Rauzy dimension and finite state dimension

Verónica Becher[1] Olivier Carton[2] Santiago Figueira[1]

[1]Universidad de Buenos Aires & CONICET

[2]IRIF, Université Paris Cité & CNRS

[1,2]Supported by LIA SINFIN

ANR/FWF Project SYMDYNAR
Roscoff – 2024

Outline

What is a dimension ?

A dimension is a function $A^{\mathbb{N}} \to \mathbb{R}$ (often $[0,1]$) which *measures* the complexity of the sequence $x \in A^{\mathbb{N}}$.

Example (Topological entropy)

$$h_{\text{top}}(x) := \lim_{\ell \to \infty} \frac{\log_2 \#(\text{Fact}(x) \cap A^\ell)}{\ell}$$

For instance

$$\begin{aligned} h_{\text{top}}(010101010101 0\cdots) &= 0 \quad \text{Periodic} \\ h_{\text{top}}(0100101001001\cdots) &= 0 \quad \text{Fibonacci} \\ h_{\text{top}}(0100011011000\cdots) &= 1 \quad \text{Champernowne} \end{aligned}$$

Outline

Outline

Expansion of real numbers (in some base b)

Fix an integer base $b \geqslant 2$. The alphabet is $A = \{0, 1, \ldots, b-1\}$.

- if $b = 2$, $A = \{0, 1\}$,
- if $b = 10$, $A = \{0, 1, 2, \ldots, 9\}$.

Each real number $\alpha \in [0, 1)$ has an expansion in base b:
$x = a_1 a_2 a_3 \cdots$ where $a_i \in A$ and

$$\alpha = \sum_{k \geqslant 1} \frac{a_k}{b^k}.$$

In the rest of this talk:

$$\begin{array}{rcl}
\text{real number } \alpha \in [0, 1) & \longleftrightarrow & \text{sequence } x \in A^{\mathbb{N}} \\
1/3 & \longleftrightarrow & 010101 \cdots = (01)^{\mathbb{N}} \\
\pi/4 & \longleftrightarrow & 1100100100001111 \cdots
\end{array}$$

Normal sequences

A normal sequence is a sequence such that all finite words of the same length occur in it with the same limiting frequency.

A sequence $x \in A^{\mathbb{N}}$ is normal if for each finite word $w \in A^*$:

$$\lim_{n \to \infty} \frac{|x[1:n]|_w}{n} = \frac{1}{(\#A)^{|w|}}$$

where
- $\#A$ is the cardinality of the alphabet A
- $|u|$ is the length of the word u.
- $|u|_w$ is the number of occurrences of w in u.

Preservation of normality by addition

Theorem (Rauzy, 1976)

For $\beta \in [0,1)$, the following conditions are equivalent:

- *β is normal,*
- *β has Rauzy dimension $(\#A - 1)/\#A$.*

Theorem (Rauzy, 1976)

For $\beta \in [0,1)$, the following conditions are equivalent:

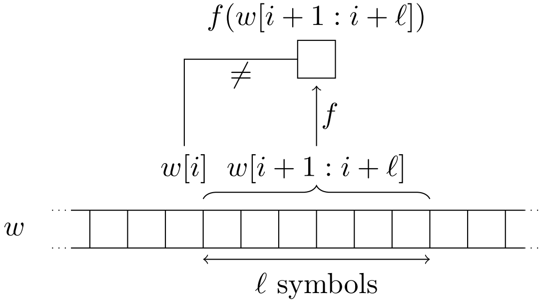
- *for each α normal, $\alpha + \beta$ is still normal,*
- *β has Rauzy dimension 0.*

Outline

Rauzy dimensions

For $w \in A^*$ and $\ell \geqslant 0$

$$\beta_\ell(w) := \min_{f:A^\ell \to A} \frac{\#\{i : w[i] \neq f(w[i+1:i+\ell])\}}{|w|}$$

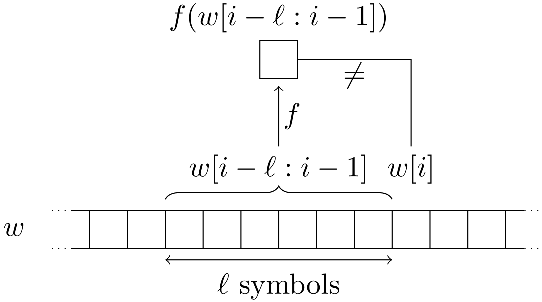


Rauzy dimensions

For $w \in A^*$ and $\ell \geqslant 0$

$$\beta_\ell(w) := \min_{f:A^\ell \to A} \frac{\#\{i : w[i] \neq f(w[i+1 : i+\ell])\}}{|w|}$$

$$\gamma_\ell(w) := \min_{f:A^\ell \to A} \frac{\#\{i : w[i] \neq f(w[i-\ell : i-1])\}}{|w|}$$



Rauzy dimensions

For $w \in A^*$ and $\ell \geqslant 0$

$$\beta_\ell(w) := \min_{f:A^\ell \to A} \frac{\#\{i : w[i] \neq f(w[i+1 : i+\ell])\}}{|w|}$$

$$\gamma_\ell(w) := \min_{f:A^\ell \to A} \frac{\#\{i : w[i] \neq f(w[i-\ell : i-1])\}}{|w|}$$

For $x \in A^{\mathbb{N}}$ and $\ell \geqslant 1$,

$$\underline{\beta}_\ell(x) := \liminf_{n\to\infty} \beta_\ell(x[1 : n]) \quad \text{and} \quad \overline{\beta}_\ell(x) := \limsup_{n\to\infty} \beta_\ell(x[1 : n])$$

$$\underline{\beta}(x) := \lim_{\ell\to\infty} \underline{\beta}_\ell(x) \quad \text{and} \quad \overline{\beta}(x) := \lim_{\ell\to\infty} \overline{\beta}_\ell(x)$$

$\underline{\gamma}(x)$ and $\overline{\gamma}(x)$ are defined similarly using γ_ℓ instead of β_ℓ.

Outline

Finite state dimensions

The finite state dimensions $\underline{\dim}$ and $\overline{\dim}$ has several equivalent definitions using either

- Measure-theoretic entropy, or
- Finite state compressibility, or
- Finite state predictors/martingales.

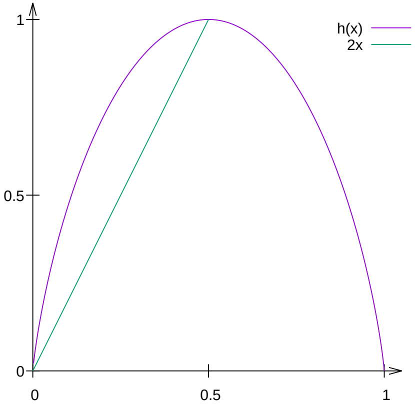
Outline

Function $\mathfrak{h}$

Let $\mathfrak{h} : [0,1] \to [0,1]$ be the classical entropy function

$$\mathfrak{h}(\alpha) := -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha)$$

whose graph is



Results

Here, we suppose $\#A = 2$.

Theorem

- *For every* $x \in A^{\mathbb{N}}$,

$$2\underline{\gamma}(x) \leqslant \underline{\dim}(x) \leqslant \mathfrak{h}(\underline{\gamma}(x)),$$
$$2\overline{\gamma}(x) \leqslant \overline{\dim}(x) \leqslant \mathfrak{h}(\overline{\gamma}(x)),$$
$$2\underline{\beta}(x) \leqslant \underline{\dim}(x) \leqslant \mathfrak{h}(\underline{\beta}(x)),$$
$$2\overline{\beta}(x) \leqslant \overline{\dim}(x) \leqslant \mathfrak{h}(\overline{\beta}(x)).$$

- *These inequalities are sharp.*

Results

Reformulation of one implication of second Rauzy's theorem:

Theorem (Rauzy, 1976)

If $\underline{\dim}(x) = 1$ *and* $\overline{\dim}(y) = 0$*, then* $\underline{\dim}(x+y) = 1$*.*

Theorem

For every $x, y \in A^{\mathbb{N}}$*,*

$$\underline{\dim}(x) - \overline{\dim}(y) \leqslant \underline{\dim}(x+y) \leqslant \underline{\dim}(x) + \overline{\dim}(y),$$
$$\overline{\dim}(x) - \overline{\dim}(y) \leqslant \overline{\dim}(x+y) \leqslant \overline{\dim}(x) + \overline{\dim}(y).$$

Measure-theoretic entropy

Recall that $|w|_u$ is the number of occurrences of u in w.

Normalized ℓ-entropy of $w \in A^*$:

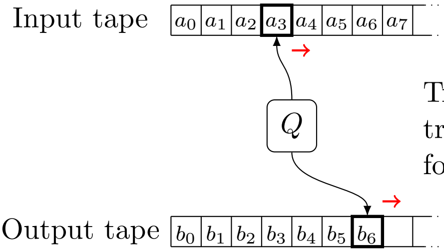
$$h_\ell(w) := -\frac{1}{\ell} \sum_{u \in A^\ell} f_u \log f_u \quad \text{where} \quad f_u := \frac{|w|_u}{|w| - |u| + 1}$$

Normalized entropy of $x \in A^{\mathbb{N}}$:

$$\underline{h}(x) := \liminf_{\ell \to \infty} \underline{h}_\ell(x) \quad \text{where} \quad \underline{h}_\ell(x) := \liminf_{n \to \infty} h_\ell(x[1:n])$$

$$\overline{h}(x) := \liminf_{\ell \to \infty} \overline{h}_\ell(x) \quad \text{where} \quad \overline{h}_\ell(x) := \limsup_{n \to \infty} h_\ell(x[1:n])$$

Finite state compressibility (by transducers)



Transducer $\mathcal{T}$ with transitions $p \xrightarrow{a|v} q$ for $a \in A$ and $v \in A^*$

The compression ratios of a run $q_0 \xrightarrow{a_1|v_1} q_1 \xrightarrow{a_2|v_2} q_2 \xrightarrow{a_3|v_3} \cdots$

$$\underline{\rho}_{\mathcal{T}}(x) := \liminf_{n\to\infty} \frac{|v_1 v_2 \cdots v_n|}{n} \quad \text{and} \quad \bar{\rho}_{\mathcal{T}}(x) := \limsup_{n\to\infty} \frac{|v_1 v_2 \cdots v_n|}{n}.$$

The compression ratios of $x \in A^{\mathbb{N}}$ are given by

$$\underline{\rho}(x) := \inf\{\underline{\rho}_{\mathcal{T}}(x) : \mathcal{T} \text{ is a one-to-one transducer}\}$$

$$\bar{\rho}(x) := \inf\{\bar{\rho}_{\mathcal{T}}(x) : \mathcal{T} \text{ is a one-to-one transducer}\}$$

Finite state predictors/martingales

A predictor π is a function $A^* \times A \to [0,1]$ such that $\sum_{a \in A} \pi(w,a) = 1$ for each $w \in A^*$.
It is a finite state predictor if $\pi(w,0)$ and $\pi(w,1)$ are computed by an automaton reading w.

The dimension of π on w:

$$\dim_\pi(w) := \sum_{i=0}^{|w|-1} \log \frac{1}{\pi(w[0:i), w[i])}.$$

The dimensions of $x \in A^{\mathbb{N}}$ are

$$\underline{\dim}(x) := \inf_{\pi \in \Pi} \liminf_{n \to \infty} \frac{\dim_\pi(x[1:n))}{n}$$

$$\overline{\dim}(x) := \inf_{\pi \in \Pi} \limsup_{n \to \infty} \frac{\dim_\pi(x[1:n))}{n}$$

where Π is the class of all finite state predictors.

Relations

The finite state dimension has several equivalent definitions using either

- Measure-theoretic entropy, or
- Finite state compressibility, or
- Finite state predictors/martingales.

Theorem (Many people)
For every $x \in A^{\mathbb{N}}$,

$$\underline{h}(x) = \underline{\rho}(x) = \underline{\dim}(x) \quad \textit{and} \quad \overline{h}(x) = \bar{\rho}(x) = \overline{\dim}(x).$$

From the previous theorem

For every $x \in A^{\mathbb{N}}$,

$$\begin{aligned}
\underline{\dim}(x) = 0 &\iff \underline{\beta}(x) = 0 \iff \underline{\gamma}(x) = 0 \\
\overline{\dim}(x) = 0 &\iff \overline{\beta}(x) = 0 \iff \overline{\gamma}(x) = 0 \\
\underline{\dim}(x) = 1 &\iff \underline{\beta}(x) = \tfrac{1}{2} \iff \underline{\gamma}(x) = \tfrac{1}{2} \\
\overline{\dim}(x) = 1 &\iff \overline{\beta}(x) = \tfrac{1}{2} \iff \overline{\gamma}(x) = \tfrac{1}{2}
\end{aligned}$$

Quoting Rauzy, are the function β and γ similar ?

Counterexample

Let x be a *generic* sequence for the following Markov chain.
$\underline{\beta}(x) = \overline{\beta}(x) < \underline{\gamma}(x) = \overline{\gamma}(x) = \frac{11}{24}$.