On Presburger arithmetic extended with multiple powers

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These problems can be encoded in the MSO theory of $\langle \mathbb{N}; <, \mathsf{Pow}_2, \mathsf{Pow}_3 \rangle$

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We can answer such questions

Presburger arithmetic

Theorem (Presburger, 1929)

The theory of the structure $\langle \mathbb{N}; 0, 1, <, + \rangle$ *is decidable*

Extensions of Presburger arithmetic

The following extensions of Presburger arithmetic are decidable:

- $\langle \mathbb{N}; 0, 1, <, +, V_2 \rangle$ (Büchi, Bruyère 1960, 1994)
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- $\langle \mathbb{N}; 0, 1, <, +, \mathrm{Pow}_2, \mathrm{Pow}_3 \rangle$ (Hieronymi, Schulz 2023)

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The $\exists\forall\exists$*-fragment of* $\langle \mathbb{N}; 0, 1, <, +, \mathsf{Pow}_2, \mathsf{Pow}_3 \rangle$ *is undecidable*

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Adding multiple powers to Presburger arithmetic

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Lemma

Sufficient to decide whether there exist $m_1, \ldots, m_k, n_1, \ldots, n_\ell$ such that

$$A\,\mathbf{z} > \mathbf{0} \wedge C\,\mathbf{z} = \mathbf{d}$$

for $\mathbf{z} = (2^{m_1}, \ldots, 2^{m_k}, 3^{n_1}, \ldots, 3^{n_\ell})^\top$ and the rest given matrices/vectors

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Apply Baker's Theorem twice:

$$\begin{aligned} 2^m - 5 \cdot 3^{n_1} \qquad\qquad\qquad &= 8 + 5 \cdot 3^{n_2} \\ 2^m - 5 \cdot 3^{n_1} + 15 \cdot 3^{n_2} &= 8 \end{aligned}$$

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$$\begin{aligned} n_1 - C_1 \cdot \log(n_1) \qquad\qquad\qquad &< n_2 \\ n_1 - C_2 \cdot \log(n_1) \cdot (n_1 - n_2) &< 0 \end{aligned}$$

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$$C_1 \cdot C_2 \cdot \log^2(n_1) > n_1$$

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This set is **semilinear**

Theorem

The solution to $C\mathbf{z} = \mathbf{d}$ is $\mathbf{z} = (2^{m_1}, \ldots, 2^{m_k}, 3^{n_1}, \ldots, 3^{n_\ell})$ forms a semilinear set which is effectively computable

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For some $\kappa, \delta > 0$, there is a solution when $2^{m_1}/3^{n_1} \in (\kappa - \delta, \kappa + \delta)$.

Assume that

$$2^{m_1}, 3^{n_1} \geq 3^{n_2} \geq 2^{m_i}, 3^{n_j} \text{ for } i \geq 2, j \geq 3$$

$$v_-^\top \begin{pmatrix} 2^{m_2} \\ \vdots \\ 2^{m_k} \\ 3^{n_2} \\ \vdots \\ 3^{n_\ell} \end{pmatrix} < a2^{m_1} - b3^{n_1} < v_+^\top \begin{pmatrix} 2^{m_2} \\ \vdots \\ 2^{m_k} \\ 3^{n_2} \\ \vdots \\ 3^{n_\ell} \end{pmatrix} \qquad B \begin{pmatrix} 2^{m_2} \\ \vdots \\ 2^{m_k} \\ 3^{n_2} \\ \vdots \\ 3^{n_\ell} \end{pmatrix} > \mathbf{0} \qquad (\star)$$

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Lemma

$(\star)$ has a solution if and only if the following system has one:

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Find $\varepsilon, \delta, \kappa, \lambda > 0$ such that when $2^{m_2}/3^{n_2} \in (\kappa - \delta, \kappa + \delta)$, $3^{n_2} > 2^{m_i}, 3^{n_j}$

$$v_-^\top \begin{pmatrix} 2^{m_2} \\ \vdots \\ 2^{m_k} \\ 3^{n_2} \\ \vdots \\ 3^{n_\ell} \end{pmatrix} < (\lambda - \varepsilon)3^{n_2} < (\lambda + \varepsilon)3^{n_2} < v_+^\top \begin{pmatrix} 2^{m_2} \\ \vdots \\ 2^{m_k} \\ 3^{n_2} \\ \vdots \\ 3^{n_\ell} \end{pmatrix} \qquad B \begin{pmatrix} 2^{m_2} \\ \vdots \\ 2^{m_k} \\ 3^{n_2} \\ \vdots \\ 3^{n_\ell} \end{pmatrix} > \mathbf{0}$$

For given $a, b, \delta, \varepsilon, \kappa, \lambda > 0$ solve

$$\begin{aligned} 3^{n_1} &\geq 3^{n_2} \\ 2^{m_2}/3^{n_2} &\in (\kappa - \delta, \kappa + \delta) \quad \text{and} \\ (\lambda - \varepsilon)3^{n_2} &< a2^{m_1} - b3^{n_1} < (\lambda + \varepsilon)3^{n_2} \end{aligned}$$

For given $\widetilde{b}, \delta, \widetilde{\varepsilon}, \kappa, \widetilde{\lambda} > 0$ solve

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For some D, there is a $n_2 \in \{N, \dots, N + D - 1\}$ and a m_2 such that $2^{m_2}/3^{n_2} \in (\kappa - \delta, \kappa + \delta)$

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What about adding functions?

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Much harder! We access the base-2 and base-3 expansions of $\log_2(3)$

$$\alpha := \log_2(3)_2 = 1.1001010111000000000110100011100111111101111010\cdots$$
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Let $w \in \{0, 1\}^*$

- Does w appear in α?

What about adding functions?

The existential fragment of $\langle \mathbb{N}; 0, 1, <, +, x \mapsto 2^x, x \mapsto 3^x \rangle$

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Find an underlying dynamical system

- expansion of a real number
- continued fraction of a real number
- automaton
- Diophantine equation
- rotation on a torus