Amorphic complexity, tameness, and nullness of constant length substitution systems

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Dynamical systems

A dynamical system (X, T) is a compact metric space (X, d) with a homeomorphism $T\colon X \to X$. For the talk we assume (X, T) is minimal (it doesn't have proper nontrivial subsystems).

Numerical invariant of dynamics: entropy

Entropy:

$$h_{top}(X) = \lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}} \frac{\log \text{sep}(X, n, \epsilon)}{n},$$

where $\text{sep}(X, n, \epsilon)$ is the number of orbits in X distinguishable up to time n at resolution ϵ.

Geometrical interpretation for subshifts $X \subset \Sigma_d^{\mathbb{Z}} = \{0, \ldots, d-1\}^{\mathbb{Z}}$:

$$h_{top}(X) = \beta \dim_{box}(X) = \beta \dim_H(X),$$

where β is a normalising constant depending on what metric exactly we chose on the full shift $\Sigma_d^{\mathbb{Z}}$.

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Low complexity as a lack of independence

Theorem (Kerr, Li), only for subshifts

1. $X \subset \mathcal{A}^{\mathbb{Z}}$ has positive entropy iff it has an independence set $S \subseteq \mathbb{Z}$ of positive density.
2. $X \subset \mathcal{A}^{\mathbb{Z}}$ is not tame iff it has an infinite independence set $S \subseteq \mathbb{Z}$.
3. $X \subset \mathcal{A}^{\mathbb{Z}}$ is not null iff it has an arbitrarily large independence set $S \subseteq \mathbb{Z}$

Set $S \subseteq \mathbb{Z}$ is an independence set for $X \subset \mathcal{A}^{\mathbb{Z}}$ if there are $a \neq b$ in $\mathcal{A}$ such that for any $\sigma : S \to \{a, b\}$ there is $x = (x_n)_n \in X$ with

$$x_n = \sigma(n) \quad \text{for } n \in S.$$

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Mean equicontinuity

Besicovitch pseudometric on (X, T) is given by

$$D_B(x,y) = \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=0}^{n-1} d(T^k(x), T^k(y)).$$

Besicovitch space $[X]_B$ is a quotient space $X/\sim$ obtained by identifying point x, y such that $D_B(x,y) = 0$

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A system is mean equicontinuous if $D_B : X \times X \to [0, \infty)$ is continuous (w.r.t. the original metric d). In this case $([X]_B, [T])$ is the MEF (maximal equicontinuous factor) of X.

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Amorphic complexity - intuition

Amorphic complexity [Fuhrmann, Gröger, Jäger (2018)] is a numerical invariant of dynamical systems based on an asymptotic notion of separation and suited for systems in low complexity regime.

If a system is not mean equicontinuous, then its amorphic complexity is infinite.

Asymptotic separation numbers

For $\delta > 0$ and $\nu \in (0, 1]$ we say that $x, y \in X$ are (δ, ν)-separated if

$$D_\delta(x, y) = \overline{\lim_{n\to\infty}} \frac{\#\left\{-n \leq k \leq n : d\left(T^k(x), T^k(y)\right) \geq \delta\right\}}{2n + 1} \geq \nu.$$

A subset $S \subseteq X$ is said to be (δ, ν)-separated if all pairs of distinct points $x, y \in S$ are (δ, ν)-separated.

The (asymptotic) separation number $\mathrm{Sep}(X, \delta, \nu)$ of X is the largest cardinality of a (δ, ν)-separated subset S of X.

We say (X, T) has finite separation numbers if all $\mathrm{Sep}(X, \delta, \nu)$ are finite for $\delta > 0, \nu \in (0, 1]$.

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Finite separation numbers - characterisation

The following conditions are equivalent:

1. (X, T) has finite separation numbers,
2. (X, T) is mean equicontinuous,
3. (X, T) is uniquely ergodic and has discrete spectrum with continuous eigenfunctions.

$$\text{null} \subseteq \text{tame} \subseteq \text{mean equicontinuous} \subseteq \text{discrete spectrum}$$

System (X, T) is called regular almost automorphic if for the factor map $\pi \colon X \to \mathrm{MEF}$, the set $\{z \in \mathrm{MEF} \colon \left|\pi^{-1}(z)\right| = 1\}$ has full Haar measure.

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Amorphic complexity - definition

Definition:

$$\underline{\mathrm{ac}}(X) = \sup_{\delta>0} \varliminf_{\nu\to 0} \frac{\log \mathrm{Sep}(X,\delta,\nu)}{-\log \nu} \quad \text{and} \quad \overline{\mathrm{ac}}(X) = \sup_{\delta>0} \varlimsup_{\nu\to 0} \frac{\log \mathrm{Sep}(X,\delta,\nu)}{-\log \nu}.$$

If the numbers coincide we put $\mathrm{ac}(X) = \underline{\mathrm{ac}}(X) = \overline{\mathrm{ac}}(X)$.

Geometric interpretation for a subshift $X \subset \Sigma_d^{\mathbb{Z}}$:

$$\underline{\mathrm{ac}}(X) = \underline{\dim}_{\mathrm{box}}([X]_B) \quad \text{and} \quad \overline{\mathrm{ac}}(X) = \overline{\dim}_{\mathrm{box}}([X]_B),$$

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Amorphic complexity so far

1. ac(Isometry) = 0,
2. ac(Sturmian) = 1,
3. ac(Denjoy) = 1,
4. upper bounds for Toeplitz subshifts,
5. (Fuhrmann, Gröger, Jäger, Kwietniak): upper bounds for some regular model sets,
6. (Baake, Gähler, Gohlke): ac(Hat tiling) $= \frac{4\log(\varphi)}{4\log(\varphi)-\log(2+\sqrt{3})} = 3.166443$
7. (Fuhrmann, Gröger): bounds for (lower\upper) amorphic complexity of minimal constant length substitution systems + closed formula over two letter alphabet,
8. Our result: closed formula for ac(constant length substitution) + relation to nullness and tameness.

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Constant length substitution shifts

Consider a substitution $\varphi\colon \mathcal{A} \to \mathcal{A}^*$ on some finite alphabet $\mathcal{A}$, e.g. the Thue–Morse substitution

$$\varphi(0) = 01, \quad \varphi(1) = 10.$$

A substitution φ is said to be of constant length k if it sends all letters to words of the same length k (e.g. the Thue–Morse substitution is of constant length 2).

With a substitution φ we associate a substitution subshift:

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Separation substitution

To state our result we need to define separation substitution and separation number of a primitive constant length substitution φ of height h=1.

The separation substitution of $\varphi\colon \mathcal{A} \to \mathcal{A}^*$ is defined on the set of all unordered pairs of distinct letters in $\mathcal{A}$.

substitution φ	separation substitution φ_s of φ
$a \to aabca$	$\binom{a}{b} \to \binom{a}{b}\binom{a}{b}\binom{a}{c}$
$b \to abacc$	$\binom{a}{c} \to \binom{a}{c}\binom{a}{b}\binom{b}{c}\binom{a}{c}$
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The separation substitution of $\varphi \colon \mathcal{A} \to \mathcal{A}^*$ is defined on the set of all unordered pairs of distinct letters in $\mathcal{A}$.

substitution φ	separation substitution φ_s of φ
$a \to aabca$	$\binom{a}{b} \to \binom{a}{b}\binom{a}{b}\binom{a}{c}$
$b \to abacc$	$\binom{a}{c} \to \binom{a}{c}\binom{a}{b}\binom{b}{c}\binom{a}{c}$
$c \to acabc$	$\binom{b}{c} \to \binom{b}{c}\binom{b}{c}$

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$a \to aabca$

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$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix}$$

$$\begin{pmatrix} a \\ c \end{pmatrix} \to \begin{pmatrix} a \\ c \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} b \\ c \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix}$$

$$\begin{pmatrix} b \\ c \end{pmatrix} \to \begin{pmatrix} b \\ c \end{pmatrix}\begin{pmatrix} b \\ c \end{pmatrix}$$

Separation substitution and separation number

separation substitution φ_s

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$$\begin{pmatrix}b\\c\end{pmatrix} \to \begin{pmatrix}b\\c\end{pmatrix}\begin{pmatrix}b\\c\end{pmatrix}$$

incidence matrix M_s of φ_s

$$\begin{pmatrix}2 & 2 & 0\\1 & 1 & 0\\0 & 1 & 2\end{pmatrix}$$

The matrix M_s has a dominant (Perron–Frobenius) eigenvalue $\lambda_s = 3$ which we call the separation number of φ.

1. $\lambda_s = 0$ or $1 \leq \lambda_s \leq k$,
2. $\lambda_s = 0$ if and only if X_φ is finite
3. $\lambda_s = k$ if and only if X_φ does not have discrete spectrum.

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Amorphic complexity of constant length substitution shifts

Theorem (Gröger, K.)

For a (pure) minimal substitution shift X of constant length k its amorphic complexity is given by

$$\mathrm{ac}(X) = \frac{\log k}{\log k - \log \lambda_s},$$

where λ_s is the separation number of φ ($(\log k)/0 = \infty$).

For a general (nonminimal) automatic system X we have

$$\mathrm{ac}(X) = \max\{\mathrm{ac}(Y) \mid Y \subset X \text{ minimal subshift}\}.$$

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Theorem (Gröger, K. + Fuhrmann, Kellendonk, Yassawi)

For an infinite minimal automatic shift X, the following are equivalent:

1. $ac(X) = 1$,
2. X is tame,
3. X is null,
4. the factor map $\pi\colon X \to MEF$ has only countably many nonregular points (countably many points $z \in MEF$ with $|\pi^{-1}(z)| \geq 2$).

(2) $\iff$ (4) was shown by Fuhrmann, Kellendonk, and Yassawi.

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Proof synopsis