Overlap coincidences for general S-adic tilings and Pisot substitution conjecture for binary case

A joint work with Jörg Thuswaldner

September 2024 @Roscoff, Yasushi Nagai (Shinshu University)

Contents

Pure point spectrum

Picture by Dirk Frettolöh

Analogies

Symbolic	Geometric
Sequences (Words)	Tilings
The product topology on $\mathscr{A}^{\mathbb{N}}$	Tiling metric
Shift σ	Translation by $x \in \mathbb{R}^d$
Subshift $X = \overline{\{\sigma^n(w) \mid n \in \mathbb{N}\}}$	Continuous hull $X_{\mathscr{T}} = \overline{\{\mathscr{T} + x \mid x \in \mathbb{R}^d\}}$

Definition of one-dimensional tiling

the label

- a tile: $T = ([a, b], l)$ Or just $T = [a, b]$

the support ($\mathrm{supp}T$)

- a patch=a collection $\mathcal{P}$ of tiles such that

$$S, T \in \mathcal{P}, S \neq T \Rightarrow (\mathrm{supp}\, S)^\circ \cap (\mathrm{supp}\, T)^\circ = \emptyset$$

- a tiling=a patch $\mathcal{T}$ such that $\mathbb{R} = \bigcup_{T \in \mathcal{T}} \mathrm{supp}\, T$

A construction of non-periodic tilings

interest: non-periodic but "ordered" tiling

$\mathcal{T} + x = \mathcal{T}$ only for $x = 0$

construction: via a substitution rule

Tiling dynamical systems

Continuous hull

$$X_{\mathscr{T}} = \overline{\{\mathscr{T} + x \mid x \in \mathbb{R}\}}$$

$\mathbb{R}$ acts on $X_{\mathscr{T}}$ via translation:

$$X_{\mathscr{T}} \times \mathbb{R} \ni (\mathscr{S}, x) \mapsto \mathscr{S} + x \in X_{\mathscr{T}}$$

Often there is one and only one invariant Borel probability measure μ

Tiling dynamical systems

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We say $\mathscr{T}$ has pure point dynamical spectrum if there exists a complete orthonormal basis for $L^2(\mu)$ consisting of eigenfunctions for the Koopman operators $U_x : f \mapsto f(\cdot - x) \quad (U_x(f)(\mathscr{S}) = f(\mathscr{S} - x))$

(f is an eigenfunction if $U_x(f) = c_x f$)

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A tiling $\mathscr{T}$ has pure point spectrum
$\Longleftrightarrow \mathscr{T}$ is almost periodic (a weak form of translational symmetry)

(Gouéré, Lenz-Spindeler-Strungaru)

Tiling dynamical systems

A tiling $\mathscr{T}$ has pure point spectrum

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$\Longleftrightarrow$the corresponding dynamical system is conjugate to a rotation of compact abelian group

Main question

Decide which non-periodic tiling has pure point dynamical spectrum.

$\Longleftrightarrow$ has a weak form of symmetry

$\Longleftrightarrow$ falls into the class of tilings which are classified by their spectra

A construction of non-periodic tilings

a substitution rule =a recipe for "expanding and subdividing"

= • $\mathcal{A}$: a finite set of tiles (the alphabet)

• ρ: the rule of expanding $P \in \mathcal{A}$ and then subdivide it

• $\lambda > 1$: an expansion factor

Example

$$\tau = \frac{1+\sqrt{5}}{2}$$:expansion factor

$$\mathcal{A} = \{\underset{T_1}{[0,\tau]}, \underset{T_2}{[0,1]}\}$$

$$\rho_F(T_1) = \{T_1, T_2 + \tau\} \qquad \rho_F(T_2) = \{T_1\}$$

A construction of non-periodic tilings

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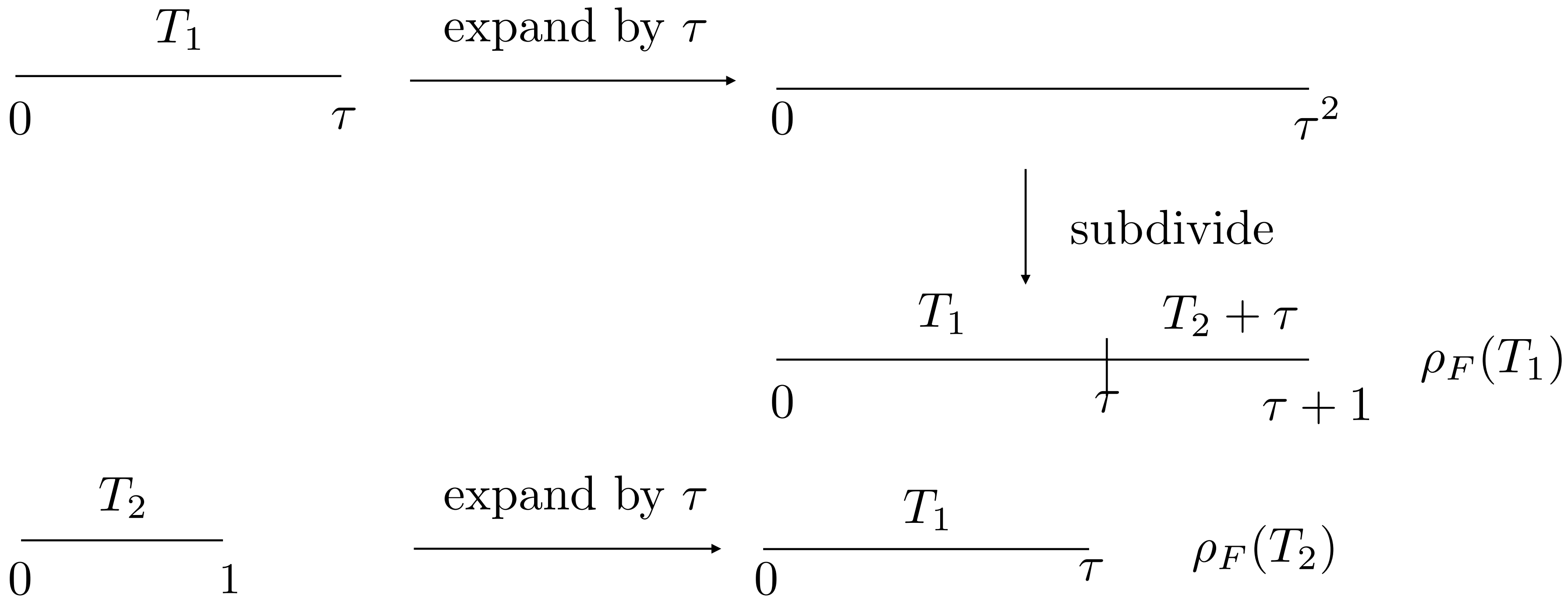

A construction of non-periodic tilings

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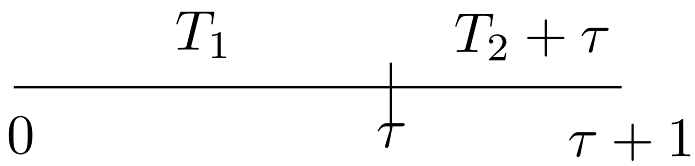
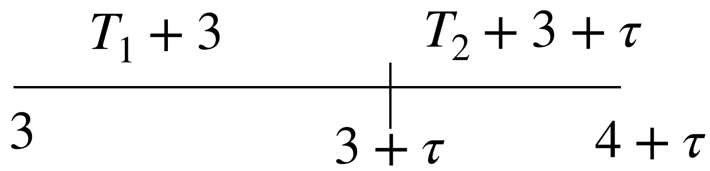
$$\rho_F(T_1) = \{T_1, T_2 + \tau\} \qquad \rho_F(T_2) = \{T_1\}$$

We extend the domain of ρ by setting $\rho(P+x) = \rho(P) + \lambda x$



$\rho_F(T_1)$



$\rho_F(T_1) + 3 = \rho_F(T_1 + 3/\tau)$

A construction of non-periodic tilings

S-adic tilings: tilings of the form $\mathcal{T} = \lim_{n\to\infty} \rho_{i_1} \circ \rho_{i_2} \circ \cdots \circ \rho_{i_{k_n}}(\mathcal{P}_n)$

$\{\rho_1, \rho_2, \ldots, \rho_{m_a}\}$: a finite family of substitutions with a common alphabet $\mathcal{A}$

$i_1, i_2, \ldots \in \{1, 2, \ldots, m_a\}$: a directive sequence

in other words: a tiling $\mathcal{T} = \mathcal{T}^{(1)}$ that admits "de-substituted tilings"

$$\mathcal{T}^{(2)}, \mathcal{T}^{(3)}, \mathcal{T}^{(4)}, \ldots$$

such that

$$\rho_{i_n}(\mathcal{T}^{(n+1)}) = \mathcal{T}^{(n)}, n = 1, 2, \ldots$$

A construction of non-periodic tilings

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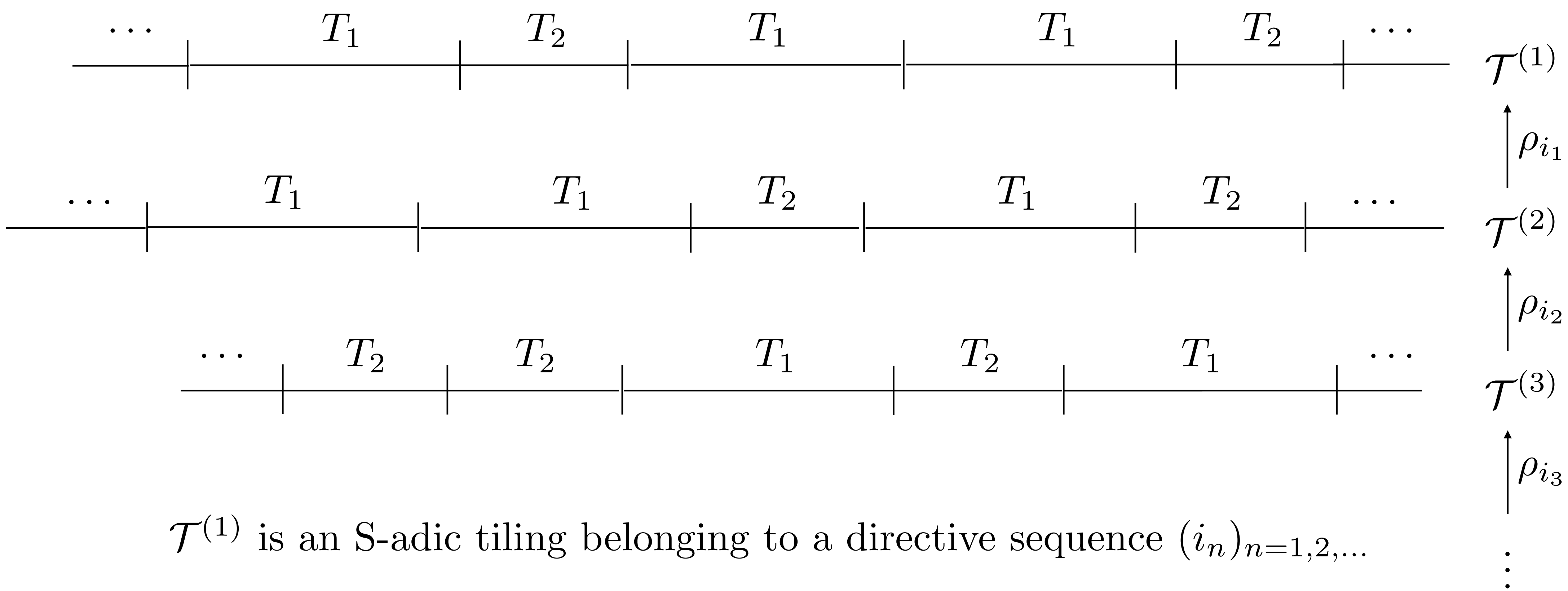


$\mathcal{T}^{(1)}$ is an S-adic tiling belonging to a directive sequence $(i_n)_{n=1,2,\ldots}$

Main question

Decide which S-adic tiling has pure point dynamical spectrum.

Pisot Conjecture:
Self-similar tilings by substitution rules with the Pisot condition have pure point spectrum

Today's result
(1)Give a sufficient condition for a given S-adic tiling to be pure point
(2)this condition is satisfied for one of the simplest classes

Contents

(1)Definitions (done)

(2)The first main result (Overlap algorithm)

(3)The Second main result (binary irreducible one-dim substitutions)

The main idea

(1)Generalize Solomyak's overlap algorithm [Solomyak 1997] to the S-adic setting

(2)apply the overlap algorithm to a class of S-adic tiligs of interest

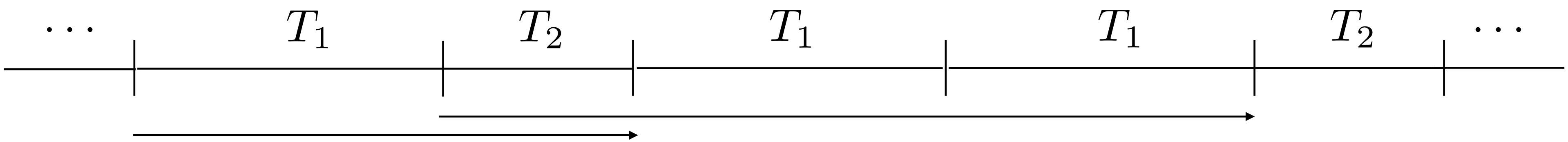
Goes back to the coincidence condition for constant-length symbolic substitution

Overlap algorithm

$$\mathcal{T}_1 \overset{\rho_1}{\leftarrow} \mathcal{T}_2 \overset{\rho_2}{\leftarrow} \mathcal{T}_3 \overset{\rho_3}{\leftarrow} \cdots,$$

where ρ_n: a substitution rule with a fixed alphabet $\mathcal{A}$ and non-fixed expansion factor λ_n

Set $\Lambda_n = \{x \in \mathbb{R} \mid \exists T \in \mathcal{T}_n (T + x \in \mathcal{T}_n)\}$



An overlap @n = a triple (S, x, T) such that $S, T \in \mathcal{T}_n$ and $x \in \Lambda_n$ with $\mathrm{int}(S + x) \cap \mathrm{int}T \neq \emptyset$

Overlap algorithm

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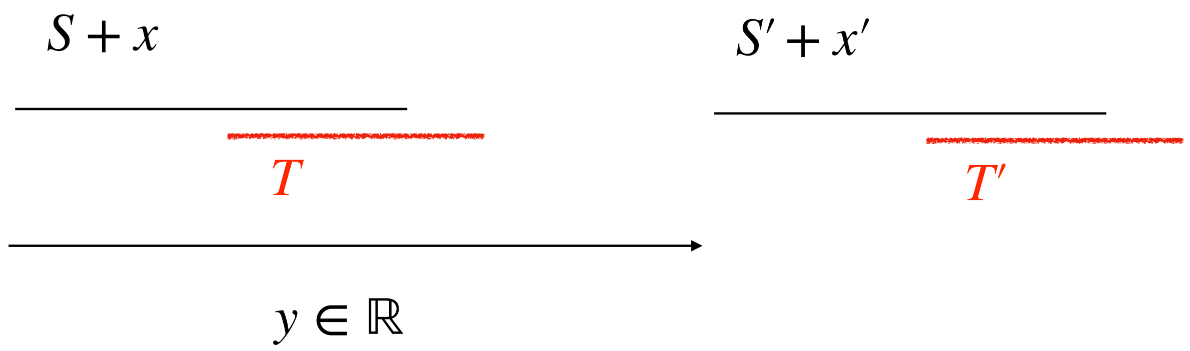

$(S, x, T) \sim (S', x', T')$

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$[S, x, T]$: the equivalence class

$V_n = \{[S, x, T] \mid (S, x, T)$: an overlap @$n\}$

Overlap algorithm

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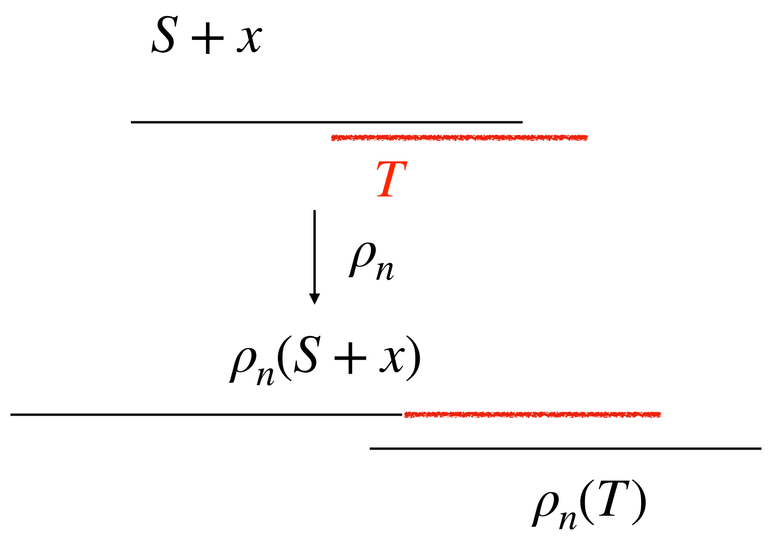
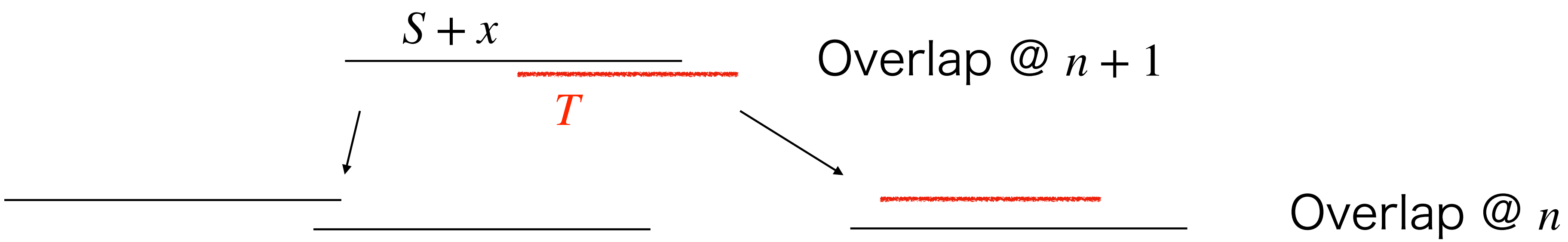


$(S, x, T)@n + 1 \to (S', x', T')@n$

if $S' \in \rho_n(S), T' \in \rho_n(T)$, and $x' = \lambda_n x$



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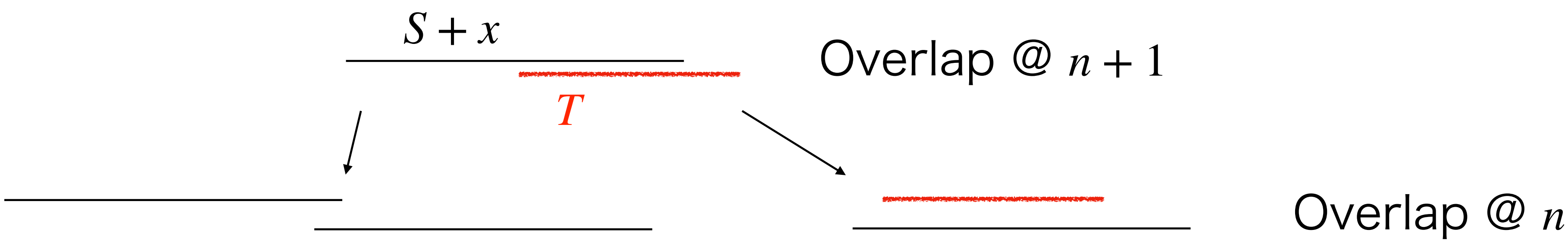
$V_{n+1} \ni v \to w \in V_n$ if there are

$(S, x, T) \in v, (S', x', T') \in w$

such that $(S, x, T) \to (S', x', T')$



Overlap algorithm

$[S, x, T]$: the equivalence class

$V_n = \{[S, x, T] \mid (S, x, T)$: an overlap $@n\}$

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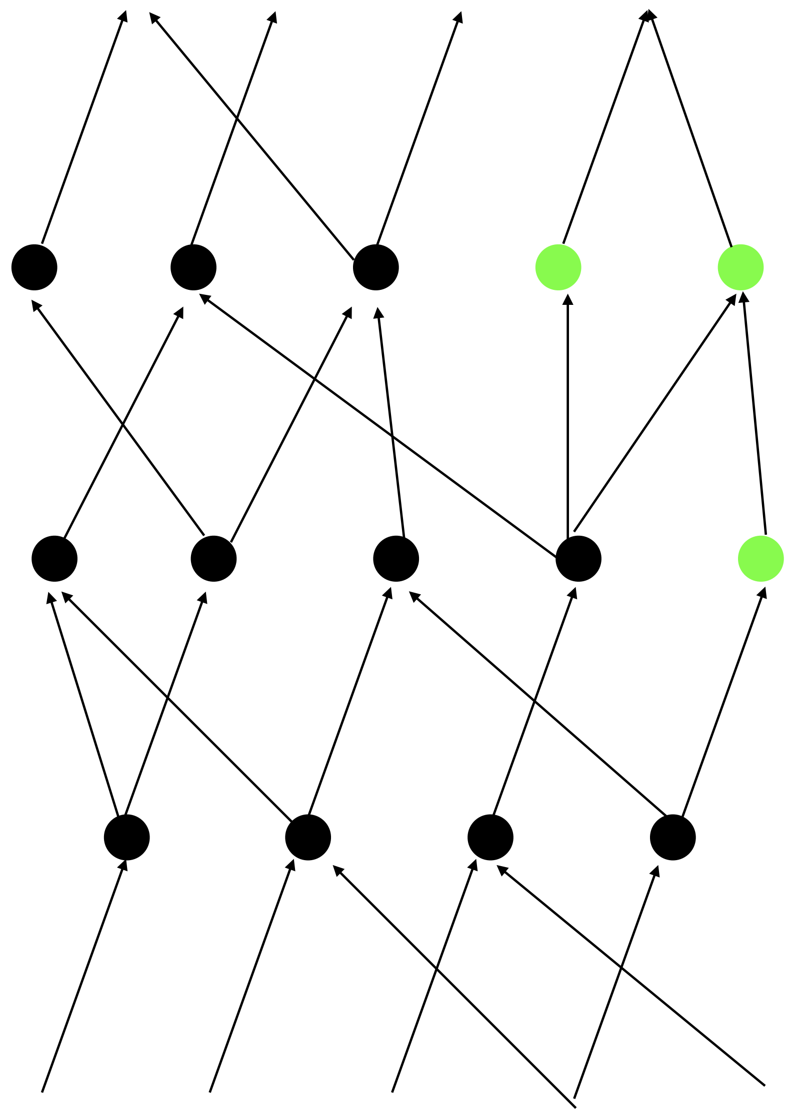


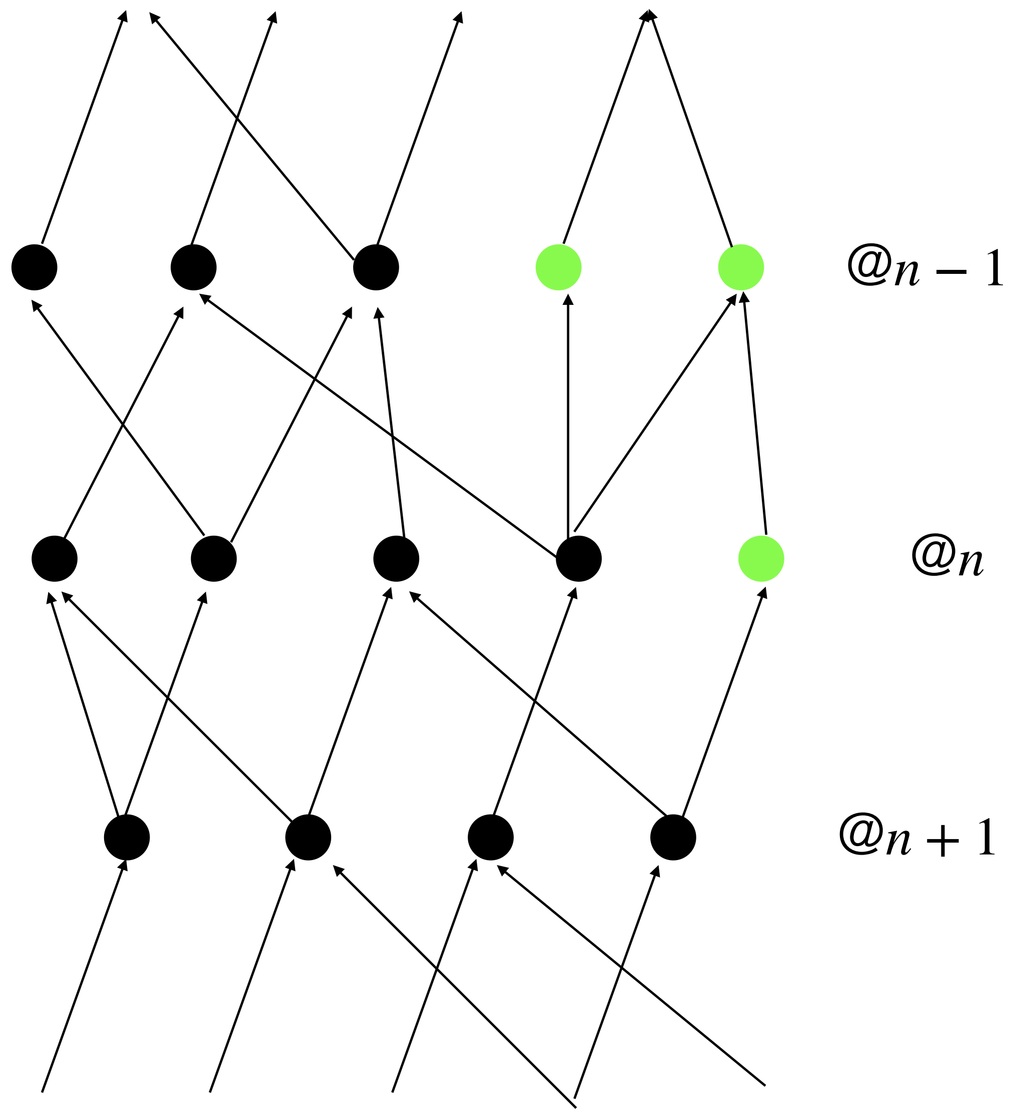
Overlap algorithm

$[S, x, T]$: the equivalence class

$V_n = \{[S, x, T] \mid (S, x, T)$: an overlap $@n\}$

An overlap (S, x, T) is a coincidence if

$S + x = T$



The first main theorem

Theorem (N-Thuswaldner)

If there are $n_1 < m_1 < n_2 < m_2 < \cdots$ such that,

for any j and $v \in V_{m_j}$, there is a path from v to a coincidence $w \in V_{n_j}$

$+$ a technical condition,

Then $\mathscr{T}_1$ has pure point dynamical spectrum

A combinatorial condition⇒an analytic condition

A remark

- [Bustos-Mañibo-Yassawi 23+]: similar criterion for one-dimensional S-adic words

Contents

(1)Definitions (done)

(2)The first main result (Overlap algorithm) (done)

(3)The Second main result (binary irreducible one-dim substitutions)

Setting

M: a 2×2 matrix with non-negative integer entries and irreducible characteristic polynomial (fix)

$\rho_1, \rho_2, \ldots, \rho_{n_s}$: one-dimensional binary geometric substitution rules with M as the substitution matrix

$$\begin{pmatrix} \text{\# of 0 in the image of 0} & \text{\# of 0 in the image of 1} \\ \text{\# of 1 in the image of 0} & \text{\# of 1 in the image of 1} \end{pmatrix} = M$$

(Take a left PF eigenvector (l_1, l_2)

$\rightsquigarrow$two tiles $[0, l_1], [0, l_2]$ form the alphabet)

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Take a directive sequence $i_1, i_2, \ldots \in \{1,2,\ldots,n_s\}$ and consider an S-adic tiling $\mathcal{T}_1$ belonging to $(i_k)_k$.

$$\mathcal{T}_1 \underset{\rho_{i_1}}{\leftarrow} \mathcal{T}_2 \underset{\rho_{i_2}}{\leftarrow} \mathcal{T}_3 \underset{\rho_{i_3}}{\leftarrow} \cdots$$

The second main theorem

Take a directive sequence $i_1, i_2, \ldots \in \{1,2,\ldots,n_s\}$ and consider an S-adic tiling $\mathscr{T}_1$ belonging to $(i_k)_k$.

$$\mathscr{T}_1 \underset{\rho_{i_1}}{\leftarrow} \mathscr{T}_2 \underset{\rho_{i_2}}{\leftarrow} \mathscr{T}_3 \underset{\rho_{i_3}}{\leftarrow} \cdots$$

Assumption

$\mathscr{T}_1$ is repetitive and has uniform patch frequency, and

The Perron-Frobenius eigenvalue λ for M is a Pisot number

Theorem

The tiling $\mathscr{T}_1$ has pure point spectrum.

(Single symbolic substitution case: Barge-Diamond 2002)

A related result

Theorem(Berthé-Minervino-Steiner-Thuswaldner 2016)

Under an assumption, generic symbolic S-adic Pisot conjecture for unimodular substitutions with two letters holds. (Matrix not fixed)

Thank you for your attention.