Introduction to Permeable Sets

G. Leobacher and A. Steinicke,
University of Graz, Montanuniversitaet Leoben

(joint work with T. Rajala, and J.M. Thuswaldner; with Z. Buczolich)

Roscoff, September 2024

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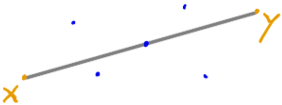

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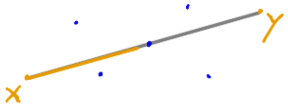

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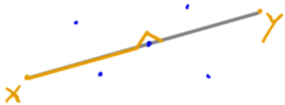

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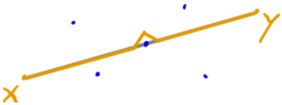

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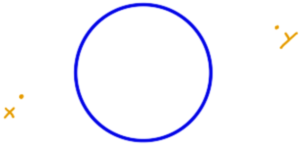

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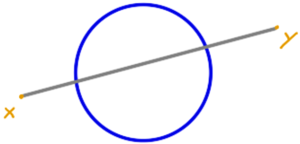

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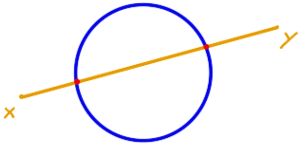

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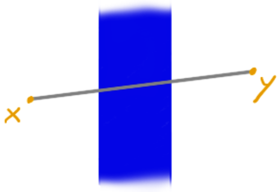

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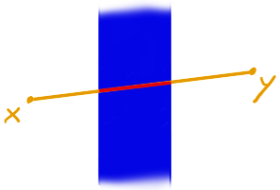

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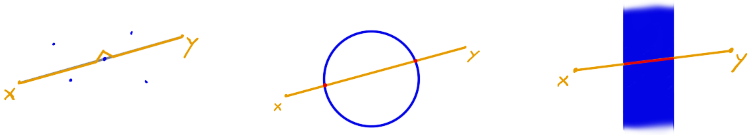

Intuitively, they have different levels of ‘permeability’.

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Definition ((Null-)permeability)

Let $\mathbb{R}^d$ be equipped with some norm $\|\cdot\|$.

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Easy consequences:

- Subsets of permeable sets are permeable
- Permeable sets have empty interior (rel. to $\mathbb{R}^d$)

More examples

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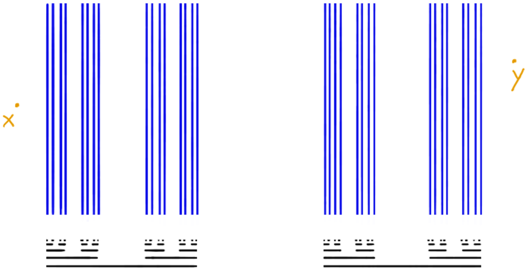

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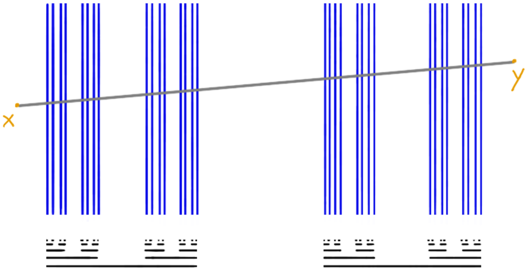

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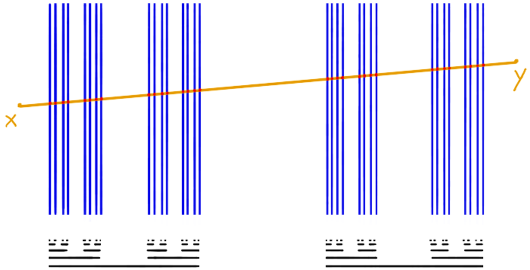

impermeable

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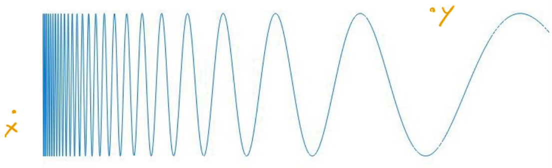
- $\left\{ \left(x, \sin(x^{-1})\right) : x > 0 \right\} \subset \mathbb{R}^2$



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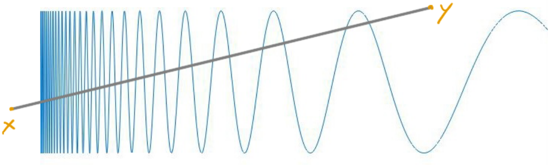

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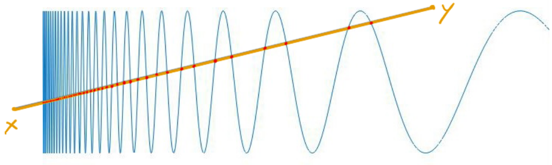

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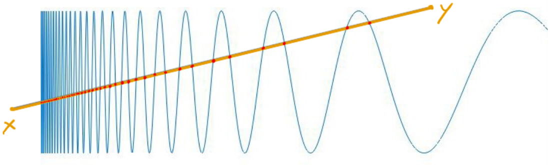

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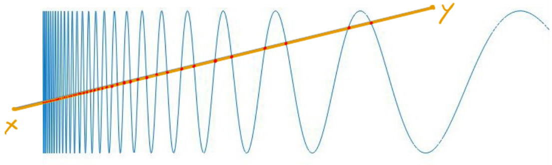

  permeable, but not null permeable
- $\left\{(x, \sin(x^{-1}), \cos(x^{-1})) : x > 0\right\} \subset \mathbb{R}^3$, null permeable

But why?

But why would one be interested in this concept?

Stochastic differential equations (SDE):

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Modeling recent phenomena such as electricity market models or dividend maximization for insurance companies lead to SDEs with discontinuities in their drift terms (that is in b).

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$G : \mathbb{R} \to \mathbb{R}$ a suitable bijective function. $Z_t = G(X_t)$.
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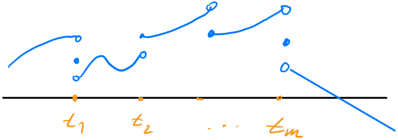

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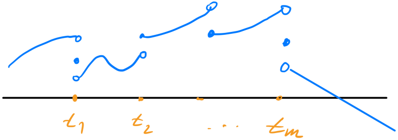

No straightforward generalization: The intervals are the 1-dimensional convex, connected, path-connected, star-shaped,... sets, the polytopes, balls etc.

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How to employ the piecewise Lipschitz property consistently?

- intrinsic metric on $\mathbb{R}^d \setminus \Theta$:

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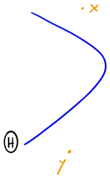

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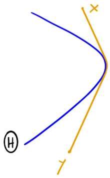

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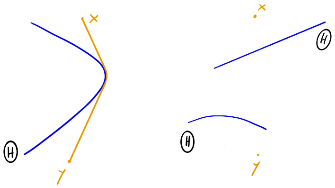

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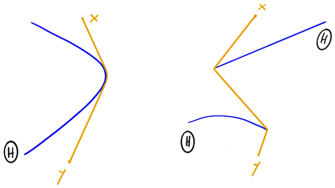

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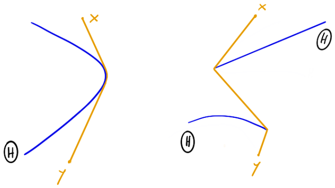
- f is intrinsically L-Lipschitz if f is Lipschitz w.r.t. ϱ on $\mathbb{R}^d \setminus \Theta$ with Lipschitz constant L.



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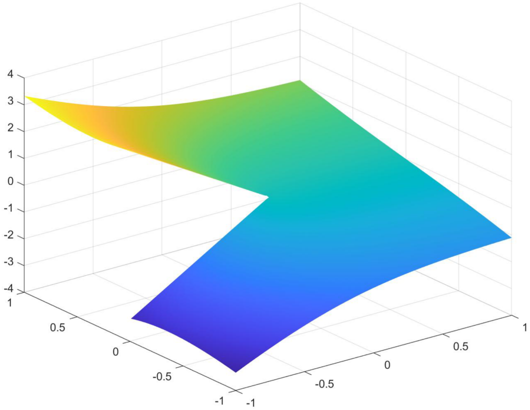

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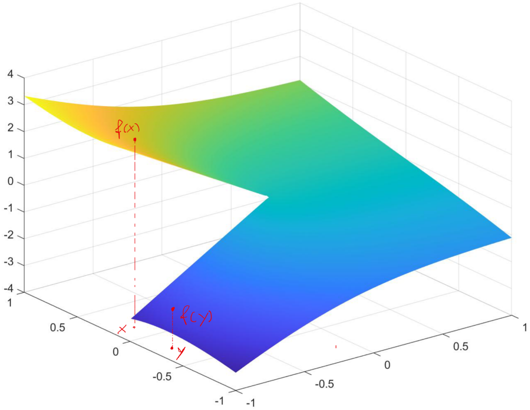

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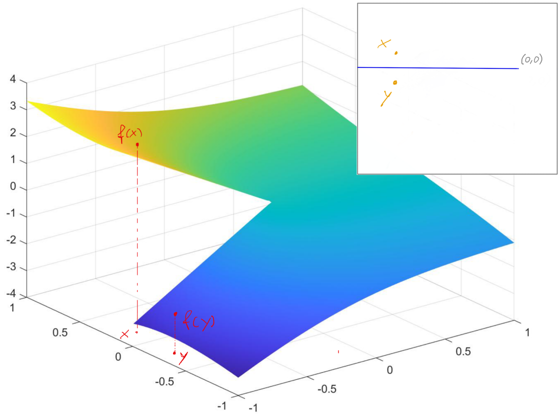

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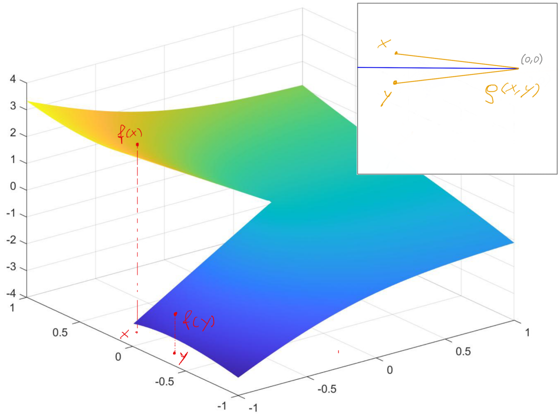

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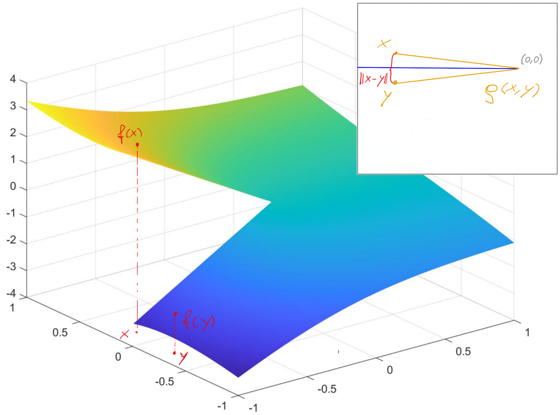

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What condition on Θ could guarantee this?

Important result

Theorem (L. & St., 2022)

Let $\Theta \subseteq \mathbb{R}^d$ be permeable. Then every continuous function $\mathbb{R}^d \to \mathbb{R}$ which is intrinsically L-Lipschitz on $\mathbb{R}^d \setminus \Theta$ is L-Lipschitz continuous on the whole of $\mathbb{R}^d$.