For a positive integer n, let $\mathcal{F}_n$ denote the set of connected components of $\mathbb{T}^d \setminus \bigcup_{i=0}^n T_{\boldsymbol{\alpha}}^{-i}(F)$.

The main observation is that if $\mathbf{x}$ and $\mathbf{y}$ are in the same connected component $U \in \mathcal{F}_n$ then, for $j = 1, \ldots, n$, the two points $T_{\boldsymbol{\alpha}}^j(\mathbf{x})$ and $T_{\boldsymbol{\alpha}}^j(\mathbf{y})$ are in the same W_i and even in the interior of the same W_i. Indeed, let $\gamma : [0, 1] \to U$ be a continuous path going from $\mathbf{x}$ to $\mathbf{y}$. If for some $j \in \{0, \ldots, n\}$, $T_{\boldsymbol{\alpha}}^j(\mathbf{x}) \in W_i$ and $T_{\boldsymbol{\alpha}}^j(\mathbf{y}) \notin W_i$, then the path $T_{\boldsymbol{\alpha}}^j \circ \gamma$ would have to intersect the boundary of W_i, but this impossible for, by definition of F, U does not meet $T_{\boldsymbol{\alpha}}^{-j}(\partial W_i)$.

It follows that if $u = (u_n)_{n\in\mathbb{N}}$ is the coding sequence of a point $\mathbf{x}_0 \in \mathbb{T}^d$ such that, for all $n \in \mathbb{N}$, $T^n(\mathbf{x}_0) \in \mathring{W}_{u_n}$, then the factor complexity of the sequence $u = (u_n)_{n\in\mathbb{N}}$ satisfies

$$p_u(n+1) \leq \operatorname{card} \mathcal{F}_n \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

When the sets W_i are finite unions of convex polytopes, using translates of the hyperplanes supporting the facets of the convex polytopes, it is easy to bound the above $\operatorname{card} \mathcal{F}_n$; this is the object of Proposition 4.2 below.

4.2. **Upper bound for the factor complexity.** We will need the following lemma, whose proof is a double induction on the dimension and n, along with a use of the formula $\binom{n}{p}+\binom{n}{p-1} = \binom{n+1}{p}$.

Lemma 4.1. [BY98, Theorem 14.2.5] *If $H_1, \ldots, H_n$ are n affine hyperplanes in $\mathbb{R}^d$, then $\mathbb{R}^d \setminus (H_1 \cup \cdots \cup H_n)$ has at most $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{d}$ connected components. Hence the number of connected components of the complement of the union of n hyperplanes in $\mathbb{R}^d$ is $O(n^d)$.*

Call $\pi_{\mathbb{T}^d} : \mathbb{R}^d \to \mathbb{T}^d$ the projection $\mathbf{x} \in \mathbb{R}^d \mapsto \mathbf{x}+\mathbb{Z}^d \in \mathbb{T}^d$. Fix a norm on $\mathbb{R}^d$ and denote by $B(\mathbf{x}, r)$ the open ball of center $\mathbf{x} \in \mathbb{R}^d$ and radius r associated with this norm.

Proposition 4.2. *Let $\boldsymbol{\alpha} \in \mathbb{R}^d$ and let $T_{\boldsymbol{\alpha}} : \mathbb{T}^d \to \mathbb{T}^d$ be the translation defined by $T_{\boldsymbol{\alpha}}(\mathbf{x}) = \mathbf{x}+\boldsymbol{\alpha} \in \mathbb{T}^d$. Suppose that*

$$\mathbb{T}^d = W_1 \cup W_2 \cup \cdots \cup W_k,$$

where the sets W_i, $i = 1, \ldots, k$ have disjoint interiors and are the projections of finite unions of (bounded) convex polytopes.

Then there exists a constant $C > 0$ such that, when $u = (u_n)_{n\in\mathbb{N}}$ is the coding sequence of a point $\mathbf{x}_0 \in \mathbb{T}^d$ satisfying $T^n(\mathbf{x}_0) \in \mathring{W}_{u_n}$ for all $n \in \mathbb{N}$, the factor complexity of u satisfies

$$p_u(n) \leq Cn^d \quad \textit{for all } n \in \mathbb{N}.$$

Proof. We follow the idea of the previous section by bounding above the number of connected components of $\mathbb{T}^d \setminus \bigcup_{j=0}^n T_{\boldsymbol{\alpha}}^{-j}(F)$, where, as previously, $F = \bigcup_{i=1}^k \partial W_i$. To achieve this goal, we use a finite union of hyperplanes whose projection contains F. Indeed, by assumption, for each i, $W_i = \bigcup_{j\in J_i} \pi_{\mathbb{T}^d}(C_{i,j})$, where J_i is a finite set and the $C_{i,j}$ are convex polytopes. Let $\mathcal{D} = \bigcup_{i=1}^k \bigcup_{j\in J_i} C_{i,j}$. Since $\partial W_i \subset \bigcup_{j\in J_i} \pi_{\mathbb{T}^d}(\partial C_{i,j})$ for each i, there exists a finite set $\mathcal{H}$ of affine hyperplanes such that $F = \bigcup_{i=1}^k \partial W_i \subset \pi_{\mathbb{T}^d}(\bigcup_{H\in\mathcal{H}} H \cap \mathcal{D})$.

Let $B(\mathbf{0}, R)$ be a ball containing $\mathcal{D}$; note that $\mathcal{D}$ is bounded as a finite union of convex polytopes. There exists a constant $N \in \mathbb{N}$, depending only on the radius R and on the dimension d, such that, for all $\mathbf{a} \in \mathbb{R}^d$, there exist at most N vectors $\mathbf{q} \in \mathbb{Z}^d$ such that $(B(\mathbf{a}, R)+\mathbf{q}) \cap [0,1]^d \neq \emptyset$. It follows that, for any integer j and for any hyperplane $H \in \mathcal{H}$, the set $E_{H,j}$ of vectors $\mathbf{q} \in \mathbb{Z}^d$ such that $(H \cap \mathcal{D} - j\boldsymbol{\alpha} + \mathbf{q}) \cap [0,1]^d \neq \emptyset$, has cardinality at most N. By definition of the set $E_{H,j}$, we have

$$T_{\boldsymbol{\alpha}}^{-j}(\pi_{\mathbb{T}^d}(H \cap \mathcal{D})) = \pi_{\mathbb{T}^d}\Bigg(\bigcup_{\mathbf{q}\in E_{H,j}} (H \cap \mathcal{D} - j\boldsymbol{\alpha} + \mathbf{q}) \cap [0,1]^d\Bigg).$$

It follows that, for any positive integer n,

$$\begin{aligned}\bigcup_{j=0}^n T_{\boldsymbol{\alpha}}^{-j}(F) &\subset \pi_{\mathbb{T}^d}\Bigg(\bigcup_{j=0}^n \bigcup_{H\in\mathcal{H}} \bigcup_{\mathbf{q}\in E_{H,j}} (H \cap \mathcal{D} - j\boldsymbol{\alpha} + \mathbf{q}) \cap [0,1]^d\Bigg) \\ &\subset \pi_{\mathbb{T}^d}\Bigg(\bigcup_{j=0}^n \bigcup_{H\in\mathcal{H}} \bigcup_{\mathbf{q}\in E_{H,j}} (H - j\boldsymbol{\alpha} + \mathbf{q}) \cap [0,1]^d\Bigg).\end{aligned}$$

Let $\mathcal{C}_n$ be the set of connected components of

$$\Omega = (0,1)^d \setminus \bigcup_{j=0}^n \bigcup_{H\in\mathcal{H}} \bigcup_{\mathbf{q}\in E_{H,j}} (H - j\boldsymbol{\alpha} + \mathbf{q}).$$

We recall that $\mathcal{F}_n$ stands for the set of connected components of $\mathbb{T}^d \setminus \bigcup_{j=0}^n T_{\boldsymbol{\alpha}}^{-j}(F)$. On the one hand, if $V \in \mathcal{C}_n$, then $\pi_{\mathbb{T}^d}(V)$ does not intersect $\bigcup_{j=0}^n T_{\boldsymbol{\alpha}}^{-j}(F)$, hence $\pi_{\mathbb{T}^d}(V)$ is contained in some $U \in \mathcal{F}_n$. On the other hand, since $\pi_{\mathbb{T}^d}(\Omega)$ is dense in $\mathbb{T}^d$, each $U \in \mathcal{F}_n$ intersects at least one $\pi_{\mathbb{T}^d}(V)$, where $V \in \mathcal{C}_n$. It follows that the map which associates with each $V \in \mathcal{C}_n$ the unique $U \in \mathcal{F}_n$ containing $\pi_{\mathbb{T}^d}(V)$ is surjective; hence

$$\operatorname{card} \mathcal{F}_n \leq \operatorname{card} \mathcal{C}_n.$$

By (4.1), if $\operatorname{card} \mathcal{C}_n \leq C n^d$ for some constant C that does not depend on n, the proposition holds. Now, the set $\mathcal{C}_n$ is included in the set of connected components of $\mathbb{R}^d$ minus the union of the following sets of hyperplanes:

- $H - j\boldsymbol{\alpha} + \mathbf{q}$ with $H \in \mathcal{H}$, $j \in \{0, \ldots, n\}$, $\mathbf{q} \in E_{H,j}$,
- $\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i = c\}$, $i = 1, \ldots, d$, $c = 0, 1$.

The number of these hyperplanes is bounded above by

$$2d + (n+1) N \operatorname{card} \mathcal{H} \leq C_1 n$$

for some constant C_1 that does not depend on n. The proposition is now a consequence of Lemma 4.1. $\square$

4.3. **Lower bound for the factor complexity.** In Section 4.2, we obtained the upper bound $p_u(n) = O(n^d)$. Obtaining a lower bound is more difficult. Roughly, the idea is to find a subset $\mathcal{G}_n$ of $\mathcal{F}_n$, the set of connected components of $\mathbb{T}^d \setminus \bigcup_{i=0}^{n} T_{\boldsymbol{\alpha}}^{-i}(F)$, such that, if $\mathbf{x}$ and $\mathbf{y}$ are in two different elements of $\mathcal{G}_n$, then there is at least one $j \in \{1, \ldots, n\}$ such that $T_{\boldsymbol{\alpha}}^{j}(\mathbf{x})$ and $T_{\boldsymbol{\alpha}}^{j}(\mathbf{y})$ are not in the same W_i. Note that the existence statement for the linear forms $f_1, \ldots, f_d$ below will be handled in Section 4.4.

Theorem 4.3. *Let $T_{\boldsymbol{\alpha}}$ be the translation $T_{\boldsymbol{\alpha}} : \mathbf{x} \in \mathbb{T}^d \mapsto \mathbf{x}+\boldsymbol{\alpha} \in \mathbb{T}^d$ assumed to be minimal. Suppose that*

$$\mathbb{T}^d = W_1 \cup W_2 \cup \cdots \cup W_k,$$

where the sets W_i, $i = 1, \ldots, k$, are closed sets with disjoint interiors. Assume that there are d independent linear forms $f_1, \ldots, f_d$ on $\mathbb{R}^d$, d points $\mathbf{a}_1, \ldots, \mathbf{a}_d \in \mathbb{R}^d$ and $r > 0$ such that, for each $i \in \{1, \ldots, d\}$,

$$\pi_{\mathbb{T}^d}(B(\mathbf{a}_i, r) \cap \{f_i < f_i(\mathbf{a}_i)\}) \subset W_{b(i)} \quad \text{and} \quad \pi_{\mathbb{T}^d}(B(\mathbf{a}_i, r) \cap \{f_i > f_i(\mathbf{a}_i)\}) \subset W_{c(i)},$$

where $b(i) \neq c(i)$ are in $\{1, \ldots, k\}$.

Then there exists $c > 0$ such that, if $u = (u_n)_{n\in\mathbb{N}}$ is the coding sequence of a point $\mathbf{x}_0 \in \mathbb{T}^d$ with $T_{\boldsymbol{\alpha}}^n(\mathbf{x}_0) \in \mathring{W}_{u_n}$ for all $n \in \mathbb{N}$, the factor complexity of the sequence $u = (u_n)_{n\in\mathbb{N}}$ satisfies

$$p_u(n) \geq cn^d \quad \text{for all } n \in \mathbb{N}.$$

The proof of the above theorem uses the following lemma.

Lemma 4.4. *Let $\boldsymbol{\alpha} \in \mathbb{R}^d$ with $(1, \boldsymbol{\alpha})$ having rationally independent coordinates. Let $H \subset \mathbb{R}^d$ be an affine hyperplane and let $K \subset \mathbb{R}^d$ be a bounded subset. Then there exists a positive integer Q, depending only on K and the direction of H, such that, for all $\mathbf{a} \in \mathbb{R}^d$, there are at most Q integers q such that $\mathbf{a}+q\boldsymbol{\alpha} \in K \cap H + \mathbb{Z}^d$.*

Proof. It is enough to prove that there exists a positive integer Q depending only on K and the direction of H such that, for all $\mathbf{a} \in \mathbb{R}^d$ and all $m \in \mathbb{Z}$, there exists at most one $k \in \mathbb{Z}$ satisfying $\mathbf{a}+(m+kQ)\boldsymbol{\alpha} \in K \cap H + \mathbb{Z}^d$.

Suppose first that $\mathbf{a} = \mathbf{0}$ and that H is a vector hyperplane, i.e., $\mathbf{0} \in H$. Let $H_{\mathbb{Q}}$ be the vector space generated by $H \cap \mathbb{Q}^d$ and let f be a nonzero rational linear form such that $H_{\mathbb{Q}} \subset \ker f$. Since $\boldsymbol{\alpha}$ is totally irrational, we have, for all $k \in \mathbb{Z} \setminus \{0\}$ and all $\mathbf{q} \in \mathbb{Z}^d$, $f(k\boldsymbol{\alpha}+\mathbf{q}) \neq 0$ and hence $k\boldsymbol{\alpha}+\mathbf{q} \notin H_{\mathbb{Q}}$. Furthermore, if $k\boldsymbol{\alpha}+\mathbf{q} \in H$ and if $k\boldsymbol{\alpha}+\mathbf{q}' \in H$ for some other point $\mathbf{q}' \in \mathbb{Z}^d$, then $\mathbf{q}-\mathbf{q}' \in H \cap \mathbb{Q}^d \subset H_{\mathbb{Q}}$.

Let q_0 be the smallest positive integer such that $q_0\boldsymbol{\alpha} \in H+\mathbb{Z}^d$, if any. If there is no such integer $q_0 > 0$, then we are done because $q = 0$ will be the only integer such that $q\boldsymbol{\alpha} \in H+\mathbb{Z}^d$. Let $\mathbf{q}_0$ be a point in $\mathbb{Z}^d$ such that $q_0\boldsymbol{\alpha}+\mathbf{q}_0 \in H$.

Let $r > 0$ be such that $K \subset B(\mathbf{0}, r)$. Set $k_0 = \lceil \frac{2r}{r_0} \rceil + 1$, where $r_0 = \mathrm{d}(q_0\boldsymbol{\alpha}+\mathbf{q}_0, H_{\mathbb{Q}}) > 0$. Set $Q = k_0 q_0$. We want to show that each class modulo Q contains at most one integer m

such that $m\boldsymbol{\alpha} \in B(\mathbf{0}, r) \cap H + \mathbb{Z}^d$. Let $m \in \mathbb{Z}$ and $\mathbf{q}_m \in \mathbb{Z}^d$ be such that $m\boldsymbol{\alpha}+\mathbf{q}_m \in B(\mathbf{0}, r) \cap H$. If $k \in \mathbb{Z} \setminus \{0\}$ and $\mathbf{q} \in \mathbb{Z}^d$ are such that $(m+kQ)\boldsymbol{\alpha}+\mathbf{q} \in H$, then

$$\mathbf{q} - \mathbf{q}_m - kk_0\mathbf{q}_0 = (m+kQ)\boldsymbol{\alpha} + \mathbf{q} - (m\boldsymbol{\alpha}+\mathbf{q}_m) - (kk_0q_0\boldsymbol{\alpha}+kk_0\mathbf{q}_0) \in H_{\mathbb{Q}}.$$

Therefore, for any $\mathbf{u} \in H_{\mathbb{Q}}$, we have

$$\begin{aligned} \|(m+kQ)\boldsymbol{\alpha} + \mathbf{q} - \mathbf{u}\| &= \|(kk_0q_0\boldsymbol{\alpha}+kk_0\mathbf{q}_0+\mathbf{q}-\mathbf{q}_m-kk_0\mathbf{q}_0) - \mathbf{u} + (m\boldsymbol{\alpha}+\mathbf{q}_m)\| \\ &\geq \|kk_0q_0\boldsymbol{\alpha} + kk_0\mathbf{q}_0 + (\mathbf{q}-\mathbf{q}_m-kk_0\mathbf{q}_0) - \mathbf{u}\| - \|m\boldsymbol{\alpha}+\mathbf{q}_m\| \\ &\geq \mathrm{d}(kk_0q_0\boldsymbol{\alpha}+kk_0\mathbf{q}_0, H_{\mathbb{Q}}) - \|m\boldsymbol{\alpha}+\mathbf{q}_m\|. \end{aligned}$$

Notice that, for all $t \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^d$, $\mathrm{d}(t\mathbf{a}, H_{\mathbb{Q}}) = |t|\,\mathrm{d}(\mathbf{a}, H_{\mathbb{Q}})$. Therefore,

$$\begin{aligned} \|(m+kQ)\boldsymbol{\alpha} + \mathbf{q} - \mathbf{u}\| &\geq |kk_0|\,\mathrm{d}(q_0\boldsymbol{\alpha}+\mathbf{q}_0, H_{\mathbb{Q}}) - \|m\boldsymbol{\alpha}+\mathbf{q}_m\| \\ &= |k|k_0r_0 - r \geq |k|\big(\tfrac{2r}{r_0}+1\big)r_0 - r > r. \end{aligned}$$

It follows that $\mathrm{d}((m+kQ)\boldsymbol{\alpha}+\mathbf{q}, \mathbf{0}) \geq \mathrm{d}((m+kQ)\boldsymbol{\alpha}+\mathbf{q}, H_{\mathbb{Q}}) > r$, which in turn implies that $(m+kQ)\boldsymbol{\alpha}+\mathbf{q} \notin B(\mathbf{0}, r)$.

Suppose now that H is an affine hyperplane. Let $\mathbf{a} \in \mathbb{R}^d$. If there is no pair $(q, \mathbf{q}) \in \mathbb{Z}\times\mathbb{Z}^d$ such that $\mathbf{a}+q\boldsymbol{\alpha}+\mathbf{q} \in K \cap H$, then we are done. Otherwise fix such a pair $(q_1, \mathbf{q}_1)$. We are going to use the first step with the bounded subset $K' = K-K$ and the vector space $\overrightarrow{H} = H-(\mathbf{a}+q_1\boldsymbol{\alpha}+\mathbf{q}_1)$, which is the direction of H. By the first step, there exists an integer Q such that for any integer m' there is at most one integer k such that $(m'+kQ)\boldsymbol{\alpha} \in K' \cap \overrightarrow{H} + \mathbb{Z}^d$. Now, for any m and k, if $\mathbf{a}+(m+kQ)\boldsymbol{\alpha} \in K \cap H + \mathbb{Z}^d$, then

$$(m-q_1+kQ)\boldsymbol{\alpha} \in K \cap H - (\mathbf{a}+q_1\boldsymbol{\alpha}+\mathbf{q}_1) + \mathbf{q}_1 + \mathbb{Z}^d,$$

and, since $K \cap H - (\mathbf{a}+q_1\boldsymbol{\alpha}+\mathbf{q}_1) \subset K' \cap \overrightarrow{H}$, we have

$$(m'+kQ)\boldsymbol{\alpha} \in K' \cap \overrightarrow{H} + \mathbb{Z}^d,$$

with $m' = m-q_1$. Therefore, given $m \in \mathbb{Z}$, there exists at most one integer k such that $\mathbf{a}+(m+kQ)\boldsymbol{\alpha} \in K \cap H + \mathbb{Z}^d$. $\square$

Proof of Theorem 4.3. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\} \subset \mathbb{R}^d$ be the dual basis of the basis $\{f_1, \ldots, f_d\}$ of linear forms. Let $\|\mathbf{x}\| = \max_{i=1,\ldots,d} |f_i(\mathbf{x})|$ denote the supremum norm associated with the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$. By reducing the positive real number r of the assumption on the W_i, we can suppose that the balls $B(\cdot,\cdot)$ are associated with the supremum norm $\|\cdot\|$. For $i = 1, \ldots, d$, set

$$U_i = \{\mathbf{y} \in \mathbb{R}^d \,:\, \|\mathbf{y}\| < \tfrac{r}{2},\ f_i(\mathbf{y}) > 0\}$$

and, for $n \in \mathbb{N}$,

$$J_{i,n} = \big\{p \in \{0, \ldots, n\} \,:\, T_{\boldsymbol{\alpha}}^{-p}(\pi_{\mathbb{T}^d}(\mathbf{a}_i)) \in \pi_{\mathbb{T}^d}(U_i)\big\}.$$

Since the translation $T_{\boldsymbol{\alpha}}$ is uniquely ergodic (by assumption on $\boldsymbol{\alpha}$), one has by ergodicity $\lim_{n\to\infty} \frac{1}{n} \operatorname{card} J_{i,n} = 2c_1$, where $2c_1$ is the Lebesgue measure of $\pi_{\mathbb{T}^d}(U_i)$. Thus there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and all $i \in \{1, \ldots, d\}$,

$$\operatorname{card} J_{i,n} \geq c_1 n$$

Fix now $n \geq n_0$. For each $p \in J_{i,n}$, let $\mathbf{q}_{i,p} \in \mathbb{Z}^d$ be such that $\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p} \in U_i$. Making use of Lemma 4.4 with the hyperplanes $H_{i,p} = \{\mathbf{x} \in \mathbb{R}^d : f_i(\mathbf{x}) = f_i(\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p})\}$ and the bounded sets $K_{i,p} = U_i$, we see that, for each $i \in \{1, \ldots, d\}$, there exists $Q_i \in \mathbb{N}$ such that, for each $p \in J_{i,n}$, there are at most Q_i integers $q \in J_{i,n}$ with $\mathbf{a}_i - q\boldsymbol{\alpha} \in H_{i,p} \cap U_i + \mathbb{Z}^d$. Set $Q = \max\{Q_1, \ldots, Q_d\}$.

Observe that, if $p, q \in J_{i,n}$ are such that $f_i(\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p}) = f_i(\mathbf{a}_i - q\boldsymbol{\alpha} + \mathbf{q}_{i,q})$, then $\mathbf{a}_i - q\boldsymbol{\alpha} + \mathbf{q}_{i,q}$ is both in U_i and $H_{i,p}$, hence $\mathbf{a}_i - q\boldsymbol{\alpha} \in H_{i,p} \cap U_i + \mathbb{Z}^d$. Therefore, for each $i \in \{1, \ldots, d\}$, we can find a subset $J'_{i,n} \subset J_{i,n}$, such that

- $\operatorname{card} J'_{i,n} \geq \frac{1}{Q} c_1 n = c_2 n$,
- $f_i(\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p}) \neq f_i(\mathbf{a}_i - q\boldsymbol{\alpha} + \mathbf{q}_{i,q})$ for all $p, q \in J'_{i,n}$ with $p \neq q$.

Set

$$U^+ = \{\mathbf{y} \in \mathbb{R}^d \,:\, \|\mathbf{y}\| < \tfrac{r}{2},\, f_i(\mathbf{y}) > 0,\, i = 1, \ldots, d\}$$

and, for $i \in \{1, \ldots, d\}$ and $p \in J'_{i,n}$, set

$$\begin{aligned} R^+_{i,p} &= U^+ \cap B(\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p}, r) \cap \{\mathbf{y} : f_i(\mathbf{y}) > f_i(\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p})\}, \\ R^-_{i,p} &= U^+ \cap B(\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p}, r) \cap \{\mathbf{y} : f_i(\mathbf{y}) < f_i(\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p})\}. \end{aligned}$$

Let $p \in J'_{i,n}$. Since $\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p} \in U_i \subset B(\mathbf{0}, \frac{1}{2} r)$, the set U^+ is included in the ball $B(\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p}, r)$. Therefore,

$$\begin{aligned} R^+_{i,p} &= U^+ \cap \{\mathbf{y} : f_i(\mathbf{y}) > f_i(\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p})\}, \\ R^-_{i,p} &= U^+ \cap \{\mathbf{y} : f_i(\mathbf{y}) < f_i(\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p})\}. \end{aligned}$$

Furthermore, if $\mathbf{y} \in R^+_{i,p}$, then, by definition of $R^+_{i,p}$, $\mathbf{y} = \mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p} + \mathbf{x}$ with $\mathbf{x} \in B(\mathbf{0}, \frac{r}{2})$ and $f_i(\mathbf{x}) > 0$, hence

$$\mathbf{y} + p\boldsymbol{\alpha} - \mathbf{q}_{i,p} = \mathbf{a}_i + \mathbf{x} \in \mathbf{a}_i + B(\mathbf{0}, \tfrac{r}{2}) \cap \{\mathbf{z} : f_i(\mathbf{z}) > 0\} = B(\mathbf{a}_i, \tfrac{r}{2}) \cap \{f_i > f(\mathbf{a}_i)\},$$

and, since by assumption $\pi_{\mathbb{T}^d}(B(\mathbf{a}_i, \frac{r}{2}) \cap \{f > f(\mathbf{a}_i)\}) \subset W_{c(i)}$,

$$T^p_{\boldsymbol{\alpha}}(\pi_{\mathbb{T}^d}(\mathbf{y})) \in W_{c(i)}.$$

In the same way, if $\mathbf{y} \in R^-_{i,p}$, then

$$T^p_{\boldsymbol{\alpha}}(\pi_{\mathbb{T}^d}(\mathbf{y})) \in W_{b(i)}.$$

For each $i \in \{1, \ldots, d\}$, the real numbers $k_{i,p} = f_i(\mathbf{a}_i - p\boldsymbol{\alpha} + \mathbf{q}_{i,p})$, $p \in J'_{i,n}$, are pairwise distinct and are in the interval $(0, \frac{r}{2})$ by definition of $J'_{i,n}$. It follows that the “coordinate” hyperplanes

$$\{\mathbf{y} \,:\, f_i(\mathbf{y}) = k_{i,p}\} = \left\{ \sum_{j=1}^{d} y_i \mathbf{e}_i \in \mathbb{R}^d \,:\, y_i = k_{i,p} \right\},$$

for $i \in \{1, \ldots, d\}$ and $p \in J'_{i,n}$, divide U^+ into a set E of nonempty coordinate parallelepipeds with cardinality $\prod_{i=1}^{d} (\operatorname{card} J'_{i,n} + 1) \geq (c_2 n)^d$. Suppose that $\mathbf{y}$ and $\mathbf{y}'$ are in two such distinct coordinate parallelepipeds. This means that there exist $i \in \{1, \ldots, d\}$ and $p \in J'_{i,n}$ such that $f_i(\mathbf{y}) > k_{i,p} > f_i(\mathbf{y}')$, which implies that $\mathbf{y} \in R^+_{i,p}$ and $\mathbf{y}' \in R^-_{i,p}$, therefore $T^p_{\boldsymbol{\alpha}}(\pi_{\mathbb{T}^d}(\mathbf{y})) \in W_{c(i)}$ and $T^p_{\boldsymbol{\alpha}}(\pi_{\mathbb{T}^d}(y') \in W_{b(i)}$. Since, for each parallelepiped $\mathcal{P} \in E$, there exists a nonnegative integer $m_{\mathcal{P}}$ such that $T_{\boldsymbol{\alpha}}^{m_{\mathcal{P}}}(\mathbf{x}_0) \in \pi_{\mathbb{T}^d}(\mathcal{P})$, the factors $u_{m_{\mathcal{P}}} \cdots u_{m_{\mathcal{P}}+n}$, $\mathcal{P} \in E$, of the sequence u encoding $\mathbf{x}_0$ are pairwise distinct. This implies that

$$p_u(n+1) \geq (c_2 n)^d. \qquad \square$$

4.4. **End of the proof of Theorem 1.1.** We now want to use Theorem 4.3 to bound below the complexity. It remains to prove the existence statement for the linear forms $f_1, \cdots, f_d$ from Theorem 4.3.

For each $i \in \{1, \ldots, d\}$, call $W_i = \pi_{\mathbb{T}^{d-1}}(P_i)$ the projection of P_i in the torus. Given $i \in \{1, \ldots, d\}$, we now describe the boundary of W_i. Since each P_i is a finite union of convex polytopes and since $\pi_{\mathbb{T}^{d-1}}^{-1}(W_i) = \bigcup_{\mathbf{x} \in \mathbb{Z}^{d-1}} (\mathbf{x}+P_i)$, the set $\pi_{\mathbb{T}^{d-1}}^{-1}(W_i)$ is locally a finite union of bounded convex polytopes. Given a nonzero linear form f on $\mathbb{R}^{d-1}$ and a real number c, denote

$$H_{f,c} = \{\mathbf{q} \in \mathbb{R}^{d-1} : f(\mathbf{q}) = c\}$$

the hyperplane defined by f and c. When it is nonempty, we call the relative interior of $(-1,1)^{d-1} \cap \pi_{\mathbb{T}^{d-1}}^{-1}(\partial W_i) \cap H_{f,c}$ a *facet* of W_i associated with f and c.

Consider the set W^* of nonzero linear forms f on $\mathbb{R}^{d-1}$ such that there exist $i \in \{1, \ldots, d\}$ and $c \in \mathbb{R}$ such that $(-1,1)^{d-1} \cap \pi_{\mathbb{T}^{d-1}}^{-1}(\partial W_i) \cap H_{f,c}$ has nonempty interior relatively to the hyperplane $H_{f,c}$.

Thanks to the following lemma whose proof is given below, each ∂W_i is the union of the projections of the closures of the facets of W_i. Observe that each P_i is a finite union of convex polytopes with nonempty interiors because P_i is the closure of its interior.

Lemma 4.5. *Let $K \subset \mathbb{R}^{d-1}$ be a finite union of convex polytopes with nonempty interiors. Then, for any $\mathbf{x} \in \partial K$, there exist a linear form f and $c \in \mathbb{R}$ such that $H_{f,c} \cap \partial K$ has nonempty interior relative to $H_{f,c}$ and such that $\mathbf{x}$ is in the closure of this relative interior.*

Moreover, there are only finitely many hyperplanes $H_{f,c}$ such that the intersection $(-1,1)^{d-1} \cap \pi_{\mathbb{T}^{d-1}}^{-1}(\partial W_j) \cap H_{f,c}$ has nonempty interior.

Let U be a facet of some W_i. Since $\pi_{\mathbb{T}^{d-1}}(U)$ is included in the boundary of $\bigcup_{j \neq i} W_j$, the closures of the facets of the W_j, $j \neq i$, cover U, and, since there are only finitely many such facets, there exist $j \neq i$ and a facet V of W_j such that $V \cap U \neq \emptyset$. Since W_i and W_j have disjoint interiors, the facets U and V must be defined by the same hyperplane. It follows that, for each $f \in W^*$, there exist $\mathbf{a} \in \mathbb{R}^{d-1}$, $i \neq j$ in $\{1, \ldots, d\}$ and $r > 0$ such that

$$\pi_{\mathbb{T}^{d-1}}(B(\mathbf{a}, r) \cap \{f < f(\mathbf{a})\}) \subset W_i \quad \text{and} \quad \pi_{\mathbb{T}^{d-1}}(B(\mathbf{a}, r) \cap \{f > f(\mathbf{a})\}) \subset W_j.$$

Thus, to use Theorem 4.3, we only have to prove that the linear forms f in W^* generate the vector space of all linear forms on $\mathbb{R}^{d-1}$. Suppose on the contrary that W^* does not generate the vector space of all linear forms. With this assumption, the intersection $\bigcap_{f \in W^*} \ker f$ contains a nonzero vector $\mathbf{v}$. Consider the set $\mathcal{H}$ of all hyperplanes $H_{f,c}$ associated with a facet of one of the W_j, and let $G = \bigcup_{H \in \mathcal{H}} H + \mathbb{Z}^{d-1} + \mathbb{Z}\boldsymbol{\alpha}$. Since G is a countable union of hyperplanes, $\mathbb{R}^{d-1} \setminus G$ is nonempty. Moreover by definition of G, if $\mathbf{a} \notin G$, then $(\mathbf{a}+\mathbb{R}\mathbf{v}) \cap G = \emptyset$, therefore $\pi_{\mathbb{T}^{d-1}}(\mathbf{a}+\mathbb{R}\mathbf{v})$ does not meet $\pi_{\mathbb{T}^{d-1}}(G)$, which in turn implies that $\pi_{\mathbb{T}^{d-1}}(\mathbf{a}+\mathbb{R}\mathbf{v})$ does not meet $\bigcup_{n \in \mathbb{Z}} T_{\boldsymbol{\alpha}}^n(\bigcup_{j=1}^d \partial W_j)$. It follows that, if

$\mathbf{x} \in \mathbb{T}^{d-1} \setminus \pi_{\mathbb{T}^{d-1}}(G)$, then all the points in $\mathbf{x}+\mathbb{R}\mathbf{v}$ have the same coding sequence, which contradicts Remark 2.5.

Proof of Lemma 4.5. Let $K = K_1 \cup \cdots \cup K_n \subset \mathbb{R}^{d-1}$ be finite union of convex polytopes and let $\mathbf{x} \in \partial K$. Let $\mathcal{F}$ be the set of closures of the facets of the convex polytopes $K_1, \ldots, K_n$, and let $\mathcal{F}_{\mathbf{x}}$ be the set of $F \in \mathcal{F}$ that contains $\mathbf{x}$. The facets in $\mathcal{F}_{\mathbf{x}}$ are defined by finitely many hyperplanes, $H_1, \ldots, H_N$. Let $r = \frac{1}{2}\,\mathrm{d}(\mathbf{x}, \cup F)$ where the union is over all the facets $F \in \mathcal{F} \setminus \mathcal{F}_{\mathbf{x}}$. The set

$$B(\mathbf{x}, r) \setminus (H_1 \cup \cdots \cup H_N)$$

is finite union of open truncated polyhedral cones of apex $\mathbf{x}$. Each of these cones is contained either in $\mathbb{R}^{d-1} \setminus K$ or in K because these cones intersect none of the boundaries of the K_i. It follows that one cone contained in K and one cone contained in K share a common facet. This facet is contained in a facet of K and its closure contains $\mathbf{x}$. □

5. Additional comments

By definition, Tijdeman sequences with parameters $C = C' = 1 - \frac{1}{2d-2}$ are fairly distributed, i.e., $\Delta_{\boldsymbol{\alpha}}(u) \leq 1 - \frac{1}{2d-2}$. In the case $d = 2$, fairly distributed sequences and Tijdeman sequences coincide, and they are Sturmian sequences; see Remark 3.4. Moreover we have seen in Remark 3.3 that some hypercubic billiard sequences are also fairly distributed (and they are also Tijdeman sequences by Remark 3.6). However, we do not expect all fairly distributed sequences to be Tijdeman sequences.

The present study raises the following natural questions.

- What happens when we consider not only a notion of discrepancy based on occurrences of letters but based on occurrences of factors?
- What happens when the frequency vector $\boldsymbol{\alpha}$ has rationally dependent coordinates, and even when all the entries of $\boldsymbol{\alpha}$ are rational?
- Are there finite sequences with discrepancy smaller than or equal to $1 - \frac{1}{2d-2}$ that cannot be prolonged into a fairly distributed sequence?

We turn to questions related to the factor complexity function $p_u(n)$. If u is a symbolic coding of a piecewise translation map associated with a minimal translation on the torus $\mathbb{T}^{d-1}$, then it is shown in [BB13] that $p_u(n) \geq (d-1)n+1$ for each n. There exist sequences u having a bounded discrepancy function $\Delta_{\boldsymbol{\alpha}}(u)$ and having also a factor complexity of smaller order than Tjideman sequences; they even have linear factor complexity whereas Tijdeman sequences have factor complexity of order $d-1$. However, the price to pay when reducing factor complexity seems to yield an increase of the discrepancy. More precisely, a construction of sequences having both finite discrepancy and linear factor complexity in dimension $d = 3$ for a.e. $\boldsymbol{\alpha}$ is provided in [BST23] with constructions based on the Cassaigne–Selmer multidimensional continued fraction algorithm. These symbolic codings thus enjoy the striking properties of Sturmian sequences combining linear factor complexity and good local discrepancy properties. But the corresponding fundamental domains have fractal boundary. They are obtained as so-called Rauzy fractals which are known to provide suitable and effective windows for cut and project schemes as well as fundamental domains for toral translations. We end with the following questions.

- When $d \geq 3$, what is the lowest bound for $\Delta_{\boldsymbol{\alpha}}(u)$ when we restrict to sequences u with linear factor complexity?
- Does the following hold for $\boldsymbol{\alpha}$ a totally irrational frequency vector: If u is such that $\Delta_{\boldsymbol{\alpha}}(u) \leq D_d$, then there exists $C_u > 0$ such that $p_u(n) \geq C_u n^{d-1}$ for all n (where d is the size of the alphabet).

References

[Ada03] B. Adamczewski, *Balances for fixed points of primitive substitutions*, Theoret. Comput. Sci. **307** (2003), no. 1, 47–75, Words.

[Ada04] ______, *Symbolic discrepancy and self-similar dynamics*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 7, 2201–2234 (2005).

[AGH00] E. Altman, B. Gaujal, and A. Hordijk, *Balanced sequences and optimal routing*, J. ACM **47** (2000), no. 4, 752–775.

[AMST94a] P. Arnoux, C. Mauduit, I. Shiokawa, and J.-I. Tamura, *Complexity of sequences defined by billiard in the cube*, Bull. Soc. Math. France **122** (1994), no. 1, 1–12.

[AMST94b] ______, *Rauzy's conjecture on billiards in the cube*, Tokyo J. Math. **17** (1994), no. 1, 211–218.

[AV22] M. Andrieu and L. Vivion, *Imbalances in hypercubic billiard words*, Journées Montoises 2022, 2022.

[BB13] N. Bédaride and J.-F. Bertazzon, *Minoration of the complexity function associated to a translation on the torus*, Monatsh. Math. **171** (2013), no. 3-4, 291–304.

[BC04] N. Brauner and Y. Crama, *The maximum deviation just-in-time scheduling problem*, Discrete Appl. Math. **134** (2004), no. 1-3, 25–50.

[BCS13] V. Berthé, J. Cassaigne, and W. Steiner, *Balance properties of Arnoux-Rauzy words*, Internat. J. Algebra Comput. **23** (2013), no. 4, 689–703.

[BJ08] N. Brauner and V. Jost, *Small deviations, JIT sequencing and symmetric case of Fraenkel's conjecture*, Discrete Math. **308** (2008), no. 11, 2319–2324.

[BR10] V. Berthé and M. Rigo (eds.), *Combinatorics, automata and number theory*, Encyclopedia of Mathematics and its Applications, vol. 135, Cambridge University Press, Cambridge, 2010.

[BST23] V. Berthé, W. Steiner, and J. M. Thuswaldner, *Multidimensional continued fractions and symbolic codings of toral translations*, J. Eur. Math. Soc. (JEMS) **25** (2023), no. 12, 4997–5057.

[BY75] M. L. Balinski and H. P. Young, *The quota method of apportionment*, Amer. Math. Monthly **82** (1975), no. 7, 701–730.

[BY77] ______, *Apportionment schemes and the quota method*, Amer. Math. Monthly **84** (1977), no. 6, 450–455.

[BY85] ______, *The apportionment of representation*, Fair allocation (Anaheim, Calif., 1985), Proc. Sympos. Appl. Math., vol. 33, Amer. Math. Soc., Providence, RI, 1985, pp. 1–29.

[BY98] J.-D. Boissonnat and M. Yvinec, *Algorithmic geometry*, Cambridge University Press, Cambridge, 1998, Translated from the 1995 French original by Hervé Brönnimann.

[CCV22] J. Cembrano, J. Correa, and V. Verdugo, *Multidimensional political apportionment*, Proceedings of the National Academy of Sciences **119** (2022), no. 15, e2109305119.

[Che09] N. Chevallier, *Coding of a translation of the two-dimensional torus*, Monatsh. Math. **157** (2009), no. 2, 101–130.

[CNP+11] D. Coppersmith, T. Nowicki, G. Paleologo, C. Tresser, and C. W. Wu, *The optimality of the online greedy algorithm in carpool and chairman assignment problems*, ACM Trans. Algorithms **7** (2011), no. 3, Art. 37, 22. MR 2820114

[DMP23] L. Dvoráková, M. Masek, and E. Pelantová, *On a class of 2-balanced sequences*, Combinatorics on Words - 14th International Conference, WORDS 2023, Umeå, Sweden, June 12-16, 2023, Proceedings (Anna E. Frid and Robert Mercas, eds.), Lecture Notes in Computer Science, vol. 13899, Springer, 2023, pp. 143–154.

[FN20] N. Pytheas Fogg and C. Noûs, *Symbolic coding of linear complexity for generic translations of the torus, using continued fractions*, arXiv:2005.12229, preprint, 2020.

[GL15] S. Grepstad and N. Lev, *Sets of bounded discrepancy for multi-dimensional irrational rotation*, Geom. Funct. Anal. **25** (2015), no. 1, 87–133.

[Gra73] R. L. Graham, *Covering the positive integers by disjoint sets of the form* $\{[n\alpha + \beta] : n = 1, 2, \ldots\}$, J. Combinatorial Theory Ser. A **15** (1973), 354–358.

[Hub00] P. Hubert, *Suites équilibrées*, Theoret. Comput. Sci. **242** (2000), no. 1-2, 91–108.

[Jul10] A. Julien, *Complexity and cohomology for cut-and-projection tilings*, Ergodic Theory Dynam. Systems **30** (2010), no. 2, 489–523.

[KW21] H. Koivusalo and J. J. Walton, *Cut and project sets with polytopal window I: Complexity*, Ergodic Theory Dynam. Systems **41** (2021), no. 5, 1431–1463.

[KW22] ———, *Cut and project sets with polytopal window II: Linear repetitivity*, Trans. Amer. Math. Soc. **375** (2022), no. 7, 5097–5149.

[Li22] X. Li, *Webster sequences, apportionment problems, and just-in-time sequencing*, Discrete Appl. Math. **306** (2022), 52–69.

[Lot02] M. Lothaire, *Algebraic combinatorics on words*, Encyclopedia of Mathematics and its Applications, vol. 90, Cambridge University Press, Cambridge, 2002.

[Mei73] H. G. Meijer, *On a distribution problem in finite sets*, Nederl. Akad. Wetensch. Proc. Ser. A **76**. Indag. Math. **35** (1973), 9–17.

[Mey72] Y. Meyer, *Algebraic numbers and harmonic analysis*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1972, North-Holland Mathematical Library, Vol. 2.

[MH38] M. Morse and G. A. Hedlund, *Symbolic dynamics*, American Journal of Mathematics **60** (1938), 815–866.

[MH40] ———, *Symbolic dynamics II. Sturmian trajectories*, Amer. J. Math. **62** (1940), 1–42.

[MN72] H. G. Meijer and H. Niederreiter, *On a distribution problem in finite sets*, Compositio Math. **25** (1972), 153–160.

[Nie72a] H. Niederreiter, *A distribution problem in finite sets*, Applications of number theory to numerical analysis (Proc. Sympos., Univ. Montréal, Montreal, Que., 1971), Academic Press, New York, 1972, pp. 237–248.

[Nie72b] ———, *On the existence of uniformly distributed sequences in compact spaces*, Compositio Math. **25** (1972), 93–99.

[Que10] M. Queffélec, *Substitution dynamical systems—spectral analysis*, second ed., Lecture Notes in Mathematics, vol. 1294, Springer-Verlag, Berlin, 2010.

[RSZ11] G. Richomme, K. Saari, and L. Q. Zamboni, *Abelian complexity of minimal subshifts*, J. Lond. Math. Soc. (2) **83** (2011), no. 1, 79–95.

[Sch68] W. M. Schmidt, *Irregularities of distribution*, Quart. J. Math. Oxford Ser. (2) **19** (1968), 181–191.

[Sch96] R. Schneider, *On the chairman assignment problem*, Discrete Math. **159** (1996), no. 1-3, 217–222.

[Tij73] R. Tijdeman, *On a distribution problem in finite and countable sets*, J. Combinatorial Theory Ser. A **15** (1973), 129–137.

[Tij80] ———, *The chairman assignment problem*, Discrete Math. **32** (1980), no. 3, 323–330. MR 593979

[Tij82] ———, *A progress report on discrepancy*, Astérisque (1982), no. 94, 175–185.

[Vui03] L. Vuillon, *Balanced words*, Bull. Belg. Math. Soc. Simon Stevin **10** (2003), no. suppl., 787–805.

[Wal82] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982.

[Wal24] J. J. Walton, *A characterisation of linear repetitivity for cut and project sets with general polytopal windows*, Indag. Math. (N.S.) (2024), In Press.

Université Paris Cité, CNRS, IRIF, F-75013 Paris, France
Email address: berthe@irif.fr

Université Paris Cité, CNRS, IRIF, F-75013 Paris, France
Email address: carton@irif.fr

Université de Haute Alsace, IRIMAS département de mathématiques, 18 rue des frères Lumière, F-68200 Mulhouse, France
Email address: nicolas.chevallier@uha.fr

Université Paris Cité, CNRS, IRIF, F-75013 Paris, France
Email address: steiner@irif.fr

School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London, E1 4NS, United Kingdom
Email address: r.yassawi@qmul.ac.uk